Modern Stochastics: Theory and Applications 2 (2015) 371–389
DOI: 10.15559/15-VMSTA44

On packing dimension preservation by distribution functions of random variables with independent $\tilde{Q}$-digits

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Received: 16 October 2015, Revised: 13 December 2015, Accepted: 13 December 2015,
Published online: 23 December 2015

Abstract The article is devoted to finding conditions for the packing dimension preservation by distribution functions of random variables with independent $\tilde{Q}$-digits.

The notion of "faithfulness of fine packing systems for packing dimension calculation" is introduced, and connections between this notion and packing dimension preservation are found.

Keywords Packing dimension of a set, Hausdorff–Besicovitch dimension of a set, faithfulness of fine packing system for packing dimension calculation, $\tilde{Q}$-expansion of real numbers, packing-dimension-preserving transformations

2010 MSC 28A78, 28A80

1 Introduction

Let (M,ρ) be a metric space. Suppose that the Hausdorff–Besicovitch dimension $\dim_H$ [8] is well defined in (M,ρ). A transformation $f : M \to M$ is called dimension-preserving transformation [13] or *DP*-transformation if

$$\dim_H\bigl(f(E)\bigr) = \dim_H(E), \quad \forall E \subset M.$$

Let $G(M,\dim_H)$ be the set of all *DP*-transformations defined on (M,ρ). It is easy to see that G forms a group w.r.t. the composition of transformations. It is well known that any bi-Lipschitz transformation belongs to this group [8]. However, G is essentially wider than the group of all bi-Lipschitz transformations. In 2004, some sufficient conditions for belonging of distribution functions of random variable with independent s-adic digits to group G was proved by G. Torbin et al. [2]. There exist a lot of DP-functions that are not bi-Lipschitz.

Sufficient conditions for distribution functions of random variables with independent s-adic digits to be DP have been found by G. Torbin [13] in 2007. These conditions were generalized for Q by G. Torbin [14] and later for Q^*- and $\tilde{Q}$-expansions by S. Albeverio, V. Koshmanenko, M. Pratsiovytyi, and G. Torbin [3, 4].

Recently, G. Torbin and M. Ibragim proved rather general sufficient conditions for distribution functions of random variables with independent $\tilde{Q}$-digits to be in DP-class. The notion of fine covering system faithfulness for $\dim_H$ calculation [5] plays an important role in the proof of these conditions. This notion gives us the possibility to consider coverings by sets from some family Φ and to be sure that a "dimension" calculated in such a way is equal to $\dim_H$. Faithfulness of the family of all s-adic cylinders (if s is fixed) have been proven by Billingsley [6] in 1961. Faithfulness of the family of Q-cylinders have been proven by M. Pratsiovytyi and A. Turbin [16] in 1992, and faithfulness of the family of Q^*-cylinders (under the condition of separation from zero of the corresponding coefficients) have been proven by S. Albeverio and G. Torbin [1] in 2005. It is necessary to remark that the last result can be easily generalized to $\tilde{Q}$-expansion under a similar condition.

In 1982, C. Tricot [15] introduced the notion of packing dimension $\dim_P$. This dimension is in some sense dual to the Hausdorff–Besicovitch dimension: the definition of $\dim_H$ of a set F is based on ε-coverings of this figure, but the definition of $\dim_P$ is based on ε-packings (the countable sets of disjoint open balls $B_k(r_k, c_k)$, $k \in \mathbb{N}$, with radii $r_k \leqslant \varepsilon$ and centers $c_k \in F$). The packing dimension has all "good" properties of a fractal dimension, such as the countable stability. Therefore, proving or disproving similar results for $\dim_P$ is important. For example, we consider the group of packing-dimension-preserving transformations (or PDP-transformations).

Definition 1.1. The transformation f is said to be a PDP-transformation if

$$\forall E \subset M, \quad \dim_P\bigl(f(E)\bigr) = \dim_P(E).$$

There are a lot of problems with proving of many conjectures for $\dim_P$ because work with packings is essentially more complicated than work with coverings [10].

These problems are solving bit by bit. For example, M. Das [7] has proven the Billingsley theorem for packing dimension; J. Li [9] obtained some sufficient conditions for distribution functions of random variables with independent $\tilde{Q}$-digits to be in PDP-class. Namely, J. Li has proven the following theorem.

Theorem 1.1. *Let F_ξ be the distribution function of a random variable ξ with independent $\tilde{Q}$-representation. If $\inf_{i,j} q_{ij} = q_* > 0$ and $\inf_{i,j} p_{ij} = p_* > 0$, then F_ξ preserves the packing dimension if and only if*

$$\limsup_{k\to\infty} \frac{h_1 + h_2 + \cdots + h_k}{b_1 + b_2 + \cdots + b_k} = 1,$$

where $h_j = -\sum_{i=1}^{n_j} p_{ij} \ln p_{ij}$ and $b_j = -\sum_{i=1}^{n_j} p_{ij} \ln q_{ij}$.

In Remark 4.2 at the and of article [9], we read: "The conditions $\inf_{i,j} q_{ij} = q_* > 0$ and $\inf_{i,j} p_{ij} = p_* > 0$ play an important role in the proof of the theorem. Open question: What can we say about the topic if we remove these conditions?"

S. Albeverio, M. Pratsiovytyi, and G. Torbin [3] removed the condition $\inf_{i,j} p_{ij} = p_* > 0$ in a similar situation for *DP*-transformations.

In case of packing dimension, the approach of [3] is complicated because it requires appropriate results about the fine packing system faithfulness for packing dimension calculation. Even the definition of the fine packing system faithfulness is a problem because centers of all balls in packings should be in the set the dimension of which is calculated.

The aim of this paper is to propose some alternative definition of the packing dimension, uncentered packing dimension or $\dim_{P(unc)}$. In the proposed definition, the condition "the centers of balls should be in the figure the dimension of which is calculated" in the definition of $\dim_P$ is replaced by "every ball should have a nonempty intersection with the figure." We prove that, in some wide class of metric spaces (including $\mathbb{R}^n$), the value of packing dimension with uncentered balls is matching to the value of classical packing dimension. Introduction of the fine packing system faithfulness notion is very simple in the case of proposed definition. It allows us to prove faithfulness (under the condition of separation from zero of the coefficients) of a $\tilde{Q}$-cylinder system and sufficient conditions for the distribution function of a random variable with independent $\tilde{Q}$-digits to be in the *PDP*-class. The corresponding theorem is the main result of the paper.

Theorem 1.2. *Let* $\inf_{i,j} q_{ij} := q_{\min}$. *Suppose that* $q_{\min} > 0$. *Let*

$$\begin{aligned} T &:= \left\{k : k \in \mathbb{N},\, p_k < \frac{q_{\min}}{2}\right\}; \\ T_k &:= T \cap \{1, 2, \dots, k\}; \\ B &:= \limsup_{k\to\infty} \frac{\sum_{j\in T_k} \ln \frac{1}{p_j}}{k}. \end{aligned}$$

Let F_ξ be the distribution function of a random variable ξ with independent $\tilde{Q}$-representation. Then F_ξ preserves the packing dimension if and only if

$$\begin{cases} \dim_P \mu_\xi = 1; \\ B = 0. \end{cases}$$

2 Packing dimension

Let us recall the definition of packing dimension in the form given, for example, in [8].

Definition 2.1. Let $E \subset M$ and $\varepsilon > 0$. A finite or countable family $\{E_j\}$ of open balls is called an *ε-packing* of a set E if

1. $|E_i| \leqslant \varepsilon$ for all i;
2. $c_i \in E,\ i \in \mathbb{N}$, where c_i is the center of the ball E_i;
3. $E_i \cap E_j = \varnothing$ for all $i, j, i \neq j$.

Remark 2.1. The empty set of balls is a packing of any set.

Definition 2.2. Let $E \subset M$, $\alpha \geqslant 0$, $\varepsilon > 0$. Then the *α-dimensional packing premeasure* of a bounded set E is defined by

$$\mathcal{P}_\varepsilon^\alpha(E) := \sup\left\{\sum_i |E_i|^\alpha\right\},$$

where the supremum is taken over all at most countable ε-packings $\{E_j\}$ of E (if $E_j = \varnothing$ for all j, then $\mathcal{P}_\varepsilon^\alpha(E) = 0$).

Definition 2.3. The *α-dimensional packing quasi-measure* of a set E is defined by

$$\mathcal{P}_0^\alpha(E) := \lim_{\varepsilon \to 0} \mathcal{P}_\varepsilon^\alpha(E).$$

Definition 2.4. The *α-dimensional packing measure* is defined by

$$\mathcal{P}^\alpha(E) := \inf\left\{\sum_j \mathcal{P}_0^\alpha(E_j) : E \subset \bigcup E_j\right\},$$

where the infimum is taken over all at most countable coverings $\{E_j\}$ of E, $E_j \subset \mathbf{M}$.

Definition 2.5. The nonnegative number

$$\dim_P(E) := \inf\{\alpha : \mathcal{P}^\alpha(E) = 0\}$$

is called the *uncentered packing dimension* of a set $E \subset M$.

3 Uncentered packing dimension

Definition 3.1. Let $E \subset M$ and $\varepsilon > 0$. A finite or countable family $\{E_j\}$ of open balls is called an *uncentered ε-packing* of a set E if

1. $|E_i| \leqslant \varepsilon$ for all i;
2. $E_i \cap E \neq \varnothing$;
3. $E_i \cap E_j = \varnothing$ for all $i, j, i \neq j$.

Remark 3.1. The empty set of balls is an uncentered packing of any set.

Definition 3.2. Let $E \subset M$, $\alpha \geqslant 0$, $\varepsilon > 0$. Then the *uncentered α-dimensional packing premeasure* of a bounded set E is defined by

$$\mathcal{P}_{\varepsilon(unc)}^\alpha(E) := \sup\left\{\sum_i |E_i|^\alpha\right\},$$

where the supremum is taken over all at most countable uncentered ε-packings $\{E_i\}$ of E.

Definition 3.3. The *uncentered α-dimensional packing quasi-measure* of a set E is defined by

$$\mathcal{P}^{\alpha}_{0(unc)}(E) := \lim_{\varepsilon\to 0} \mathcal{P}^{\alpha}_{\varepsilon(unc)}(E).$$

Definition 3.4. *Uncentered α-dimensional packing measure* is defined by

$$\mathcal{P}^{\alpha}_{(unc)}(E) := \inf\left\{\sum_j \mathcal{P}^{\alpha}_{0(unc)}(E_j) : E \subset \bigcup E_j\right\},$$

where the infimum is taken over all at most countable coverings $\{E_j\}$ of E, $E_j \subset \mathbf{M}$.

Remark 3.2. If $(M, \rho) = \mathbb{R}^1$ and $\alpha = 1$, then the α-dimensional packing measure and uncentered α-dimensional packing measure are the Lebesgue measure.

Definition 3.5. The nonnegative number

$$\dim_{P(unc)}(E) := \inf\left\{\alpha : \mathcal{P}^{\alpha}_{(unc)}(E) = 0\right\}.$$

is called the *uncentered packing dimension* of a set $E \subset M$.

Theorem 3.1. *Let (M, ρ) be a metric space. Let $C \in \mathbb{N}$. If for all $r > 0$ and for any open ball I with $|I| = 8r$, there exist at most $N(I)$ balls I_i, $i \in \{1, \dots, N(I)\}$ such that $I_i \subset I$, $i \in \{1, \dots, N(I)$, $|I_i| = r$, $i \in \{1, \dots, N(I)\}$, and $N(I) \leq C$. Then*

$$\dim_{P(unc)}(E) = \dim_P(E).$$

Proof. *Step 1.* Let us prove the inequality $\dim_{P(unc)}(E) \geqslant \dim_P(E)$.

By the definitions and supremum property we have

$$\mathcal{P}^{\alpha}_{r(unc)}(E) \geqslant \mathcal{P}^{\alpha}_{r}(E).$$

By the limit property of inequalities we have

$$\mathcal{P}^{\alpha}_{0(unc)}(E) \geqslant \mathcal{P}^{\alpha}_{0}(E).$$

Hence,

$$\mathcal{P}^{\alpha}_{(unc)}(E) \geqslant \mathcal{P}^{\alpha}(E).$$

Let $\dim_{P(unc)}(E) = \alpha_0$. By the definition of $\dim_{P(unc)}(E)$ we have

$$\forall\varepsilon > 0, \quad \mathcal{P}^{\alpha_0+\varepsilon}_{(unc)}(E) = 0.$$

Therefore,

$$\forall\varepsilon > 0, \quad \mathcal{P}^{\alpha_0+\varepsilon}_{0}(E) = 0,$$

and, consequently,

$$\dim_P(E) \leqslant \alpha_0.$$

Hence, it follows that $\dim_{P(unc)}(E) \geqslant \dim_P(E)$, which is our claim.

Step 2. Let us show that $\dim_{P(unc)}(E) \leqslant \dim_P(E)$.

If $\dim_{P(unc)}(E) = 0$, then the statement is true.

Let us consider the case $\dim_{P(unc)}(E) \neq 0$. Fix $0 < t < s < \dim_{P(unc)}(E)$. Since $s < \dim_{P(unc)}(E)$, we have

$$\mathcal{P}^s_{(unc)}(E) = +\infty,$$
$$\mathcal{P}^s_{0(unc)}(E) = +\infty.$$

Therefore,

$$\forall r > 0,\ \mathcal{P}^s_{r(unc)}(E) = +\infty.$$

From this and from the supremum property, it follows that there is an uncentered packing $V := \{E_i\}$ of the set E with

$$\sum_i |E_i|^s > 1. \tag{1}$$

Let us divide the packing V into classes

$$V_k := \left\{E_i : 2^{-k-1} \leqslant |E_i| < 2^{-k}\right\}.$$

Let n_k be the number of balls V_k. We will show that

$$\exists k_0 : n_{k_0} \geqslant 2^{k_0 t}\left(1 - 2^{t-s}\right).$$

To obtain a contradiction, suppose that

$$n_k < 2^{kt}\left(1 - 2^{t-s}\right) \quad \text{for all } k.$$

Then

$$\sum_i |E_i|^s < \sum_k 2^{-ks} \cdot n_k < \sum_k 2^{-ks} \cdot 2^{kt}\left(1 - 2^{t-s}\right) = \left(1 - 2^{t-s}\right) \cdot \sum_k \left(2^{t-s}\right)^k = 1,$$

which contradicts our assumption (1).

Therefore, such k_0 exists. Let us consider V_{k_0}. We denote by $A_1, A_2, \ldots, A_{n_{k_0}}$ the balls in V_{k_0}, that is,

$$V_{k_0} = \{A_1, A_2, \ldots, A_{n_{k_0}}\}.$$

Fix $r := 2^{-k_0-1}$. Then the radius of any A_i is less than r. Let T_i be a point of A_i such that $T_i \in A_i \cap E$. Let V' be the set of balls with the centers T_i and radius r, that is,

$$V' = \left\{A'_i : A'_i = B(T_i, r)\right\}.$$

Fix

$$V^* = \left\{A^*_i : A^*_i = B(T_i, 4r)\right\}.$$

Let us divide the set V' into classes $K_1, K_2, \ldots, K_l$ as follows.

1. Let us take a ball $A'_{j_1} = A'_1$ and put it in K_1 together with all other balls $A'_i \in V'$ such that $A'_i \cap A'_{j_1} \neq \varnothing$.
2. Let us take an arbitrary ball $A'_{j_2} \in V' \setminus K_1$ and put it in K_2 together with all other balls $A'_i \in V' \setminus K_1$ such that $A'_i \cap A'_{j_2} \neq \varnothing$.

3. Let us continue this way until $V' \setminus (K_1 \cup K_2 \cup \dots \cup K_l) \neq \varnothing$. Since the number of elements in a set V' is a finite, we can find such a number l.

Now suppose that the balls A'_i and A'_j intersect each other. In other words, $\rho(T_i, T_j) \leqslant 2r$. Therefore, $A_j \subset A_i^*$.

The radius of A_j is greater than $r/2$. By the theorem condition, there are no more than C disjoint balls with radius $r/2$ in a ball with radius $4r$

Therefore, there are no more than C balls in any class K_i.

Moreover, in the case $i < m$, the balls A'_{j_i} and A'_{j_m} do not intersect each other. Indeed, suppose otherwise. Then A'_{j_m} is in a class K_i or in a class with number less than i.

Hence,

$$V'' = \left\{A'_{j_1}, A'_{j_2}, \dots, A'_{j_l}\right\}$$

is a centered packing of a set E, and the t-volume of this packing is less than the t-volume of the uncentered packing V_{k_0} no more than C times. Therefore,

$$\sum_{V''} \left|A'_{j_i}\right|^t \geqslant n_{k_0} \cdot \frac{2^{-k_0 t}}{C} \geqslant 2^{k_0 t}\left(1 - 2^{t-s}\right) \cdot \frac{2^{-k_0 t}}{C} = \frac{1 - 2^{t-s}}{C}.$$

From this it follows that

$$\mathcal{P}^t_{2^{-k_0}}(E) \geqslant \frac{1 - 2^{t-s}}{C}.$$

By the inequality $2^{-k_0} < r$ we get

$$\mathcal{P}^t_r(E) \geqslant \frac{1 - 2^{t-s}}{C} \quad \text{for all } r > 0.$$

Consequently, as $r \to 0$, we get the inequality

$$\mathcal{P}^t_0(E) \geqslant \frac{1 - 2^{t-s}}{C}.$$

Let us show that $\mathcal{P}^t(E) \geqslant \frac{1-2^{t-s}}{C}$. Recall the definition

$$\mathcal{P}^t(E) = \inf\left\{\sum_j \mathcal{P}^t_0(E_j) : E \subset \bigcup E_j\right\},$$

where the infimum is taken over all at most countable coverings E_j of a set E.

Let $\{E_j\}$ be an at most countable covering of E. Since $\dim_{P(unc)}(E) > s$, there is j_0 such that $\dim_{P(unc)}(E_{j_0}) > s$ (by the countable stability of the packing dimension $\dim_{P(unc)}$). In other words, we have

$$\mathcal{P}^s_{(unc)}(E_{j_0}) = +\infty,$$
$$\mathcal{P}^s_{0(unc)}(E_{j_0}) = +\infty.$$

We conclude by the part of the theorem already proved for E that

$$\mathcal{P}^t_0(E_{j_0}) \geqslant \frac{1 - 2^{t-s}}{C}$$

and

$$\sum_j \mathcal{P}_0^t(E_j) \geqslant \frac{1-2^{t-s}}{C}.$$

But the previous inequality is true for an arbitrary covering $\{E_j\}$ of a set E and for the infimum for all coverings. Therefore,

$$\mathcal{P}^t(E) \geqslant \frac{1-2^{t-s}}{C}$$

and

$$\dim_P(E) \geqslant t.$$

Since t-$\dim_{P(unc)}(E)$ can be approximated by 0, we get $\dim_P(E) \geqslant \dim_{P(unc)}(E)$, which completes the proof. □

Corollary 3.1. *If* $M = \mathbb{R}^n$, *then* $\dim_{P(unc)}(E) = \dim_P(E)$.

Proof. Let B_{8r} be a ball with radius $8r$, B_r be a ball with radius r, and λ be the n-dimensional Lebesgue measure. Then

$$\lambda(B_{8r}) = 8^n \cdot \lambda(B_r).$$

Therefore, we can put no more than $C = 8^n$ disjoint balls with radii r in a ball with radius $8r$, which completes the proof. □

3.1 Packing dimension with respect to the family of sets

Let Φ be a family of balls in a metric space (M, ρ).

Definition 3.6. Let $E \subset M$, $\alpha \geqslant 0$, $\varepsilon > 0$. Then the *α-dimensional packing premeasure* of a bounded set E with respect to Φ is defined by

$$\mathcal{P}_\varepsilon^\alpha(E, \Phi) := \sup\left\{\sum_i |E_i|^\alpha\right\},$$

where the supremum is taken over all uncentered ε-packings $\{E_i\} \subset \Phi$ of E (if $\{E_i\} = \varnothing$, then $\mathcal{P}_\varepsilon^\alpha(E, \Phi) = 0$).

Definition 3.7. The *α-dimensional packing quasi-measure* of a set E w.r.t. Φ is defined by

$$\mathcal{P}_0^\alpha(E, \Phi) := \lim_{\varepsilon\to 0} \mathcal{P}_\varepsilon^\alpha(E, \Phi).$$

Definition 3.8. The *α-dimensional packing measure* w.r.t. Φ is defined by

$$\mathcal{P}^\alpha(E, \Phi) := \inf\left\{\sum_j \mathcal{P}_0^\alpha(E_j, \Phi) : E \subset \bigcup E_j\right\},$$

where the infimum is taken over all at most countable coverings $\{E_j\}$ of E, $E_j \subset \mathbf{M}$.

Definition 3.9. The nonnegative number

$$\dim_P(E, \Phi) := \inf\{\alpha : \mathcal{P}^\alpha(E, \Phi) = 0\}$$

is called the *packing dimension* of a set $E \subset M$ w.r.t. Φ.

Remark 3.3. In the definition of $\dim_P(E, \Phi)$, we used uncentered packing. But we will denote this dimension without index (*unc*) because:

1. We will work in $\mathbb{R}^n$. In this space, centered and uncentered packing dimensions are equal;
2. The centered packing dimension w.r.t. some family of balls is not defined.

Theorem 3.2.

$$\dim_P(E, \Phi) \leqslant \dim_{P(unc)}(E).$$

Proof. Let Φ_0 be the family of all open balls of M. Then

$$\mathcal{P}^\alpha_{r(unc)}(E) = \mathcal{P}^\alpha_r(E, \Phi_0).$$

Since $\Phi \subseteq \Phi_0$, by the supremum property we have

$$\mathcal{P}^\alpha_r(E, \Phi) \leqslant \mathcal{P}^\alpha_r(E, \Phi_0).$$

By the inequality for packing premeasures it follows that

$$\dim_P(E, \Phi) \leqslant \dim_{P(unc)}(E),$$

which proves the theorem. □

4 Faithfulness of the open balls families for packing dimension calculation

Definition 4.1. Suppose that some open balls family Φ satisfies the following condition: for all $E \subset M$, $\dim_{P(unc)}(E, \Phi) = \dim_{P(unc)}(E)$. Then Φ is said to be *faithful for uncentered packing dimension calculation.*

Remark 4.1. The notion of faithfulness is introduced for the Hausdorff–Besicovitch dimension $\dim_H$ [11]. It is clear that

$$\forall \Phi \subset 2^M, \dim_H(E, \Phi) \geqslant \dim_H(E).$$

Theorem 4.1 (The sufficient condition for the open-ball family to be faithful for packing dimension calculation)**.** *Suppose that*

1. Φ *is a family of intervals from* $[0; 1]$*;*
2. $\exists C > 0 : \forall (a; b) \subset [0; 1]$, $\exists \Delta(a; b) \in \Phi$ *such that:*
   (a) $\frac{a+b}{2} \in \Delta(a, b)$*;*
   (b) $\Delta(a, b) \subset (a; b)$*;*
   (c) $\frac{b-a}{|\Delta(a,b)|} \geqslant C$.

Then Φ is a faithful open-ball family for packing dimension calculation.

Proof. Let E be any set, $\alpha \geqslant 0$, and $r > 0$. Let $\{E_i\} = \{(a_i; b_i)\}$ be a family of disjoint intervals such that $\frac{a_i+b_i}{2} \in E$ and $b_i - a_i < r$.

Then the following inequality holds:

$$\sum_i |E_i|^\alpha \leqslant \sum_i \left|\Delta(a_i, b_i)\right|^\alpha \cdot C^\alpha.$$

Taking the supremum (over all sets of intervals $\{E_i\}$ satisfying the previous conditions), we have

$$\mathcal{P}_r^\alpha(E) \leqslant \sup_{\{E_i\}} \left|\Delta(a_i, b_i)\right|^\alpha \cdot C^\alpha.$$

Any set of intervals $\{\Delta(a_i, b_i)\}$ satisfies the conditions from the $\mathcal{P}^\alpha_{r(unc)}(E, \Phi)$ definition. So,

$$\sup_{\{E_i\}} \left|\Delta(a_i, b_i)\right|^\alpha \cdot C^\alpha \leqslant \mathcal{P}^\alpha_{r(unc)}(E, \Phi) \cdot C^\alpha.$$

Therefore,

$$\mathcal{P}_r^\alpha(E) \leqslant \mathcal{P}^\alpha_{r(unc)}(E, \Phi) \cdot C^\alpha.$$

Taking the limit of both sides, we have

$$\mathcal{P}_0^\alpha(E) \leqslant \mathcal{P}^\alpha_{0(unc)}(E, \Phi) \cdot C^\alpha.$$

Taking the infimum over all possible coverings of the set E, we have

$$\mathcal{P}^\alpha(E) \leqslant \mathcal{P}^\alpha_{(unc)}(E, \Phi) \cdot C^\alpha$$

and

$$\dim_P(E) \leqslant \dim_{P(unc)}(E, \Phi).$$

Since $[0; 1] \subset \mathbb{R}^1$, it follows that

$$\dim_P(E) = \dim_{P(unc)}(E)$$

and

$$\dim_{P(unc)}(E) \leqslant \dim_{P(unc)}(E, \Phi).$$

Using

$$\dim_{P(unc)}(E) \geqslant \dim_{P(unc)}(E, \Phi) \quad \text{for all } \Phi,$$

we obtain that Φ is a faithful open-ball family for the packing dimension calculation. □

5 Sufficient conditions for $\tilde{Q}$-expansion cylindric interval family to be faithful

The $\tilde{Q}$-expansion of real numbers is a generalization of s-expansion and Q-expansion and was described, for example, in [4].

Theorem 5.1. *Let Φ be the system of cylindric intervals of some $\tilde{Q}$-expansion. Suppose that*

$$\inf_{i,j} q_{ij} = q_{\min} > 0.$$

Then Φ is a faithful ball family for packing dimension calculation.

Proof. Let $\tilde{Q}_0$ be the set of $\tilde{Q}$-rational points, and E' be any subset of $[0; 1]$. Let $E = E' \setminus \tilde{Q}_0$. Since $\tilde{Q}_0$ is countable, it follows that

$$\dim_{P(unc)}(\tilde{Q}_0) = 0, \qquad \dim_{P(unc)}(\tilde{Q}_0, \Phi) = 0,$$

and

$$\dim_{P(unc)}\big(E'\big) = \dim_{P(unc)}(E), \qquad \dim_{P(unc)}\big(E', \Phi\big) = \dim_{P(unc)}(E, \Phi).$$

The proof is completed by showing that

$$\dim_{P(unc)}(E) = \dim_{P(unc)}(E, \Phi)$$

for every set $E \subset [0; 1]$ if E does not contain $\tilde{Q}$-rational points.

Let $(a; b) \subset [0; 1]$. Let $\Delta(a, b)$ be the $\tilde{Q}$-cylindric interval of the minimal rank such that

$$\frac{a+b}{2} \in \Delta(a; b) \subset (a; b).$$

Denote the rank of $\Delta(a, b)$ by k. Since this rank is minimal, it follows that $(a; b)$ is a subset of one or two cylinders with rank $k-1$. Let us denote the cylinder with rank $k-1$ that contains $\Delta(a, b)$ by Δ'. If the second cylinder exists, then we denote it by Δ''.

Let us consider the following two cases.

Case 1. The Δ'' does not exist. Then

$$\big|\Delta(a, b)\big| \leqslant b - a \leqslant \big|\Delta'\big|,$$

and, therefore,

$$\big|\Delta(a, b)\big| \geqslant (b-a) \cdot q_{\min}.$$

Case 2. The Δ'' exists. Then

$$\big|\Delta'\big| \cdot 2 \geqslant b - a \quad \Rightarrow \quad \big|\Delta(a, b)\big| \geqslant (b-a) \cdot \frac{q_{\min}}{2}.$$

Summary of the two cases. For every interval $(a; b)$, there exists a $\tilde{Q}$-cylindric interval $\Delta(a, b)$ such that $\frac{a+b}{2} \in \Delta(a; b)$ and

$$\big|\Delta(a, b)\big| \geqslant (b-a) \cdot \frac{q_{\min}}{2}.$$

It follows that the family Φ satisfies the conditions of Theorem 4.1 and is faithful for packing dimension calculation. □

Corollary 5.1. *Let Φ be a family of Q^*-cylinders under the condition* $\inf_{i,j} q_{ij} > 0$. *Then Φ is faithful for packing dimension calculation.*

Corollary 5.2. *Let Φ be a family of Q-cylinders. Then Φ is faithful for packing dimension calculation.*

Corollary 5.3. *Let Φ be a family of s-adic cylinders. Then Φ is faithful for packing dimension calculation.*

6 Proof of the main result

To prove the main result, we need the following two lemmas.

Lemma 6.1. *Let $\tilde{Q}$ be the matrix $\|q_{ik}\|$, $i \in \mathbb{N}$, $k \in \{0, 1, \dots, N_k - 1\}$. If*

$$\lim_{i\to\infty} \frac{\ln q_{i_k k}}{\ln(q_{i_1 1} q_{i_2 2} \dots q_{i_{k-1}(k-1)})} = 0$$

for every sequence (i_k), then the open-ball family Φ of the respective expansion cylinder interiors is faithful for packing dimension calculation.

Proof. Let us fix a set $E \subset [0; 1]$. Let us fix any numbers $m \in \mathbb{N}$, $\delta > 0$ and consider the following sets:

$$W_{m,\delta} = \left\{ x \in E : \frac{\ln q_{i_k k}(x)}{\ln(q_{i_1 1}(x) q_{i_2 2}(x) \dots q_{i_{k-1}(k-1)}(x))} < \delta,\ \forall k \geqslant m \right\}.$$

Fix some value m and consider any set $W_{m,\delta}$ corresponding to this value. There exists $\varepsilon > 0$ such that $|c_m| \geqslant \varepsilon$ for any cylinder c_m of rank m. Consider the centered ε-packing of the set $W_{m,\delta}$ by intervals E_j.

For every interval E_j, there exists a cylindric interval $\Delta(E_j)$ such that:

1. $\Delta(E_j) \subset E_j$;
2. $\Delta(E_j)$ contains the middle point x_j of the E_j;
3. $\Delta(E_j)$ has the minimal possible rank. We denote this rank by i_j.

We will say that the cylinder $\Delta'(E_j)$ is the "father" of $\Delta(E_j)$ if $\Delta'(E_j) \supset \Delta(E_j)$ and the rank of $\Delta'(E_j)$ is equal to $i_j - 1$. It is obvious that $|\Delta'(E_j)| \geqslant \frac{|E_j|}{2}$. Therefore,

$$|E_j| \leqslant \frac{2|\Delta(E_j)|}{q_{i_{k_j} k_j}(x_j)}, \quad \text{where } x_j \in W_{m,\delta}.$$

Let us estimate the α-volume of packing of the set E by intervals E_j:

$$\sum_k |E_j|^\alpha \leqslant \sum_k \left|\Delta(E_j)\right|^\alpha \cdot \left(\frac{2}{q_{i_{k_j} k_j}(x_j)}\right)^\alpha.$$

This inequality is equivalent to

$$\sum_k |E_j|^\alpha \leqslant \sum_k \left|\Delta(E_j)\right|^{\alpha-\delta} \cdot \left|\Delta(E_j)\right|^\delta \cdot \left(\frac{2}{q_{i_{k_j} k_j}(x_j)}\right)^\alpha.$$

Let us estimate the expression

$$\begin{aligned} &\ln\left(\left|\Delta(E_j)\right|^\delta \cdot \left(\frac{2}{q_{i_{k_j} k_j}(x_j)}\right)^\alpha \right) \\ &\quad = \delta \ln\bigl(q_{i_1 1}(x) q_{i_2 2}(x) \dots q_{i_{k_j-1}(k_j-1)}(x)\bigr) + \alpha \ln 2 - \alpha \ln q_{i_{k_j} k_j}(x_j). \end{aligned}$$

Since $x_j \in W_{m,\delta}$, it follows that

$$\delta \ln\bigl(q_{i_1 1}(x) q_{i_2 2}(x) \dots q_{i_{k_j-1}(k_j-1)}\bigr) \leqslant \ln q_{i_{k_j} k_j}(x_j).$$

Therefore,

$$\ln\left(\left|\Delta(E_j)\right|^{\delta} \cdot \left(\frac{2}{q_{i_{k_j} k_j}(x_j)}\right)^{\alpha}\right) \leqslant \alpha \ln 2 + (1-\alpha) \ln q_{i_{k_j} k_j}(x_j) \leqslant \alpha \ln 2.$$

Thus, we have

$$\sum_j |E_j|^{\alpha} \leqslant 2 \sum_j \left|\Delta(E_j)\right|^{\alpha-\delta}.$$

Take the suprema over all possible centered packings $\{E_j\}$ of both parts of the previous inequality:

$$\mathcal{P}^{\alpha}_{\varepsilon}(W_{m,\delta}) \leqslant 2\mathcal{P}^{\alpha-\delta}_{\varepsilon(unc)}(W_{m,\delta}, \Phi).$$

Take the limit as $\varepsilon \to 0$:

$$\mathcal{P}^{\alpha}_{0}(W_{m,\delta}) \leqslant 2\mathcal{P}^{\alpha-\delta}_{0(unc)}(W_{m,\delta}, \Phi).$$

We obtain that

$$\mathcal{P}^{\alpha}(W_{m,\delta}) \leqslant 2\mathcal{P}^{\alpha-\delta}_{(unc)}(W_{m,\delta}, \Phi).$$

Denote $\alpha_0 = \dim_P(W_{m,\delta})$. Then for all $\alpha < \alpha_0$, the left part is equal to infinity. Thus, for all $\alpha < \alpha_0$, the right part is equal to infinity too. It follows that

$$\dim_{P(unc)}(W_{m,\delta}, \Phi) \geqslant \alpha - \delta$$

and

$$\dim_{P(unc)}(W_{m,\delta}, \Phi) \geqslant \dim_P(W_{m,\delta}) - \delta.$$

Using the definition of $W_{m,\delta}$, we get

$$E = \bigcup_{m=1}^{\infty} W_{m,\delta}.$$

Now, by packing dimension countable stability,

$$\dim_{P(unc)}(E, \Phi) \geqslant \dim_P(E) - \delta.$$

Since δ can be arbitrarily small,

$$\dim_{P(unc)}(E, \Phi) \geqslant \dim_P(E).$$

To complete the proof, it remains to note that E is any subset of $[0; 1]$. Thus, Φ is faithful. $\square$

Lemma 6.2. *Let Φ be a family of $\tilde{Q}$-expansion cylinders under the condition* $\inf q_{ij} > 0$. *Let F_ξ be a distribution function of a random variable ξ with independent $\tilde{Q}$-digits. Assume that the following condition holds:*

$$\lim_{n\to\infty} \frac{\ln \lambda(F(\Delta_n(x)))}{\ln \lambda(\Delta_n(x))} = 1, \quad \forall x \in [0; 1], \tag{2}$$

where $\Delta_n(x)$ is the n-rank cylinder that contains x.

Then $\Phi' = F(\Phi)$ is faithful for packing dimension calculation.

Proof. Φ' is the family of cylinders for some $\tilde{Q}$-expansion. Denote this expansion by $\tilde{Q}'$ and the corresponding numbers q_{ij} by q'_{ij}.

It is not clear that condition $\inf q'_{ij} > 0$ holds, so we cannot Theorem 5.1.

Let us show that the conditions of Lemma 6.1 hold for this expansion. We have

$$F\big(\Delta^{\tilde{Q}}_{a_1 a_2 \dots a_n}(x)\big) = \Delta^{\tilde{Q}'}_{a_1 a_2 \dots a_n}(x)$$

and

$$\ln \lambda\big(F\big(\Delta_n(x)\big)\big) = \ln\big(q'_{1a_1} q'_{2a_2} \dots q'_{na_n}\big).$$

Denote

$$M = \limsup_{i\to\infty} \frac{\ln q'_{ij_i}}{\ln(q'_{1j_1} q'_{2j_2} \dots q'_{(i-1)j_{i-1}})}.$$

To estimate M, we need the following equation:

$$\lim_{n\to 0} \frac{\ln \lambda(F(\Delta_n(x)))}{\ln \lambda(\Delta_n(x))} = \lim_{n\to 0} \frac{\ln(q'_{1j_1} q'_{2j_2} \dots q'_{(i-1)j_{i-1}}) + \ln q'_{ij_i}}{\ln(q_{1j_1} q_{2j_2} \dots q_{(i-1)j_{i-1}}) + \ln q_{ij_i}}.$$

Dividing the nominator and denominator of the last fraction by $\ln(q_{1j_1} q_{2j_2} \dots q_{(i-1)j_{i-1}})$, we obtain

$$\lim_{n\to 0} \frac{1 + \frac{\ln q'_{ij_i}}{\ln(q'_{1j_1} q'_{2j_2} \dots q'_{(i-1)j_{i-1}})}}{\frac{\ln(q_{1j_1} q_{2j_2} \dots q_{(i-1)j_{i-1}})}{\ln(q'_{1j_1} q'_{2j_2} \dots q'_{(i-1)j_{i-1}})} + \frac{\ln q_{ij_i}}{\ln(q'_{1j_1} q'_{2j_2} \dots q'_{(i-1)j_{i-1}})}} = \frac{1+M}{1+0} = 1 \quad \Rightarrow \quad M = 0.$$

It follows that $\tilde{Q}'$ satisfies the conditions of Lemma 6.1, and therefore Φ' is faithful. $\square$

Proof of the main result. Let us show that if F_ξ is *PDP*, then $\dim_P(\mu_\xi) = 1$.

Assume the converse. Then there exists a set E_α such that $\mu_\xi(E_\alpha) = 1$ and $\dim_P(E_\alpha) = \alpha$. Consider $F_\xi(E_\alpha)$. Since $\mu_\xi(E_\alpha) = 1$, we have $\lambda(E_\alpha) = 1$, and thus $\dim_P(F_\xi(E_\alpha)) = 1$.

We obtain the following inequality:

$$\dim_P\big(F_\xi(E_\alpha)\big) = 1 \neq \alpha = \dim_P(E_\alpha),$$

and this contradicts the assumption that F_ξ is *PDP*. Therefore, we will show that if F_ξ is *PDP*, then $\dim_P(\mu_\xi) = 1$.

The next part of the proof consists of two steps:

1. If $\dim_P(\mu_\xi) = 1$ and $B = 0$, then F_ξ is *PDP*;
2. If $\dim_P(\mu_\xi) = 1$ and $B \neq 0$, then F_ξ is not *PDP*.

Let ε be some positive number such that $\varepsilon < \frac{1}{2}q_{\min}$. Consider the following sets:

$$T^+_{\varepsilon,k} = \bigl\{j : j \in \mathbb{N},\ j \leqslant k, |p_{ij} - q_{ij}| \leqslant \varepsilon,\ i \in \{0, 1, \dots, s-1\}\bigr\},$$

$$T^-_{\varepsilon,k} = \{1, 2, \dots, k\} \setminus T^+_{\varepsilon,k},$$

$$T = \left\{k : k \in \mathbb{N},\ p_k < \frac{1}{2}q_{\min}\right\},$$

$$T_k = T \cap \{1, 2, \dots, k\},$$

$$T_{\varepsilon,k} = T^-_{\varepsilon,k} \setminus T_k.$$

Step 1. Let us show that if $\dim_P(\mu_\xi) = 1$ and $B = 0$, then F_ξ is *PDP*. Since $B = 0$, we see that

$$\lim_{k\to\infty} \frac{\sum_{j\in T_k} \ln p_j}{k \ln q_{\min}} = 0.$$

Consider the fraction

$$\frac{\ln \mu_\xi(\Delta_{a_1a_2\dots a_k(x)})}{\ln \lambda(\Delta_{a_1a_2\dots a_k(x)})} = \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j} + \sum_{j\in T_{\varepsilon,k}} \ln p_{a_j(x)j} + \sum_{j\in T_k} \ln p_{a_j(x)j}}{\sum_j \ln q_{a_j(x)j}}.$$

Split this fraction into three terms. Consider the first term

$$\frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j}}{\sum_j \ln q_{a_j(x)j}}.$$

It is easy to prove that

$$\begin{aligned}\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j} &\geqslant \sum_{j\in T^+_{\varepsilon,k}} \ln(q_{a_j(x)j} - \varepsilon)\\ &= \sum_{j\in T^+_{\varepsilon,k}} \left(\ln q_{a_j(x)j} + \ln\left(\frac{q_{a_j(x)j} - \varepsilon}{q_{a_j(x)j}}\right)\right)\\ &\geqslant \sum_{j\in T^+_{\varepsilon,k}} (\ln q_{a_j(x)j}) + |T^+_{\varepsilon,k}| \cdot \frac{2\varepsilon}{q_{\min}},\end{aligned}$$

where $|T^+_{\varepsilon,k}|$ is the number of elements in $T^+_{\varepsilon,k}$. On the other hand,

$$\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j} \leqslant \sum_{j\in T^+_{\varepsilon,k}} (\ln q_{a_j(x)j}) - |T^+_{\varepsilon,k}| \cdot \frac{2\varepsilon}{q_{\min}}.$$

Also,

$$1 + \frac{|T^+_{\varepsilon,k}| \cdot 2\varepsilon}{q_{\min} \cdot \sum_{j=0}^k \ln q_{a_j(x)j}} \leqslant \lim_{k\to\infty} \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j}}{\sum_{j=0}^k \ln q_{a_j(x)j}} \leqslant 1 - \frac{|T^+_{\varepsilon,k}| \cdot 2\varepsilon}{q_{\min} \cdot \sum_{j=0}^k \ln q_{a_j(x)j}}$$

(note that $\frac{|T^+_{\varepsilon,k}|\cdot 2\varepsilon}{q_{\min}\cdot\sum_{j=0}^k \ln q_{a_j(x)j}} < 0$).

Since $q_{\min} \leqslant q_{ij} \leqslant q_{\max}$ and $|T^+_{\varepsilon,k}| \leqslant k$, we have

$$1 + \frac{k \cdot 2\varepsilon}{q_{\min} \cdot k \ln q_{\max}} \leqslant \lim_{k\to\infty} \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j}}{\sum_{j=0}^{k} \ln q_{a_j(x)j}} \leqslant 1 - \frac{k \cdot 2\varepsilon}{q_{\min} \cdot k \ln q_{\max}}$$

and

$$1 + \frac{2\varepsilon}{q_{\min} \cdot \ln q_{\max}} \leqslant \lim_{k\to\infty} \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j}}{\sum_{j=0}^{k} \ln q_{a_j(x)j}} \leqslant 1 - \frac{2\varepsilon}{q_{\min} \cdot \ln q_{\max}}.$$

Since ε can be arbitrarily small, it follows that

$$\lim_{k\to\infty} \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j}}{\sum_{j=0}^{k} \ln q_{a_j(x)j}} = 1.$$

Similarly,

$$|T_{\varepsilon,k}| \ln\left(\frac{q_{\min}}{2}\right) \leqslant \sum_{j\in T_{\varepsilon,k}} \ln p_{a_j(x)j} \leqslant |T_{\varepsilon,k}| \ln\left(\frac{2-q_{\min}}{2}\right).$$

Therefore,

$$\frac{\sum_{j\in T_{\varepsilon,k}} \ln p_{a_j(x)j}}{k \ln q_{\min}} \leqslant \frac{|T_{\varepsilon,k}| \ln(\frac{q_{\min}}{2})}{k \ln q_{\min}} \leqslant \frac{|T_{\varepsilon,k}|(\ln(q_{\min}) + \ln(1/2))}{k \ln q_{\min}},$$

and the second term tends to zero as $k \to \infty$.

Consider the third term

$$\frac{\sum_{j\in T_k} \ln p_{a_j(x)j}}{\sum_j \ln q_{a_j(x)j}}.$$

It can be estimated by

$$\frac{\sum_{j\in T_k} \ln p_j}{k \ln q_{\min}},$$

and this value tends to zero as $k \to \infty$ because $B = 0$.

We obtain that

$$\lim_{k\to\infty} \frac{\mu_\xi(\Delta_{a_1a_2\ldots a_k(x)})}{\lambda(\Delta_{a_1a_2\ldots a_k(x)})} = 1.$$

Denote by Φ the cylinder family of given $\tilde{Q}$-expansion. Denote the image of Φ by $\Phi' = F_\xi(\Phi)$.

Using the Billingsley theorem for packing dimension [12], we have

$$\dim_P(E, \Phi) = 1 \cdot \dim_P\big(F_\xi(E), \Phi'\big) \quad \forall E \subset [0; 1].$$

To prove that $\dim_P(E) = \dim_P(F_\xi(E))$, it suffices to prove that Φ and Φ' are faithful.

Faithfulness of Φ is already proved. Faithfulness of Φ' was proved in Lemma 1 and Lemma 2. So, we have that $\dim_P(E) = \dim_P(F_\xi(E))$ and F_ξ is a *PDP*-transformation.

Step 2. Let us show that if $\dim_P(\mu_\xi) = 1$ and $B > 0$, then F_ξ is not *PDP*.

Similarly to step 1, consider the fraction

$$\frac{\mu(\Delta_{a_1a_2\ldots a_k(x)})}{\lambda(\Delta_{a_1a_2\ldots a_k(x)})} = \frac{\sum_{j\in T^+_{\varepsilon,k}} \ln p_{a_j(x)j} + \sum_{j\in T_{\varepsilon,k}} \ln p_{a_j(x)j} + \sum_{j\in T_k} \ln p_{a_j(x)j}}{\sum_j \ln q_{a_j(x)j}}$$

and split it into three terms. It is easy to see that the first term tends to 1 and the second term tends to 0 (as $k \to \infty$). Consider the third term.

Since $B > 0$, there exists a subsequence (k_m) such that

$$\lim_{m\to\infty} \frac{\sum_{j\in T_{k_m}} \ln \frac{1}{p_j}}{k_m} = B.$$

Consider the set

$$L = \left\{ x : x = \Delta_{a_1a_2\ldots a_k\ldots};\ \begin{cases} a_k \in \{0, 1, \ldots, s-1\} \text{ if } k \notin T \\ a_k = n_k, \text{ if } k \in T, \text{ where } p_{n_k k} = \min\limits_i p_{ik} \end{cases} \right\}.$$

Since the digits are in infinitely many places, it follows that $\lambda(L) = 0$. But combining

$$\lim_{m\to\infty} \frac{|T_{k_m}|}{k_m} = 0$$

and the formula for $\dim_P(\mu_\xi)$, we have $\dim_P(L) = 1$. It follows that

$$\forall x \in L \quad \lim_{m\to\infty} \frac{\ln \mu(\Delta_{a_1a_2\ldots a_{k_m}(x)})}{\ln \lambda(\Delta_{a_1a_2\ldots a_{k_m}(x)})} = 1 + B.$$

Thus, for any $\delta > 0$, there exists $m(\delta)$ such that for all $m > m(\delta)$, we have

$$1 + B - \delta \leqslant \frac{\ln \mu(\Delta_{a_1a_2\ldots a_{k_m}(x)})}{\ln \lambda(\Delta_{a_1a_2\ldots a_{k_m}(x)})} \leqslant 1 + B + \delta.$$

Thus, we have

$$\liminf_{k\to\infty} \frac{\ln \mu(\Delta_{a_1a_2\ldots a_k(x)})}{\ln \lambda(\Delta_{a_1a_2\ldots a_k(x)})} \geqslant 1 + B - \delta,$$

and (using the Billingsley theorem for $\dim_P$)

$$\dim_{P-\mu}(L) \cdot (1 + B - \delta) \leqslant \dim_P(L),$$

that is,

$$\dim_P\bigl(F_\xi(L)\bigr) \leqslant \frac{1}{1 + B - \delta}.$$

Since the last inequality holds for any δ, it follows that

$$\dim_P\bigl(F_\xi(L)\bigr) \leqslant \frac{1}{1 + B},$$

and F_ξ is not a *PDP*-transformation. $\square$

Corollary 6.1. *Let* $\inf_{i,j} q_{ij} := q_{\min}$*. Suppose that* $q_{\min} > 0$*. Let*

$$\begin{aligned} T &:= \left\{ k : k \in \mathbb{N},\, p_k < \frac{q_{\min}}{2} \right\}; \\ T_k &:= T \cap \{1, 2, \dots, k\}; \\ B &:= \limsup_{k\to\infty} \frac{\sum_{j\in T_k} \ln \frac{1}{p_j}}{k}. \end{aligned}$$

Let F_ξ *be the distribution function of a random variable* ξ *with independent* Q^**-representation. Then* F_ξ *preserves the packing dimension if and only if*

$$\begin{cases} \dim_P \mu_\xi = 1; \\ B = 0. \end{cases}$$

Corollary 6.2. *Let* $s \in \mathbb{N}$*,* $s \geqslant 2$*;*

$$\begin{aligned} T &:= \left\{ k : k \in \mathbb{N},\, p_k < \frac{1}{2s} \right\}; \\ T_k &:= T \cap \{1, 2, \dots, k\}; \\ B &:= \limsup_{k\to\infty} \frac{\sum_{j\in T_k} \ln \frac{1}{p_j}}{k}. \end{aligned}$$

Let F_ξ *be the distribution function of a random variable* ξ *with independent* s*-adic digits. Then* F_ξ *preserves the packing dimension if and only if*

$$\begin{cases} \dim_P \mu_\xi = 1; \\ B = 0. \end{cases}$$

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