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Generalized fractional Brownian motion

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Abstract We introduce a new Gaussian process, a generalization of both fractional and subfractional Brownian motions, which could serve as a good model for a larger class of natural phenomena. We study its main stochastic properties and some increments characteristics. As an application, we deduce the properties of nonsemimartingality, Hölder continuity, nondifferentiablity, and existence of a local time.

Keywords Generalized fractional and subfractional Brownian motion, stationarity, Markovity, semimartingality

1 Introduction

It is well known that the two-sided fractional Brownian motion (tsfBm) with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$, defined on a probability space $(\Omega, F, \mathbb{P})$, with the covariance function

$$Cov\big(B_t^H, B_s^H\big) = \frac{1}{2}\big(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big). \tag{1}$$

When $H = \frac{1}{2}$, B^H is a two-sided Brownian motion (tsBm). The self-similarity and stationarity of the increments are two main properties because of which fBm enjoyed success as a modeling tool in many areas, such as finance, hydrology, biology, and telecommunications.

In [3], the authors suggested another kind of extension of the Bm, called the subfractional Brownian motion (sfBm), which preserves most of the properties of the fBm, but not the stationarity of the increments. It is a centered Gaussian process $\xi^H = \{\xi_t^H, t \in [0, \infty)\}$ with the covariance function

$$S(t,s) = t^{2H} + s^{2H} - \frac{1}{2}\big((t+s)^{2H} + |t-s|^{2H}\big), \tag{2}$$

with $H \in (0,1)$. The case $H = \frac{1}{2}$ corresponds to the Bm.

The sfBm is intermediate between Brownian motion and fractional Brownian motion in the sense that it has properties analogous to those of fBm, but the increments on nonoverlapping intervals are more weakly correlated, and their covariance decays polynomially at a higher rate. So the sfBm does not generalize the fBm.

One extension of the sfBm was introduced in [9] and called the generalized subfractional Brownian motion (GsfBm). It is a centered Gaussian process starting from zero with covariance function

$$G(t,s) = \big(t^{2H} + s^{2H}\big)^K - \frac{1}{2}\big((t+s)^{2HK} + |t-s|^{2HK}\big),$$

with $H \in (0,1)$ and $K \in [1,2)$. The case $K = 1$ corresponds to the sfBm.

Another type of a generalized form of the sfBm was introduced in [11] and [4] as a linear combination of a finite number of independent subfractional Brownian motions. It was called the mixed subfractional Brownian motion (msfBm). The msfBm is a centered mixed self-similar[1] Gaussian process and does not have stationary increments. Both GsfBm and msfBm do not generalize the fBm.

On the other hand, in the literature, we find many different kinds of extensions of the fBm, such as the multifractional Brownian motion [7], the mixed fractional Brownian motion [10], and the bifractional Brownian motion [5]. But none of them generalizes the sfBm.

In this paper, we introduce a new stochastic process, which is an extension of both subfractional Brownian motion and fractional Brownian motion. This process is completely different from all the other extensions existing in the literature; we call it the generalized fractional Brownian motion. More precisely, let us take two real constants a and b such that $(a,b) \neq (0,0)$ and $H \in (0,1)$.

Definition 1. A generalized fractional Brownian motion (gfBm) with parameters a, b, and H, is a process $Z^H = \{Z_t^H(a,b); t \geq 0\} = \{Z_t^H; t \geq 0\}$ defined on the probability space $(\Omega, F, \mathbb{P})$ by

$$\forall t \in \mathbb{R}_+, \qquad Z_t^H = Z_t^H(a,b) = aB_t^H + bB_{-t}^H, \tag{3}$$

where $(B_t^H)_{t\in\mathbb{R}}$ is a two-sided fractional Brownian motion with parameter H.

If $a = 1, b = 0$, then $Z^H = \{B_t^H; t \geq 0\}$, that is, Z^H is a fractional Brownian motion. If $a = b = \frac{1}{\sqrt{2}}$, then it is easy to see, either by a direct calculation using (1) or by Lemma 1, that the covariance of process Z^H is precisely $S(t,s)$ given by (2).

[1] The mixed self-similarity property was introduced by Zili [10].

So, in this case, Z^H is a subfractional Brownian motion. If $a = b = \frac{1}{\sqrt{2}}$ and $H = \frac{1}{2}$ or if $a = 1, b = 0$, and $H = \frac{1}{2}$, G^H is clearly a standard Brownian motion.

So the gfBm is, at the same time, a generalization of the fractional Brownian motion, of the subfractional Brownian motion, and of course of the standard Brownian motion. This is an important motivation for the introduction of such a process since it allows to deal with a larger class of modeled natural phenomena, including those with stationary or nonstationary increments.

This paper contains two sections. In the first one, we investigate the main stochastic properties of the gfBm. We show in particular that the gfBm is a Gaussian, self-similar, and non-Markov process (except in the case where $H = 1/2$). The second section is devoted to the investigation of some characteristics of increments of gfBms. As an application, we deduce the properties of nonsemimartingality, Hölder continuity, nondifferentiablity, and existence of a local time.

2 The main properties

By the Gaussianity of $B^H$and by Eq. (1) we easily get the following lemma.

Lemma 1. *The gfBm $(Z_t^H(a,b))_{t\in\mathbb{R}_+}$ satisfies the following properties:*

- *Z^H is a centered Gaussian process.*
- $\forall s \in \mathbb{R}_+,\ \forall t \in \mathbb{R}_+,$

$$\begin{aligned}&\operatorname{Cov}\bigl(Z_t^H(a,b), Z_s^H(a,b)\bigr)\\&\quad= \frac{1}{2}(a+b)^2\bigl(s^{2H}+t^{2H}\bigr) - ab(t+s)^{2H} - \frac{a^2+b^2}{2}|t-s|^{2H}.\end{aligned}$$

- $\forall t \in \mathbb{R}_+,$

$$E\bigl(Z_t^H(a,b)^2\bigr) = \bigl(a^2+b^2-\bigl(2^{2H}-2\bigr)ab\bigr)t^{2H}.$$

The following lemma deals with the self-similarity property.

Lemma 2. *The gfBm is a self-similar process.*

Proof. This follows from the fact that, for fixed $h > 0$, the processes $\{Z_{ht}^H(a,b); t \geq 0\}$ and $\{h^H Z_t^H(a,b); t \geq 0\}$ are Gaussian, centered, and have the same covariance function. □

Let us now study the non-Markov property of the gfBm.

Proposition 1. *For all $H \in (0;1)\setminus\{\frac{1}{2}\}$ and $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $(Z_t^H(a,b))_{t\in\mathbb{R}_+}$ is not a Markov process.*

Proof. We will only prove the proposition in the case where $b \neq 0$; the result with $b = 0$ is known (see [10] and the references therein). The process Z^H is a centered Gaussian, and for all $t > 0$,

$$\operatorname{Cov}\bigl(Z_t^H, Z_t^H\bigr) = \bigl(a^2+b^2-\bigl(2^{2H}-2\bigr)ab\bigr)t^{2H} > 0.$$

Then, if Z^H were a Markov process, according to [8], for all $s < t < u$, we would have

$$\operatorname{Cov}(Z_s^H, Z_u^H)\operatorname{Cov}(Z_t^H, Z_t^H) = \operatorname{Cov}(Z_s^H, Z_t^H)\operatorname{Cov}(Z_t^H, Z_u^H). \tag{4}$$

In the particular case where $1 < s = \sqrt{t} < t < u = t^2$, we have

$$\begin{aligned}
&\left[\frac{1}{2}(a+b)^2\big(1+t^{-3H}\big) - ab\big(1+t^{-3/2}\big)^{2H} - \frac{a^2+b^2}{2}\big(1-t^{-3/2}\big)^{2H}\right] \\
&\quad\times\big[a^2+b^2) - \big(2^{2H}-2\big)ab\big] \\
&\quad= \left[\frac{1}{2}(a+b)^2\big(1+t^{-H}\big) - ab\big(1+t^{-1/2}\big)^{2H} - \frac{a^2+b^2}{2}\big(1-t^{-1/2}\big)^{2H}\right] \\
&\quad\times\left[\frac{1}{2}(a+b)^2\big(1+t^{-2H}\big) - ab\big(1+t^{-1}\big)^{2H} - \frac{a^2+b^2}{2}\big(1-t^{-1}\big)^{2H}\right].
\end{aligned}$$

So

$$\begin{aligned}
&\left[\frac{1}{2}(a+b)^2\big(1+t^{-3H}\big) - ab\big(1+2Ht^{-3/2}+H(2H-1)t^{-3}+o\big(t^{-3}\big)\big)\right. \\
&\quad\left.-\frac{a^2+b^2}{2}\big(1-2Ht^{-3/2}+H(2H-1)t^{-3}+o\big(t^{-3}\big)\big)\right] \\
&\quad\times\big[a^2+b^2-\big(2^{2H}-2\big)ab\big] \\
&\quad=\left[\frac{1}{2}(a+b)^2\big(1+t^{-H}\big) - ab\big(1+2Ht^{-1/2}+H(2H-1)t^{-1}+o\big(t^{-1}\big)\big)\right. \\
&\quad\left.-\frac{a^2+b^2}{2}\big(1-2Ht^{-1/2}+H(2H-1)t^{-1}+o\big(t^{-1}\big)\big)\right] \\
&\quad\times\left[\frac{1}{2}(a+b)^2\big(1+t^{-2H}\big) - ab\big(1+2Ht^{-1}+H(2H-1)t^{-2}+o\big(t^{-2}\big)\big)\right. \\
&\quad\left.-\frac{a^2+b^2}{2}\big(1-2Ht^{-1}+H(2H-1)t^{-2}+o\big(t^{-2}\big)\big)\right].
\end{aligned}$$

First case: $0 < H < \frac{1}{2}$, $a+b \neq 0$. By Taylor's expansion we get, as $t \to \infty$,

$$\frac{1}{2}(a+b)^2\big[a^2+b^2-\big(2^{2H}-2\big)ab\big]t^{-3H} \approx \frac{1}{4}(a+b)^4 t^{-3H},$$

which is true if and only if

$$\frac{(a-b)^2}{2} - \big(2^{2H}-2\big)ab = 0.$$

However, it is easy to check that $\frac{(a-b)^2}{2} - (2^{2H}-2)ab > 0$ for fixed b and every real a.

Second case: $0 < H < \frac{1}{2}$ and $a+b=0$. By Taylor's expansion we get, as $t \to \infty$,

$$\big[a^2+b^2-\big(2^{2H}-2\big)ab\big]t^{-3/2} \approx \big(-2Hab+\big(a^2+b^2\big)H\big)t^{-3/2},$$

which is true if and only if $a=b=0$, which is false.

Third case: $\frac{1}{2} < H < 1$, $a - b \neq 0$. By Taylor's expansion we get, as $t \to \infty$,

$$H(a-b)^2\big[a^2+b^2-\big(2^{2H}-2\big)ab\big]t^{-3/2} \approx H^2(a-b)^4t^{-3/2},$$

which is true if and only if

$$a^2(1-H)+b^2(1-H)+ab\big(2-2^{2H}+2H\big)=0.$$

However, it is easy to check that $a^2(1-H)+b^2(1-H)+ab(2-2^{2H}+2H)>0$ for fixed b and every real a.

Fourth case: $\frac{1}{2} < H < 1$ and $a-b=0$. By Taylor's expansion we get, as $t \to \infty$,

$$\frac{1}{2}(a+b)^2\big[a^2+b^2-\big(2^{2H}-2\big)ab\big]t^{-3H} \approx \frac{1}{4}(a+b)^4t^{-3H},$$

which is true if and only if $2-2^{2H}=0$, which is false since $H \neq \frac{1}{2}$. The proof of Lemma 1 is complete. □

3 Study of increments and some applications

Let us start by the following lemma, in which we characterize the second moment increments of the gfBm.

Lemma 3. *For all $(s,t) \in \mathbb{R}_+^2$ such that $s \leq t$:*

1. $$E\big(Z_t^H(a,b)-Z_s^H(a,b)\big)^2 = \big(a^2+b^2\big)|t-s|^{2H} - 2^{2H}ab\big(|t|^{2H}+|s|^{2H}\big)+2ab|t+s|^{2H};$$

2. $$\gamma(a,b,H)(t-s)^{2H} \leq E\big(Z_t^H(a,b)-Z_s^H(a,b)\big)^2 \leq \nu(a,b,H)(t-s)^{2H},$$

where

$$\gamma(a,b,H)=\big(a^2+b^2-2ab\big(2^{2H-1}-1\big)\big)\mathbf{1}_{\mathcal{C}}(a,b,H)+\big(a^2+b^2\big)\mathbf{1}_{\mathcal{D}}(a,b,H),$$

and

$$\nu(a,b,H)=\big(a^2+b^2\big)\mathbf{1}_{\mathcal{C}}(a,b,H)+\big(a^2+b^2-2ab\big(2^{2H-1}-1\big)\big)\mathbf{1}_{\mathcal{D}}(a,b,H),$$

$$\mathcal{C}=\left\{(a,b,H)\in\mathbb{R}^2\setminus\big\{(0,0)\big\}\times]0,1[;\ \left(H>\frac{1}{2}, ab\geq 0\right)\ \textit{or}\ \left(H<\frac{1}{2}, ab\leq 0\right)\right\},$$

$$\mathcal{D}=\left\{(a,b,H)\in\mathbb{R}^2\setminus\big\{(0,0)\big\}\times]0,1[;\ \left(H>\frac{1}{2}, ab\leq 0\right)\ \textit{or}\ \left(H<\frac{1}{2}, ab\geq 0\right)\right\}.$$

Moreover, the constants in the inequalities of statement 2 are the best possible.

Proof. The first statement is a direct consequence of Eqs. (1) and (3). So we will just check the second one. We will do that in the case where $H > \frac{1}{2}$ and $ab \geq 0$; the proof in the remaining cases is similar.

Since the function $x \longmapsto x^{2H}$ is convex on $\mathbb{R}_+$, we have

$$2^{2H-1}\big(t^{2H}+s^{2H}\big)-(t+s)^{2H} \geq 0,$$

which yields that

$$E\big(Z_t^H(a,b)-Z_s^H(a,b)\big)^2 \leq \big(a^2+b^2\big)|t-s|^{2H}.$$

To get the lower bound, we consider the function

$$f_\lambda : x \longmapsto 2^{2H-1}\big((s+x)^{2H}+s^{2H}\big)-(2s+x)^{2H}-\lambda x^{2H},$$

with $\lambda > 0$. We have $f_\lambda(0)=0$. So to get the stated lower bound, it suffices to check that, for $\lambda = 2^{2H-1}-1$, f_λ is a decreasing function. But here we show more; we look for the lower bound ν (resp. the upper bound γ) of the set of reals $\lambda > 0$ such that f_λ decreases (resp. f_λ increases) for $x > 0$, which will give us the maximum $\gamma \geq 0$ and the minimum $\nu > 0$ such that

$$\gamma(t-s)^{2H} \leq 2^{2H-1}\big(t^{2H}+s^{2H}\big)-(t+s)^{2H} \leq \nu(t-s)^{2H}.$$

The function f_λ is differentiable in $\mathbb{R}_+^\star$, and for all $x > 0$,

$$f'_\lambda(x) = 2H\big[2^{2H-1}(s+x)^{2H-1}-(2s+x)^{2H-1}-\lambda x^{2H-1}\big].$$

So

$$\begin{aligned} f'_\lambda(x) < 0 \quad &\Longleftrightarrow \quad 2^{2H-1}(s+x)^{2H-1}-(2s+x)^{2H-1} < \lambda x^{2H-1} \\ &\Longleftrightarrow \quad \frac{2^{2H-1}(s+x)^{2H-1}-(2s+x)^{2H-1}}{x^{2H-1}} < \lambda. \end{aligned}$$

Denote $g(x) = \frac{2^{2H-1}(s+x)^{2H-1}-(2s+x)^{2H-1}}{x^{2H-1}}$, that is,

$$g(x) = 2^{2H-1}\left(\frac{s}{x}+1\right)^{2H-1}-\left(2\frac{s}{x}+1\right)^{2H-1}.$$

Putting $X = \frac{s}{x}$, we can write $g(x) = h(X)$ where

$$h(X) = 2^{2H-1}(X+1)^{2H-1}-(2X+1)^{2H-1}.$$

The function h is continuous and strictly decreasing. Consequently,

$$h\big(]0,+\infty[\big) = \Big]\lim_{X\to+\infty} h;\ \lim_{X\to 0} h\Big[=]0; 2^{2H-1}-1[.$$

So, $\sup_{x>0} g = 2^{2H-1}-1$ and $\inf_{x>0} g = 0$. We deduce that the lower bound $\nu > 0$ and the upper bound $\gamma > 0$ such that

$$\gamma(t-s)^{2H} \leq 2^{2H-1}\big(t^{2H}+s^{2H}\big)-(t+s)^{2H} \leq \nu(t-s)^{2H}$$

are $\nu = 2^{2H-1}-1$ and $\gamma = 0$. $\square$

The first main application of Lemma 3 is the following result.

Lemma 4. *For every* $H \in]0, 1[\setminus \{\frac{1}{2}\}$*, the gfBm is not a semimartingale.*

Proof. By Lemma 2.1 of [2] this result is a direct consequence of Lemma 3. □

The following result is the second important consequence of Lemma 3. It deals with the continuity and nondifferentiablity of the gfBm sample paths.

Lemma 5. *Let* $H \in (0, 1)$.

1. *The gfBm* Z^H *admits a version whose sample paths are almost surely Hölder continuous of order strictly less than* H.

2. *For every* $H \in]0; 1[^N$,

$$\lim_{\epsilon \to 0^+} \sup_{t \in [t_0-\epsilon, t_0+\epsilon]} \left| \frac{Z^H(t) - Z^H(t_0)}{t - t_0} \right| = +\infty, \tag{5}$$

with probability one for every $t_0 \in \mathbb{R}$.

Proof. The first statement follows by the Kolmogorov criterion from Lemma 3.

The second one is easily obtained by using the specific expression of the second moment of the increments of the gfBm (10) and by following exactly the same strategies as in the proofs of Lemma 4.1 and Lemma 4.2 in [11]. □

As the third consequence of Lemma 3, we see that if $b \neq 0$, then the gfBm does not have stationary increments, but this property is replaced by the inequalities appeared in statement 2 of Lemma 3. In order to understand how far is the gfBm from a process with stationary increments, we will compare the gfBm increments and the fBm increments. To reach this goal, let us first recall that, in the fractional Brownian motion case ($a = 1, b = 0$), we have, for all $p \in \mathbb{N}$ and $n \geq 0$,

$$\begin{aligned} \mathbb{E}\big(\big(B^H_{p+1} - B^H_p\big)\big(B^H_{p+n+1} - B^H_{p+n}\big)\big) &= \mathbb{E}\big(B^H_1\big(B^H_{n+1} - B^H_n\big)\big) \\ &= \frac{1}{2}\big[(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\big] \\ &= R_B(0, n). \end{aligned}$$

Denote

$$R_Z(p, p+n) = \mathbb{E}\big(\big(Z^H_{p+1} - Z^H_p\big)\big(Z^H_{p+n+1} - Z^H_{p+n}\big)\big), \quad p \geq 1. \tag{6}$$

In the following proposition, we compute the term $R_Z(p, p+n)$, showing how different it is from $R_B(0, n)$.

Proposition 2. *For every* $n \geq 1$*, we have, as* $p \to \infty$*,*

$$R_Z(p, p+n) = \big(a^2 + b^2\big)R_B(0, n) - ab\big(2^{2H-1}H(2H-1)\big)p^{2(H-1)}\big(1 + o(1)\big),$$

and, consequently,

$$\lim_{p \to \infty} R_Z(p, p+n) = \big(a^2 + b^2\big)R_B(0, n). \tag{7}$$

Proof. An easy calculus allows us to get

$$\begin{aligned}R_Z(p,p+n) &= \frac{a^2+b^2}{2}\big[(n+1)^{2H}-2n^{2H}+(n-1)^{2H}\big]\\&\quad -ab\big[(2p+n+2)^{2H}-2(2p+n+1)^{2H}+(2p+n)^{2H}\big]\\&=\big(a^2+b^2\big)R_B(0,n)-abf_p(n),\end{aligned}\tag{8}$$

for every $n \geq 1$, where $f_p(n)=(2p+n+2)^{2H}-2(2p+n+1)^{2H}+(2p+n)^{2H}$.

By Taylor's expansion we have, as $p \to \infty$,

$$\begin{aligned}f_p(n) &= (2p)^{2H}\left[\left(1+\frac{n+2}{2p}\right)^{2H}-2\left(1+\frac{n+1}{2p}\right)^{2H}+\left(1+\frac{n}{2p}\right)^{2H}\right]\\&=\big(2^{2H-1}H(2H-1)\big)p^{2(H-1)}\big(1+o(1)\big).\end{aligned}$$

Since $H<1$, the last term tends to 0 as p goes to infinity. □

Remark 1. If $a \neq 0$ and $b \neq 0$, then the gfBm increments are not stationary. The meaning of the proposition is that they converge to a stationary sequence.

Now, we are interested in the behavior of the gfBm increments with respect to n (as $n \to \infty$) and, in particular, in the long-range dependence of the process Z^H.

Definition 2. We say that the increments of a stochastic process X are long-range dependent if for every integer $p \geq 1$, we have

$$\sum_{n\geq 1} R_X(p,p+n)=\infty,$$

where $R_X(p,p+n)=\mathbb{E}((X_{p+1}-X_p)(X_{p+n+1}-X_{p+n}))$.

Theorem 1. *For every* $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$, *the increments of* $Z^H(a,b)$ *are long-range dependent if and only if* $H>\frac{1}{2}$ *and* $a\neq b$.

Proof. For every integer $p \geq 1$, by Taylor's expansion, as $n\to\infty$, we have

$$\begin{aligned}R_Z(p,p+n) &= \frac{a^2+b^2}{2}n^{2H}\left[\left(1+\frac{1}{n}\right)^{2H}-2+\left(1-\frac{1}{n}\right)^{2H}\right]\\&\quad -abn^{2H}\left[\left(1+\frac{2p+2}{n}\right)^{2H}-2\left(1+\frac{2p+1}{n}\right)^{2H}+\left(1+\frac{2p}{n}\right)^{2H}\right]\\&=H(2H-1)n^{2H-2}(a-b)^2\\&\quad -4H(2H-1)(H-1)ab(2p+1)n^{2H-3}\big(1+o(1)\big).\end{aligned}$$

If $a\neq b$, we see that as $n\to\infty$, $R_Z(p,p+n)\approx H(2H-1)n^{2H-2}(a-b)^2$. Then

$$\sum_{n\geq 1}R_Z(p,p+n)=\infty \quad\Longleftrightarrow\quad 2H-2>-1 \quad\Longleftrightarrow\quad H>\frac{1}{2}.$$

If $a = b$, then, as $n \to \infty$, $R_Z(p, p+n) \approx 4H(2H-1)(1-H)a^2(2p+1)n^{2H-3}$. For every $H \in]0, 1[$, we have $2H - 3 < -1$ and, consequently,

$$\sum_{n \geq 1} R_Z(p, p+n) < \infty. \qquad \square$$

Remark 2. We know that the subfractional Brownian motion increments $Z^H(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ are short-range dependent for every $H \in]0, 1[$. From the theorem we see another important motivation of the investigation of the general process $Z^H(a, b)$ with a and b not necessary equal; it allows us to exhibit models taking into account not only short-range dependence of the increments, but also phenomena of long-range dependence if it exists.

Lemma 3 has also an immediate application to local time of the gfBm.

Corollary 1. *For* $0 < H < 1$, *on each* (*time*)-*interval* $[0, T] \subset [0, +\infty[$, *the process* $(Z_t^H)_{0 \leq t \leq T}$ *admits a local time* $L^H([0, T], x)$, *which satisfies*

$$\int_{\mathbb{R}} L^H\big([0, T], x\big)^2\, dx < \infty.$$

Proof. First, denote, for $s, t \in \mathbb{R}_+$ and $s \neq t$, by $\varphi_{Z_t^H - Z_s^H}$ the characteristic function of the random variable $Z_t^H - Z_s^H$ and by $p^H(x; t, s)$ its probability density function. They are expressed by

$$\varphi_{Z_t^H - Z_s^H}(u) = \mathbb{E}\big[\exp\big(iu\big(Z_t^H - Z_s^H\big)\big)\big] = \exp\bigg(-\frac{u^2}{2}\mathbb{E}\big(Z_t^H - Z_s^H\big)^2\bigg), \tag{9}$$

and

$$p^H(x; s, t) = 1/\sqrt{2\pi\mathbb{E}\big(Z_t^H - Z_s^H\big)^2}\, \exp\big(-x^2/2\mathbb{E}\big(Z_t^H - Z_s^H\big)^2\big). \tag{10}$$

It is clear that

$$\int_{-\infty}^{\infty} \big|\varphi_{Z_t^H - Z_s^H}(u)\big|\, du < \infty,$$

and by Lemma 3 and the fact that $0 < H < 1$, for every $T > 0$,

$$\int_0^T \int_0^T p^H(0; s, t)\, ds\, dt \leq c \int_0^T \int_0^T |t - s|^{-H}\, ds\, dt < \infty,$$

where c is positive constant.

So, by Lemma 3.2 of [1] the local time of $(Z_t^H)_{0 \leq t \leq T}$ exists almost surely, and it is square integrable as a function of u. $\square$

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