# Functional limit theorems for additive and multiplicative schemes in the Cox–Ingersoll–Ross model

**Yuliia Mishura, Yevheniia Munchak**\*

*Taras Shevchenko National University of Kyiv, Volodymyrska str. 64, 01601, Kyiv, Ukraine*

myus@univ.kiev.ua (Yu. Mishura), yevheniamunchak@gmail.com (Ye. Munchak)

**Abstract**  In this paper, we consider the Cox–Ingersoll–Ross (CIR) process in the regime where the process does not hit zero. We construct additive and multiplicative discrete approximation schemes for the price of asset that is modeled by the CIR process and geometric CIR process. In order to construct these schemes, we take the Euler approximations of the CIR process itself but replace the increments of the Wiener process with iid bounded vanishing symmetric random variables. We introduce a "truncated" CIR process and apply it to prove the weak convergence of asset prices. We establish the fact that this "truncated" process does not hit zero under the same condition considered for the original nontruncated process.

## 1  Introduction

The problem of convergence of discrete-time financial models to the models with continuous time is well developed; see, e.g., [6, 7, 9, 11, 14, 17, 19]. The reason for such an interest can be explained as follows: from the analytical point of view, it

---

\*Corresponding author.

is much simpler to deal with continuous-time models although all real-world models operate in the discrete time. In what concerns the rate of convergence, there can be different approaches to its estimation. Some of this approaches are established in [6, 7, 23–27]. In this paper, we consider the Cox–Ingersoll–Ross process and its approximation on a finite time interval. The CIR process was originally proposed by Cox, Ingersoll, and Ross [8] as a model for short-term interest rates. Nowadays, this model is widely used in financial modeling, for example, as the volatility process in the Heston model [16]. The strong global approximation of CIR process is studied in several articles. Strong convergence (without a rate or with a logarithmic rate) of several discretization schemes is shown by [1, 4, 12, 15, 18]. In [1], a general framework for the analysis of strong approximation of the CIR process is presented along with extensive simulation studies. Nonlogarithmic convergence rates are obtained in [2]. In [10], the author extends the CIR model of the short interest rate by assuming a stochastic reversion level, which better reflects the time dependence caused by the cyclical nature of the economy or by expectations concerning the future impact of monetary policies. In this framework, the convergence of the long-term return by using the theory of generalized Bessel-square processes is studied. In [28], the authors propose an empirical method that utilizes the conditional density of the state variables to estimate and test a term structure model with known price formula using data on both discount and coupon bonds. The method is applied to an extension of a two-factor model due to Cox, Ingersoll, and Ross. Their results show that estimates based solely on bills imply unreasonably large price errors for longer maturities. The process is also discussed in [5].

In this article, we focus on the regime where the CIR process does not hit zero and study weak approximation of this process. In the first case, the sequence of prelimit markets is modeled as the sequence of the discrete-time additive stochastic processes, whereas in the second case, the sequence of multiplicative stochastic processes is modeled. The additive scheme is widely used, for example, in the papers [1, 4, 13]. The papers [10, 28] are recent examples of modeling a stochastic interest rate by the multiplicative model of CIR process. In [10], the authors say that the model has the "strong convergence property," whereas they refer to models as having the "weak convergence property" when the returns converge to a constant, which generally depends upon the current economic environment and that may change in a stochastic fashion over time. We construct a discrete approximation scheme for the price of asset that is modeled by the Cox–Ingersoll–Ross process. In order to construct these additive and multiplicative processes, we take the Euler approximations of the CIR process itself but replace the increments of the Wiener process with iid bounded vanishing symmetric random variables. We introduce a "truncated" CIR process and use it to prove the weak convergence of asset prices.

The paper is organized as follows. In Section 2, we present a complete and "truncated" CIR process and establish that the "truncated" CIR process can be described as the unique strong solution to the corresponding stochastic differential equation. We establish that this "truncated" process does not hit zero under the same condition as for the original nontruncated process. In Section 3, we present discrete approximation schemes for both these processes and prove the weak convergence of asset prices for the additive model. In the next section, we prove the weak convergence of

asset prices for the multiplicative model. Appendix contains additional and technical results.

## 2 Original and "truncated" Cox–Ingersoll–Ross processes and some of their properties

Let $\Omega_{\mathcal{F}} = (\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathsf{P})$ be a complete filtered probability space, and $W = \{W_t, \mathcal{F}_t, t \geq 0\}$ be an adapted Wiener process. Consider a Cox–Ingersoll–Ross process with constant parameters on this space. This process is described as the unique strong solution of the following stochastic differential equation:

$$dX_t = (b - X_t)dt + \sigma \sqrt{X_t}dW_t, \quad X_0 = x_0 > 0, \ t \geq 0, \tag{1}$$

where $b > 0$, $\sigma > 0$. The integral form of the process $X$ has the following form:

$$X_t = x_0 + \int_0^t (b - X_s)ds + \sigma \int_0^t \sqrt{X_s}dW_s.$$

According to the paper [8], the condition $\sigma^2 \leq 2b$ is necessary and sufficient for the process $X$ to get positive values and not to hit zero. Further, we will assume that this condition is satisfied.

For the proof of functional limit theorems, we will need a modification of the Cox–Ingersoll–Ross process with bounded coefficients. This process is called a truncated Cox–Ingerssol–Ross process. Let $C > 0$. Consider the following stochastic differential equation with the same coefficients $b$ and $\sigma$ as in (1):

$$dX_t^C = (b - X_t^C \wedge C)dt + \sigma \sqrt{(X_t^C \vee 0) \wedge C}dW_t, \quad X_0 = x_0 > 0, \ t \geq 0. \tag{2}$$

**Lemma 2.1.** *For any $C > 0$, (2) has a unique strong solution.*

**Proof.** Since the coefficients $\sigma(x) = \sigma \sqrt{(x \vee 0) \wedge C}$ and $b(x) = b - (x \wedge C)$ satisfy the conditions of Theorem A.3 and also the growth condition (14), a global strong solution $X_t^C$ exists uniquely for every given initial value $x_0$. $\qquad\square$

**Remark 2.1.** Denote $\sigma_{-\epsilon} = \inf\{t : X_t^C = -\epsilon\}$ with $\epsilon > 0$ such that $-\epsilon + b > 0$. Suppose that $\mathsf{P}(\sigma_{-\epsilon} < \infty) > 0$. Then for any $r < \sigma_{-\epsilon}$ such that $X_t^C < 0$ for $t \in (r, \sigma_{-\epsilon})$, we would have, with positive probability,

$$dX_t^C = (b - X_t^C \wedge C)dt > 0$$

on the interval $(r, \sigma_{-\epsilon})$, and hence $t \to X_t^C$ would increase in this interval. This is obviously impossible. Therefore, $X_t^C$ is nonnegative and can be written as

$$dX_t^C = (b - X_t^C \wedge C)dt + \sigma \sqrt{X_t^C \wedge C}dW_t, \quad X_0^C = x_0 > 0, \ t \geq 0. \tag{3}$$

The integral form of the process $X^C$ is as follows:

$$X_t^C = x_0 + \int_0^t (b - X_s^C \wedge C)ds + \sigma \int_0^t \sqrt{X_s^C \wedge C} dW_s.$$

**Lemma 2.2.** *Let $2b \geq \sigma^2$ and $C > b \vee 1$. Then the trajectories of the process $X^C$ are positive with probability 1.*

**Proof.** In order to prove that the process $X^C$ is positive, we will use the proof similar to that given in [22, p. 308] for the complete Cox–Ingersoll–Ross process with corresponding modifications. Note that the coefficients $g(x) := \sigma\sqrt{x \wedge C}$ and $f(x) := b - x \wedge C$ of (3) are continuous and $g^2(x) > 0$ on $x \in (0, \infty)$. Fix $\alpha$ and $\beta$ such that $0 < \alpha < x_0 < \beta$. Due to the nonsingularity of $g$ on $[\alpha, \beta]$, there exists a unique solution $F(x)$ of the ordinary differential equation

$$f(x)F'(x) + \frac{1}{2}g^2(x)F''(x) = -1, \quad \alpha < x < \beta,$$

with boundary conditions $F(\alpha) = F(\beta) = 0$, and this solution is nonnegative, which follows from its representation through a nonnegative Green function given in [21, p. 343]. Define the stopping times

$$\tau_\alpha = \inf\{t \geq 0 : X_t^C \leq \alpha\} \quad \text{and} \quad \tau_\beta = \inf\{t \geq 0 : X_t^C \geq \beta\}.$$

By the Itô formula, for any $t > 0$,

$$\mathsf{E}\, F\big(X^C(t \wedge \tau_\alpha \wedge \tau_\beta)\big) = F(x_0) - \mathsf{E}(t \wedge \tau_\alpha \wedge \tau_\beta). \tag{4}$$

This formula and nonnegativity of $F$ imply that

$$\mathsf{E}(t \wedge \tau_\alpha \wedge \tau_\beta) \leq F(x_0)$$

and, as $t \to \infty$,

$$\mathsf{E}(\tau_\alpha \wedge \tau_\beta) \leq F(x_0) < \infty.$$

This means that $X^C$ exits from every compact subinterval of $[\alpha, \beta] \subset (0, \infty)$ in finite time. It follows from the boundary conditions and equality $\mathsf{P}(\tau_\alpha \wedge \tau_\beta < \infty) = 1$ that $\lim_{t\to\infty} \mathsf{E}\, F(X^C(t \wedge \tau_\alpha \wedge \tau_\beta)) = 0$, and then from (4) we have

$$\mathsf{E}(\tau_\alpha \wedge \tau_\beta) = F(x_0).$$

Let us now define the function

$$V(x) = \int_1^x \exp\left\{-\int_1^y \frac{2f(z)}{g^2(z)}dz\right\}dy, \quad x \in (0, \infty),$$

which has a continuous strictly positive derivative $V'(x)$, and the second derivative $V''(x)$ exists and satisfies $V''(x) = -\frac{2f(x)}{g^2(x)}V'(x)$. The Itô formula shows that, for any $t > 0$,

$$V\big(X^C(t \wedge \tau_\alpha \wedge \tau_\beta)\big) = V(x_0) + \int_0^{t \wedge \tau_\alpha \wedge \tau_\beta} V'(X_u^C)g(X_u^C)dW(u)$$

and

$$\mathsf{E}\, V\big(X^C(t \wedge \tau_\alpha \wedge \tau_\beta)\big) = V(x_0).$$

Taking the limit as $t \to \infty$, we get

$$V(x_0) = \mathsf{E}\, V\big(X^C(\tau_\alpha \wedge \tau_\beta)\big) = V(\alpha)\,\mathsf{P}(\tau_\alpha < \tau_\beta) + V(\beta)\,\mathsf{P}(\tau_\beta < \tau_\alpha),$$

and hence

$$\mathsf{P}(\tau_\alpha < \tau_\beta) = \frac{V(\beta) - V(x_0)}{V(\beta) - V(\alpha)} \quad \text{and} \quad \mathsf{P}(\tau_\beta < \tau_\alpha) = \frac{V(x_0) - V(\alpha)}{V(\beta) - V(\alpha)}. \tag{5}$$

Consider the integral

$$V(x) = \int_1^x \exp\left\{ - \int_1^y \frac{2(b - z \wedge C)}{\sigma^2(z \wedge C)}dz \right\}dy.$$

First, consider the case $x < 1$. Then

$$V(x) = \int_1^x \exp\left\{ - \int_1^y \frac{2(b - z)}{\sigma^2 z}dz \right\}dy = \int_1^x y^{-\frac{2b}{\sigma^2}} \exp\left\{ \frac{2(y - 1)}{\sigma^2} \right\}dy,$$

and if $\sigma^2 \leq 2b$, then

$$\lim_{x \downarrow 0} V(x) = -\infty.$$

Now let $x$ increase and tend to infinity. Denote $C_1 = \int_1^C \exp\{\frac{2(y-1)}{\sigma^2}\}y^{-\frac{2b}{\sigma^2}}dy$. Then, for $x > C$,

$$V(x) = \int_1^C \exp\left\{ - \int_1^y \frac{2(b - z)}{\sigma^2 z}dz \right\}dy$$

$$+ \int_C^x \exp\left\{ - \int_1^C \frac{2(b - z)}{\sigma^2 z}dz - \int_C^y \frac{2(b - C)}{\sigma^2 C}dz \right\}dy$$

$$= \int_1^C \exp\left\{ \frac{2(y - 1)}{\sigma^2} \right\}y^{-\frac{2b}{\sigma^2}}dy + C^{-\frac{2b}{\sigma^2}} \exp\left\{ \frac{2(C - 1)}{\sigma^2} \right\}$$

$$\times \int_C^x \exp\left\{ - \frac{2(b - C)}{\sigma^2 C}(y - C) \right\}dy$$

$$= C_1 + C^{-\frac{2b}{\sigma^2}+1} \frac{\sigma^2}{2(C-b)} \exp\left\{\frac{2(C-1)}{\sigma^2}\right\}$$
$$\times \left(\exp\left\{\frac{2(C-b)}{\sigma^2 C}(x-C)\right\} - 1\right),$$

and thus $\lim_{x\uparrow\infty} V(x) = \infty$. Define

$$\tau_0 = \lim_{\alpha\downarrow 0} \tau_\alpha \quad \text{and} \quad \tau_\infty = \lim_{\beta\uparrow\infty} \tau_\beta$$

and put $\tau = \tau_0 \wedge \tau_\infty$. From (5) we get

$$\mathsf{P}\left(\inf_{0\leq t<\tau} X_t^C \leq \alpha\right) \geq \mathsf{P}(\tau_\alpha < \tau_\beta) = \frac{1 - V(x_0)/V(\beta)}{1 - V(\alpha)/V(\beta)},$$

and, as $\beta \uparrow \infty$, we get that, for any $\alpha > 0$, $\mathsf{P}(\inf_{0\leq t<\tau} X_t^C \leq \alpha) = 1$, whence, finally, $\mathsf{P}(\inf_{0\leq t<\tau} X_t^C = 0) = 1$. Similarly, $\mathsf{P}(\sup_{0\leq t<\tau} X_t^C = \infty) = 1$. Assume now that $\mathsf{P}(\tau < \infty) > 0$. Then

$$\mathsf{P}\left(\lim_{t\to\tau} X_t^C \text{ exists and equals } 0 \text{ or } \infty\right) > 0.$$

So the events $\{\inf_{0\leq t<\tau} X_t^C = 0\}$ and $\{\sup_{0\leq t<\tau} X_t^C = \infty\}$ cannot both have proba-bility 1. This contradiction shows that $\mathsf{P}(\tau < \infty) = 0$, whence

$$\mathsf{P}(\tau = \infty) = \mathsf{P}\left(\inf_{0\leq t<\tau} X_t^C = 0\right) = \mathsf{P}\left(\sup_{0\leq t<\tau} X_t^C = \infty\right) = 1$$

if $2b \geq \sigma^2$.                                                                                       □

Now, let $T > 0$ be fixed.

**Lemma 2.3.**
$$\mathsf{P}\{\exists t \in [0,T] : X_t \neq X_t^C\} \to 0$$

*as $C \to \infty$.*

**Proof.** Obviously, it suffices to show that

$$\mathsf{P}\left\{\sup_{t\in[0,T]} |X_t| \geq C\right\} \to 0 \quad \text{as } C \to \infty.$$

It is well known (see, e.g., [29]) that $\frac{4}{\sigma^2(1-e^{-t})} X_t$ follows a noncentral $\chi^2$ distribu-tion with (in general) noninteger degree of freedom $\frac{4b}{\sigma^2}$ and noncentrality parameter $\frac{4}{\sigma^2(1-e^{-t})} x_0 e^{-t}$. The first and second moments for any $t \geq 0$ are given by

$$\mathsf{E}\, X_t = x_0 e^{-t} + b(1 - e^{-t}),$$

$$\mathsf{E}(X_t)^2 = x_0(2b + \sigma^2)e^{-t} + (x_0^2 - x_0\sigma^2 - 2x_0 b)e^{-2t} + \left(\frac{b\sigma^2}{2} + b^2\right)(1 - e^{-t})^2.$$

Therefore, there exists a constant $B > 0$ such that $\mathsf{E}\, X_t^2 \leq B$, whence $\mathsf{E}\, X_t \leq B^{1/2}$, $0 \leq t \leq T$.

Using the Doob inequality, we estimate

$$
\mathsf{P}\left\{ \sup_{t\in[0,T]} |X_t| \geq C \right\} \leq \frac{1}{C^2}\, \mathsf{E} \sup_{t\in[0,T]} X_t^2
$$

$$
= \frac{1}{C^2}\, \mathsf{E} \sup_{t\in[0,T]} \left\{ \left( X_0 + \int_0^t (b - X_s)ds + \sigma \int_0^t \sqrt{X_s}\, dW_s \right)^2 \right\}
$$

$$
\leq \frac{3}{C^2}\left\{ X_0^2 + T\, \mathsf{E}\left( \int_0^T |b - X_s|ds \right)^2 + \sigma^2\, \mathsf{E} \sup_{t\in[0,T]} \left( \int_0^t \sqrt{X_s}\, dW_s \right)^2 \right\}
$$

$$
\leq \frac{3}{C^2}\left\{ X_0^2 + T\, \mathsf{E} \int_0^T (b - X_s)^2 ds + 4\sigma^2\, \mathsf{E} \int_0^T X_s ds \right\} \leq \frac{B_1}{C^2}
$$

for some constant $B_1 > 0$. The lemma is proved. $\qquad\square$

## 3  Discrete approximation schemes for complete and "truncated" Cox–Ingersoll–Ross processes

Consider the following discrete approximation scheme for the process $X$. Assume that we have a sequence of the probability spaces $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathsf{P}^{(n)})$, $n \geq 1$. Let $\{q_k^{(n)}, n \geq 1, 0 \leq k \leq n\}$ be the sequence of symmetric iid random variables defined on the corresponding probability space and taking values $\pm\sqrt{\frac{T}{n}}$, that is, $\mathsf{P}^n(q_k^{(n)} = \pm\sqrt{\frac{T}{n}}) = \frac{1}{2}$. Let further $n > T$. We construct discrete approximation schemes for the stochastic processes $X$ and $X^C$ as follows. Consider the following approximation for the complete process:

$$
X_0^{(n)} = x_0 > 0, \qquad X_k^{(n)} = X_{k-1}^{(n)} + \frac{(b - X_{k-1}^{(n)})T}{n} + \sigma q_k^{(n)} \sqrt{X_{k-1}^{(n)}},
$$

$$
Q_k^{(n)} := X_k^{(n)} - X_{k-1}^{(n)} = \frac{(b - X_{k-1}^{(n)})T}{n} + \sigma q_k^{(n)} \sqrt{X_{k-1}^{(n)}}, \quad 1 \leq k \leq n, \qquad (6)
$$

and the corresponding approximations for $X^C$ given by

$$
X_0^{(n,C)} = x_0 > 0,
$$

$$
X_k^{(n,C)} = X_{k-1}^{(n,C)} + \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} + \sigma q_k^{(n)} \sqrt{X_{k-1}^{(n,C)} \wedge C},
$$

$$
Q_k^{(n,C)} := X_k^{(n,C)} - X_{k-1}^{(n,C)}
$$

$$
= \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} + \sigma q_k^{(n)} \sqrt{X_{k-1}^{(n,C)} \wedge C}, \, 1 \leq k \leq n. \qquad (7)
$$

The following lemma confirms the correctness of the construction of these approximations.

**Lemma 3.1.** *Let, $n > 2T$.*

   1) *If $2b \geq \sigma^2$, then all values given by* (6) *and* (7) *are positive.*

   2) *We have*

$$\mathsf{P}\{\exists k, 0 \leq k \leq n : X_k^{(n)} \neq X_k^{(n,C)}\} \to 0 \tag{8}$$

   *as $C \to \infty$.*

**Proof.** 1) We apply the method of mathematical induction. When $k = 1$,

$$X_1^{(n)} = x_0 + \frac{(b - x_0)T}{n} + \sigma q_1^{(n)} \sqrt{x_0}.$$

Let us show that

$$x_0 + \frac{(b - x_0)T}{n} + \sigma q_1^{(n)} \sqrt{x_0} > 0. \tag{9}$$

We denote $\alpha := \sqrt{x_0}$ and reduce (9) to the quadratic inequality

$$\alpha^2 \left(1 - \frac{T}{n}\right) \pm \sigma \sqrt{\frac{T}{n}} \alpha + \frac{bT}{n} > 0,$$

which obviously holds because the discriminant $D = \frac{\sigma^2 T}{n} - \frac{4bT}{n}(1 - \frac{T}{n}) < 0$ when $\sigma^2 \leq 2b$ and $n > 2T$. So, $X_1^{(n)} > 0$. Assume now that $X_k^{(n)} > 0$. It can be shown by applying the same transformation that when $\sigma^2 \leq 2b$ and $n > 2T$, the values $X_{k+1}^{(n)} > 0$.

   It can be proved similarly that the values given by (7) are positive.

   2) $X_k^{(n)}$ can be represented as

$$X_k^{(n)} = x_0 + \sum_{i=1}^{k}(b - X_{i-1}^{(n)})\frac{T}{n} + \sigma \sum_{i=1}^{k} q_i^{(n)} \sqrt{X_{i-1}^{(n)}}$$

$$= X_{k-1}^{(n)} + \frac{(b - X_{k-1}^{(n)})T}{n} + \sigma q_k^{(n)} \sqrt{X_{k-1}^{(n)}}. \tag{10}$$

Compute

$$\mathsf{E}(X_i^{(n)})^2 = \mathsf{E}\left(X_{i-1}^{(n)}\left(1 - \frac{T}{n}\right) + \frac{bT}{n} + \sigma\sqrt{X_{i-1}^{(n)}}q_i^{(n)}\right)^2$$

$$= \mathsf{E}\left(X_{i-1}^{(n)}\left(1 - \frac{T}{n}\right) + \frac{bT}{n}\right)^2 + \frac{\sigma^2 T}{n}\mathsf{E}\,X_{i-1}^{(n)}$$

$$= \left(\frac{bT}{n}\right)^2 + \left[\frac{\sigma^2 T}{n} + \frac{2bT}{n}\left(1 - \frac{T}{n}\right)\right]\mathsf{E}\,X_{i-1}^{(n)}$$

$$+ \left(1 - \frac{T}{n}\right)^2 \mathsf{E}(X_{i-1}^{(n)})^2. \tag{11}$$

Assume that $\mathsf{E}(X_j^{(n)})^2 \leq \beta^2$, $1 \leq j \leq i - 1$, for some $\beta > 0$. Then $\mathsf{E}\, X_j^{(n)} \leq \beta$, $1 \leq j \leq i - 1$. We get the quadratic inequality of the form

$$\left(1 - \frac{T}{n}\right)^2 \beta^2 + \left[\frac{\sigma^2 T}{n} + \frac{2bT}{n}\left(1 - \frac{T}{n}\right)\right]\beta + \left(\frac{bT}{n}\right)^2 < \beta^2$$

or, equivalently,

$$\left(\left(1 - \frac{T}{n}\right)^2 - 1\right)\beta^2 + \left[\frac{\sigma^2 T}{n} + \frac{2bT}{n}\left(1 - \frac{T}{n}\right)\right]\beta + \left(\frac{bT}{n}\right)^2 < 0,$$

which obviously holds when $\beta > \frac{\sigma^2 + 2b + \sqrt{\sigma^4 + 4b\sigma^2 + 8b^2}}{\frac{3}{2}}$. So, for all $1 \leq i \leq n$,

$$\mathsf{E}\, X_i^{(n)} \leq \frac{\sigma^2 + 2b + \sqrt{\sigma^4 + 4b\sigma^2 + 8b^2}}{\frac{3}{2}} \vee x_0 =: \gamma.$$

Using the Burkholder inequality, we estimate

$$0 \leq \mathsf{E} \sup_{0 \leq k \leq n} \left(X_k^{(n)}\right)^2 \leq 2(x_0 + bT)^2 + 2\sigma^2 \mathsf{E} \sup_{0 \leq k \leq n} \left(\sum_{i=1}^n q_i^{(n)} \sqrt{X_{i-1}^{(n)}}\right)^2$$

$$\leq 2(x_0 + bT)^2 + 8\sigma^2 \mathsf{E}\left(\sum_{i=1}^n q_i^{(n)} \sqrt{X_{i-1}^{(n)}}\right)^2$$

$$\leq 2(x_0 + bT)^2 + 8\sigma^2 \gamma T.$$

Therefore,

$$\mathsf{P}\{\exists k, 0 \leq k \leq n : X_k^{(n)} \neq X_k^{(n,C)}\} = \mathsf{P}\left\{\sup_{0 \leq k \leq n} X_k^{(n)} \geq C\right\} \leq C^{-2} \mathsf{E} \sup_{0 \leq k \leq n} \left(X_k^{(n)}\right)^2$$

$$\leq 2C^{-2}(x_0 + bT)^2 + 8\sigma^2 C^{-2} \gamma T,$$

whence the proof follows. $\qquad\qquad\square$

Consider the sequences of step processes corresponding to these schemes:

$$X_t^{(n)} = X_k^{(n)} \text{ for } \frac{kT}{n} \leq t < \frac{(k+1)T}{n}$$

and

$$X_t^{(n,C)} = X_k^{(n,C)} \text{ for } \frac{kT}{n} \leq t < \frac{(k+1)T}{n}.$$

Thus, the trajectories of the processes $X^{(n)}$ and $X^{(n,C)}$ have jumps at the points $kT/n$, $k = 0, \ldots, n$, and are constant on the interior intervals. Consider the filtrations $\mathcal{F}_k^n = \sigma(X_t^{(n)}, t \leq \frac{kT}{n})$. The processes $X^{(n,C)}$ are adapted with respect to them. Therefore, we can consider the same filtrations for all discrete approximation schemes. So, we can identify $\mathcal{F}_t^n$ with $\mathcal{F}_k^n$ for $\frac{kT}{n} \leq t < \frac{(k+1)T}{n}$.

**Remark 3.1.** Now we can rewrite relation (8) as follows:

$$\mathsf{P}\{\exists t, t \in [0, T] : X_t^{(n)} \neq X_t^{(n,C)}\} \to 0$$

as $C \to \infty$.

Denote by $\mathbb{Q}$ and $\mathbb{Q}^n$, $n \geq 1$, the measures corresponding to the processes $X$ and $X^{(n)}$, $n \geq 1$, respectively, and by $\mathbb{Q}^C$ and $\mathbb{Q}^{n,C}$, $n \geq 1$, the measures corresponding to the processes $X^C$ and $X^{(n,C)}$, $n \geq 1$, respectively. Denote by $\overset{W}{\longrightarrow}$ the weak convergence of measures corresponding to stochastic processes. We apply Theorem 3.2 from [23] to prove the weak convergence of measures $\mathbb{Q}^{n,C}$ to the measure $\mathbb{Q}^C$. This theorem can be formulated as follows.

**Theorem 3.1.** *Assume that the following conditions are satisfied:*

(i) *For any $\epsilon > 0$,*
$$\lim_n \mathsf{P}\Big(\sup_{1 \leq k \leq n} \big|Q_k^{(n,C)}\big| \geq \epsilon\Big) = 0;$$

(ii) *For any $\epsilon > 0$ and $a \in (0, 1]$,*
$$\lim_n \mathsf{P}^n\Bigg(\sup_{t \in \mathbb{T}}\Bigg| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E}\big(Q_k^{(n,C)}\mathbb{I}_{|Q_k^{(n,C)}| \leq a}\big|\mathcal{F}_{k-1}^n\big)$$
$$- \int_0^t \big(b - X_s^{(n,C)} \wedge C\big)ds\Bigg| \geq \epsilon\Bigg) = 0;$$

(iii) *For any $\epsilon > 0$ and $a \in (0, 1]$,*
$$\lim_n \mathsf{P}^n\Bigg(\sup_{t \in \mathbb{T}}\Bigg| \sum_{1 \leq k \leq [\frac{nt}{T}]} \Big(\mathsf{E}\big((Q_k^{(n,C)})^2\mathbb{I}_{|Q_k^{(n,C)}| \leq a}\big|\mathcal{F}_{k-1}^n\big)$$
$$- \big(\mathsf{E}\big(Q_k^{(n,C)}\mathbb{I}_{|Q_k^{(n,C)}| \leq a}\big|\mathcal{F}_{k-1}^n\big)\big)^2\Big) - \sigma^2\int_0^t \big(X_s^{(n,C)} \wedge C\big)ds\Bigg| \geq \epsilon\Bigg) = 0;$$

*Then $\mathbb{Q}^{n,C} \overset{W}{\longrightarrow} \mathbb{Q}^C$.*

Using Theorem 3.1, we prove the following result.

**Theorem 3.2.** $\mathbb{Q}^{n,C} \overset{W}{\longrightarrow} \mathbb{Q}^C$.

**Proof.** According to Theorem 3.1, we need to check conditions (i)–(iii). Relation (7) implies that $\sup_{0 \leq k \leq n} |Q_k^{(n,C)}| \leq \frac{b+CT}{n} + \sigma\sqrt{\frac{TC}{n}}$. Hence, there exists a constant $C_2 > 0$ such that $\sup_{0 \leq k \leq n} |Q_k^{(n,C)}| \leq \frac{C_2}{\sqrt{n}}$. This means that condition (i) is satisfied.

Furthermore, in order to establish (ii), we consider any fixed $a > 0$ and $n \geq 1$ such that $\frac{C_2}{\sqrt{n}} \leq a$, that is, $n \geq (\frac{C_2}{a})^2$. For such $n$,

$$\mathsf{E}\big(Q_k^{(n,C)}\mathbb{I}_{|Q_k^{(n,C)}| \leq a}\big|\mathcal{F}_{k-1}^n\big) = \mathsf{E}\big(Q_k^{(n,C)}\big|\mathcal{F}_{k-1}^n\big)$$
$$= \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} + \sigma\, \mathsf{E}\, q_k^{(n)}\sqrt{X_{k-1}^{(n,C)} \wedge C}$$
$$= \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n}. \tag{12}$$

For any $\epsilon > 0$, we have

$$\lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \leq k \leq [\frac{nt}{T}]} \mathsf{E}(Q_k^{(n,C)} \mathbb{I}_{|Q_k^{(n,C)}| \leq a} | \mathcal{F}_{k-1}^n) - \int_0^t (b - (X_s^{(n,C)} \wedge C)) ds \right| \geq \epsilon \right)$$

$$= \lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \leq k \leq [\frac{nt}{T}]} \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} - \sum_{0 \leq k \leq [\frac{nt}{T}]-1} (b - (X_k^{(n,C)} \wedge C)) \frac{T}{n} \right. \right.$$

$$\left. \left. - (b - (X_{[\frac{nt}{T}]}^{(n,C)} \wedge C)) \left( t - \frac{[\frac{nt}{T}]T}{n} \right) \right| \geq \epsilon \right)$$

$$= \lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| (b - (X_{[\frac{nt}{T}]}^{(n,C)} \wedge C)) \left( t - \frac{[\frac{nt}{T}]T}{n} \right) \right| \geq \epsilon \right) = 0,$$

and hence condition (ii) is satisfied. Now let us check condition (iii). We have

$$\mathsf{E}((Q_k^{(n,C)})^2 \mathbb{I}_{|Q_k^{(n,C)}| \leq a} | \mathcal{F}_{k-1}^n) = \mathsf{E}((Q_k^{(n,C)})^2 | \mathcal{F}_{k-1}^n)$$

$$= \left( \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} \right)^2 + 2 \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} \sigma \, \mathsf{E} \, q_k^{(n)} \sqrt{X_{k-1}^{(n,C)} \wedge C}$$

$$+ \sigma^2 \, \mathsf{E}(q_k^{(n)})^2 (X_{k-1}^{(n,C)} \wedge C)$$

$$= \left( \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} \right)^2 + \sigma^2 \frac{T}{n} (X_{k-1}^{(n,C)} \wedge C).$$

Therefore, for any $\epsilon > 0$,

$$\lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \leq k \leq [\frac{nt}{T}]} \left( \mathsf{E}((Q_k^{(n,C)})^2 \mathbb{I}_{|Q_k^{(n,C)}| \leq a} | \mathcal{F}_{k-1}^n) \right. \right. \right.$$

$$\left. \left. \left. - (\mathsf{E}(Q_k^{(n,C)} \mathbb{I}_{|Q_k^{(n,C)}| \leq a} | \mathcal{F}_{k-1}^n))^2 \right) - \sigma^2 \int_0^t (X_s^{(n,C)} \wedge C) ds \right| \geq \epsilon \right)$$

$$= \lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \leq k \leq [\frac{nt}{T}]} \left( \left( \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} \right)^2 + \sigma^2 \frac{T}{n} (X_{k-1}^{(n,C)} \wedge C) \right. \right. \right.$$

$$\left. \left. \left. - \left( \frac{(b - (X_{k-1}^{(n,C)} \wedge C))T}{n} \right)^2 \right) - \sum_{0 \leq k \leq [\frac{nt}{T}]-1} \left( \sigma^2 \frac{T}{n} (X_k^{(n,C)} \wedge C) \right) \right. \right.$$

$$\left. \left. - \sigma^2 (X_{[\frac{nt}{T}]}^{(n,C)} \wedge C) \left( t - \frac{[\frac{nt}{T}]T}{n} \right) \right| \geq \epsilon \right)$$

$$= \lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left( \sigma^2 (X_{[\frac{nt}{T}]}^{(n,C)} \wedge C) \left( t - \frac{[\frac{nt}{T}]T}{n} \right) \right) \geq \epsilon \right) = 0.$$

The theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Theorem 3.3.** $\mathbb{Q}^n \xrightarrow{W} \mathbb{Q}, n \to \infty.$

**Proof.** According to Theorem A.1 and Theorem 3.2, it suffices to prove that

$$\lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\Big\{\sup_{0\le t\le T}\big|X_t^{(n)}-X_t^{(n,C)}\big|\ge\epsilon\Big\}=0.$$

However, due to Remark 3.1,

$$\lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\Big\{\sup_{0\le t\le T}\big|X_t^{(n)}-X_t^{(n,C)}\big|\ge\epsilon\Big\}$$

$$\le\lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\big\{\exists t,t\in[0,T]:X_t^{(n)}\ne X_t^{(n,C)}\big\}=0.\qquad\square$$

## 4 Multiplicative scheme for Cox–Ingersoll–Ross process

In this section, we construct a multiplicative discrete approximation scheme for the process $e^{X_t}$, $t\in[0,T]$, where $X_t$ is the CIR process given by (2). We construct the following multiplicative process based on the discrete approximation scheme (6)–(7). We introduce limit and prelimit processes as follows:

$$S_t^{n,C}=\exp\{x_0\}\prod_{1\le k\le[\frac{tn}{T}]}\big(1+Q_k^{(n,C)}\big),\quad t\in\mathbb{T},$$

$$S_t^{C}=\exp\Big\{X_t^C-\frac{\sigma^2}{2}\int_0^t(X_t^C\wedge C)dt\Big\},\quad t\in\mathbb{T},$$

$$S_t^{n}=\exp\{x_0\}\prod_{1\le k\le[\frac{tn}{T}]}\big(1+Q_k^{(n)}\big),\quad t\in\mathbb{T},$$

$$S_t=\exp\Big\{X_t-\frac{\sigma^2}{2}\int_0^t X_t dt\Big\},\quad t\in\mathbb{T},$$

$$\widetilde{S}_t^{n}=\exp\{x_0\}\prod_{1\le k\le[\frac{tn}{T}]}\Big[\big(1+Q_k^{(n,C)}\big)\exp\Big\{\frac{\sigma^2}{2n}X_k^{(n)}\Big\}\Big],\quad t\in\mathbb{T},$$

and

$$\widetilde{S}_t=\exp\{X_t\},\quad t\in\mathbb{T}.$$

Denote by $\mathbb{G}^C$, $\mathbb{G}^{n,C}$, $\mathbb{G}$, $\mathbb{G}^n$, $\widetilde{\mathbb{G}}$, and $\widetilde{\mathbb{G}}^n$, $n\ge1$, the measures corresponding to the processes $S_t^C$, $S_t^{n,C}$, $S_t$, $S_t^n$, $\widetilde{S}_t$, and $\widetilde{S}_t^n$, $n\ge1$, respectively.

We apply Theorem 3.3 from [23] to prove the weak convergence of measures. This theorem can be formulated as follows.

**Theorem 4.1.** *Let the following conditions hold:*

(i) $\sup_{1\le k\le n}|Q_k^{(n,C)}|\xrightarrow{\mathsf{P}}0$, $n\to\infty$;

(ii) *For any $a\in(0,1]$,*

$$\lim_{D\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}^n\Big(\sum_{1\le k\le n}\mathsf{E}\big((Q_k^{(n,C)})^2\mathbb{I}_{|Q_k^{(n,C)}|\le a}\big|\mathcal{F}_{k-1}^n\big)\ge D\Big)=0;$$

(iii) *For any $a \in (0, 1]$,*

$$\lim_{D \to \infty} \overline{\lim_{n \to \infty}} \, \mathsf{P}^n \left( \sum_{1 \le k \le n} |\mathsf{E}(Q_k^{(n,C)} \mathbb{I}_{|Q_k^{(n,C)}| \le a} | \mathcal{F}_{k-1}^n)| \ge D \right) = 0;$$

(iv) *For any $\epsilon > 0$ and $a \in (0, 1]$,*

$$\lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \le k \le [\frac{nt}{T}]} \mathsf{E}(Q_k^{(n,C)} \mathbb{I}_{|Q_k^{(n,C)}| \le a} | \mathcal{F}_{k-1}^n) \right. \right.$$
$$\left. \left. - \int_0^t (b - X_s^{(n,C)} \wedge C) ds \right| \ge \epsilon \right) = 0;$$

(v) *For any $\epsilon > 0$ and $a \in (0, 1]$,*

$$\lim_n \mathsf{P}^n \left( \sup_{t \in \mathbb{T}} \left| \sum_{1 \le k \le [\frac{nt}{T}]} \mathsf{E}((Q_k^{(n,C)})^2 \mathbb{I}_{|Q_k^{(n,C)}| \le a} | \mathcal{F}_{k-1}^n) \right. \right.$$
$$\left. \left. - \sigma^2 \int_0^t (X_s^{(n,C)} \wedge C) ds \right| \ge \epsilon \right) = 0.$$

*Then*

$$\mathbb{G}^{n,C} \xrightarrow{W} \mathbb{G}^C.$$

We prove the following result using Theorem 4.1.

**Theorem 4.2.** $\mathbb{G}^{n,C} \xrightarrow{W} \mathbb{G}^C.$

**Proof.** According to Theorem 4.1, we need to check conditions (i)–(v). It was established in the proof of Theorem 3.2 that conditions (i) and (iv) are satisfied. Let us show that condition (ii) holds. It was also established in the proof of Theorem 3.2 that $\sup_{0 \le k \le n} |Q_k^{(n,C)}| \le \frac{C_2}{\sqrt{n}}$. So, for all $a \in (0, 1]$, starting from some number $n$, we have

$$\sum_{1 \le k \le n} \mathsf{E}((Q_k^{(n,C)})^2 \mathbb{I}_{|Q_k^{(n,C)}| \le a} | \mathcal{F}_{k-1}^n)$$
$$= \sum_{1 \le k \le n} \mathsf{E}((Q_k^{(n,C)})^2 | \mathcal{F}_{k-1}^n) \le \sum_{1 \le k \le n} \frac{C_2^2}{n} \le C_2^2,$$

whence condition (ii) holds. Now, (12) implies that, for all $a \in (0, 1]$, starting from some number $n$, we have

$$|\mathsf{E}(Q_k^{(n,C)} \mathbb{I}_{|Q_k^{(n,C)}| \le a} | \mathcal{F}_{k-1}^n)| = |\mathsf{E}(Q_k^{(n,C)} | \mathcal{F}_{k-1}^n)| \le \frac{C_3}{n},$$

whence condition (iii) holds.

Let us check condition (v). For any $\epsilon > 0$ and $a \in .(0, 1]$, we have

$$\lim_n \mathsf{P}^n\left(\sup_{t\in\mathbb{T}}\left|\sum_{1\leq k\leq[\frac{nt}{T}]}\mathsf{E}((Q_k^{(n,C)})^2\mathbb{I}_{|Q_k^{(n,C)}|\leq a}|\mathcal{F}_{k-1}^n) - \sigma^2\int_0^t(X_s^{(n,C)}\wedge C)ds\right| \geq \epsilon\right)$$

$$= \lim_n \mathsf{P}^n\left(\sup_{t\in\mathbb{T}}\left|\sum_{1\leq k\leq[\frac{nt}{T}]}\left(\left(\frac{(b-(X_{k-1}^{(n,C)}\wedge C))T}{n}\right)^2 + \sigma^2\frac{T}{n}(X_{k-1}^{(n,C)}\wedge C)\right)\right.\right.$$

$$\left.\left. - \sum_{0\leq k\leq[\frac{nt}{T}]-1}\left(\sigma^2\frac{T}{n}(X_k^{(n,C)}\wedge C)\right) - \sigma^2(X_{[\frac{nt}{T}]}^{(n,C)}\wedge C)\left(t - \frac{[\frac{nt}{T}]T}{n}\right)\right| \geq \epsilon\right)$$

$$\leq \lim_n \mathsf{P}^n\left(\sup_{t\in\mathbb{T}}\left(\frac{(|b|+C)^2Tt}{n} + \sigma^2(X_{[\frac{nt}{T}]}^{(n,C)}\wedge C)\left(t - \frac{[\frac{nt}{T}]T}{n}\right)\right) \geq \epsilon\right) = 0.$$

The theorem is proved. $\qquad\square$

**Theorem 4.3.** $\mathbb{G}^n \xrightarrow{W} \mathbb{G}$, $n \to \infty$.

**Proof.** The proof immediately follows from Theorem A.1, Theorem 4.2, and Remark 3.1. Indeed,

$$\lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\left\{\sup_{0\leq t\leq T}|X_t^{(n)} - X_t^{(n,C)}| \geq \epsilon\right\}$$

$$\leq \lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\left\{\exists t, t \in [0,T] : X_t^{(n)} \neq X_t^{(n,C)}\right\} = 0. \qquad\square$$

**Remark 4.1.** The weak convergence

$$\widetilde{\mathbb{G}}^n \xrightarrow{W} \widetilde{\mathbb{G}}, \quad n \to \infty,$$

can be proved in a similar way.

# A  Additional results

We state here Theorem 4.2 from [3]:

**Theorem A.1.** *Suppose that we have sets of processes* $\{X^{(n,C)}, n \geq 1, C > 0\}$, $\{X^C, C > 0\}$, $\{X^{(n)}, n \geq 1\}$ *and a stochastic process* $X$ *on the interval* $[0, T]$. *Let* $\mathbb{Q}^{n,C}$, $\mathbb{Q}^C$, $\mathbb{Q}^n$, *and* $\mathbb{Q}$ *be their corresponding measures. Suppose that, for any* $C > 0$, $\mathbb{Q}^{n,C} \xrightarrow{W} \mathbb{Q}^C$, $n \to \infty$, *and that* $\mathbb{Q}^C \xrightarrow{W} \mathbb{Q}$ *as* $C \to \infty$. *Suppose further that, for any* $\epsilon > 0$,

$$\lim_{C\to\infty}\overline{\lim_{n\to\infty}}\,\mathsf{P}\left\{\sup_{0\leq t\leq T}|X_t^{(n,C)} - X_t^{(n)}| \geq \epsilon\right\} = 0.$$

*Then* $\mathbb{Q}^n \xrightarrow{W} \mathbb{Q}$, $n \to \infty$.

Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be continuous functions. Consider the stochastic differential equation

$$dX(t) = \sigma\big(X(t)\big)dW(t) + b\big(X(t)\big)dt, \tag{13}$$

where $W = (W(t))$ is a Wiener process.

**Theorem A.2.** *[20, p. 177]. If $\sigma$ and $b$ are continuous functions satisfying the condition*

$$\big|\sigma(x)\big|^2 + \big|b(x)\big|^2 \le K\big(1 + |x|^2\big) \tag{14}$$

*for some positive constant $K$, then for any solution of (13) such that $\mathsf{E}(|X(0)|^2) < \infty$, we have $\mathsf{E}(|X(t)|^2) < \infty$ for all $t > 0$.*

**Theorem A.3.** *[20, p. 182]. Suppose that $\sigma$ and $b$ are bounded functions. Assume further that the following conditions are satisfied:*

(i) *there exists a strictly increasing function $\rho(u)$ on $[0, \infty)$ such that*

$$\rho(0) = 0, \quad \int_{0+} \rho^{-2}(u)du = \infty, \quad and \quad \big|\sigma(x) - \sigma(y)\big| \le \rho\big(|x - y|\big)$$

*for all $x, y \in \mathbb{R}$.*

(ii) *there exists an increasing and concave function $k(u)$ on $[0, \infty)$ such that*

$$k(0) = 0, \quad \int_{0+} k^{-1}(u)du = \infty, \quad and \quad \big|b(x) - b(y)\big| \le k\big(|x - y|\big)$$

*for all $x, y \in \mathbb{R}$.*

*Then the pathwise uniqueness of solutions holds for (13), and hence it has a unique strong solution.*

## References

[1] Alfonsi, A.: On the discretization schemes for the CIR (and Bessel squared) processes. Monte Carlo Methods Appl. **11**, 355–384 (2005). MR2186814. doi:10.1163/156939605777438569

[2] Berkaoui, A., Bossy, M., Diop, A.: Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence. ESAIM, Probab. Stat. **12**, 1–11 (2008). MR2367990. doi:10.1051/ps:2007030

[3] Billingsley, P.: Convergence of Probability Measures. John Wiley and Sons, New York, London, Sydney, Toronto (1968). MR0233396

[4] Bossy, M., Diop, A.: An efficient discretization scheme for one dimensional SDEs with a diffusion coefficient function of the form $|x|^a$, $a \in [1/2, 1)$. INRIA, Sophia Antipolis (2007)

[5] Brigo, D., Mercurio, F.: Interest Rate Models – Theory and Practice. With Smile, Inflation and Credit, 2nd edn. Springer Finance. Springer, Berlin (2006). MR2255741

[6] Broadie, M., Glasserman, P., Kou, S.G.: Connecting discrete continuous path-dependent options. Finance Stoch. **3**, 52–82 (1999). MR1805321. doi:10.1007/s007800050052

[7] Chang, L., Palmer, K.: Smooth convergence in the binomial model. Finance Stoch. **11**, 91–105 (2007). MR2284013. doi:10.1007/s00780-006-0020-6

[8] Cox, J.C., Ingersol, J.., Ross, S.: A theory of the term structure of interest rates. Econometrica **53**, 385–407 (1985). MR0785475. doi:10.2307/1911242

[9] Cutland, N., Kopp, E., Willinger, W.: From discrete to continuous financial models: New convergence results for option pricing. Math. Finance **3**, 101–123 (1993)

[10] Deelstra, G.: Long-term returns in stochastic interest rate models: Applications. ASTIN Bull. **30**, 123–140 (2000). MR1945550. doi:10.2143/AST.30.1.504629

[11] Deelstra, G., Delbaen, F.: Long-term returns in stochastic interest rate models: Different convergence results. Appl. Stoch. Models Data Anal. **13**, 401–407 (1997). MR1628650. doi:10.1002/(SICI)1099-0747(199709/12)13:3/4<401::AID-ASM334>3.0.CO;2-L

[12] Deelstra, G., Delbaen, F.: Convergence of discretized stochastic (interest rate) processes with stochastic drift term. Appl. Stoch. Models Data Anal. **14**, 77–84 (1998). MR1641781. doi:10.1002/(SICI)1099-0747(199803)14:1<77::AID-ASM338>3.0.CO;2-2

[13] Dereich, S., Neuenkirch, A., Szpruch, L.: An Euler-type method for the strong approximation of the Cox--Ingersoll–Ross process. Proc. R. Soc. A, Math. Phys. Eng. Sci. **468**, 1105–1115 (2012). MR2898556. doi:10.1098/rspa.2011.0505

[14] Duffie, D., Protter, P.: From discrete- to continuous-time finance: Weak convergence of the financial gain process. Math. Finance **2**, 1–15 (1992)

[15] Gyongy, I., Rasonyi, M.: A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. Stoch. Process. Appl. **121**, 2189–2200 (2011). MR2822773. doi:10.1016/j.spa.2011.06.008

[16] Heston, S.: A closed form solution for options with stochastic volatility, with applications to bonds and currency options. Rev. Financ. Stud. **6**, 327–343 (1993)

[17] Heston, S., Zhou, G.: On the rate of convergence of discrete-time contingent claims. Math. Finance **10**, 53–75 (2000). MR1743973. doi:10.1111/1467-9965.00080

[18] Higham, D.J., Mao, X.: Convergence of Monte Carlo simulations involving the mean-reverting square root process. J. Comput. Finance **8**, 35–61 (2005)

[19] Hubalek, F., Schachermayer, W.: When does convergence of asset price processes imply convergence of option prices? Math. Finance **8**, 385–403 (1998). MR1645093. doi:10.1111/1467-9965.00060

[20] Ikeda, N., Watanabe, W.: Stochastic Differential Equations and Diffusion Processes, 2nd edn. North-Holland Mathematical Library, vol. 24. North-Holland, Amsterdam (1989). MR1011252

[21] Karatzas, I., Shreve, S.: Brownian Motion and Stochastic Calculus. Springer (1991). MR1121940. doi:10.1007/978-1-4612-0949-2

[22] Mao, X.: Stochastic Differential Equations and Applications. Horwood Pub. (2008). MR2380366. doi:10.1533/9780857099402

[23] Mishura, Y.: Diffusion approximation of recurrent schemes for financial markets, with application to the Ornstein–Uhlenbeck process. Opusc. Math. **35**, 99–116 (2015). MR3282967. doi:10.7494/OpMath.2015.35.1.99

[24] Mishura, Y.: The rate of convergence of option prices on the asset following geometric Ornstein–Uhlenbeck process. Lith. Math. J. **55**, 134–149 (2015). MR3323287. doi:10.1007/s10986-015-9270-3

[25] Mishura, Y.: The rate of convergence of option prices when general martingale discrete-time scheme approximates the Black–Scholes model. Banach Cent. Publ., Adv. Math. Finance **104**, 151–165 (2015). MR3363984. doi:10.4064/bc104-0-8

[26] Mishura, Y., Munchak, Y.: Application of method of pseudomoments to the problem of estimation of rate of convergence of option prices. Theory Probab. Math. Stat. **92**, 110–124 (2015) (in Ukrainian)

[27] Mishura, Y., Munchak, Y.: The rate of convergence of prices of options issued on the assets following Bernoulli jumps discretization scheme of geometric Ornstein–Uhlenbeck process. Theory Probab. Math. Stat. **93** (2015) (in Ukrainian). MR3363984. doi:10.4064/bc104-0-8

[28] Pearson, N.D., Sun, T.S.: Exploiting the conditional density in estimating the term structure: An application to the Cox, Ingersoll and Ross model. J. Finance **49**, 1279–1304 (1994)

[29] Zhu, L.: Limit theorems for a Cox–Ingersoll–Ross process with Hawkes jumps. J. Appl. Probab. **51**, 699–712 (2014). MR3256221. doi:10.1239/jap/1409932668