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Characterization of the generalized Gamma–Lindley q-distribution by truncated moments of order statistics

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Abstract The generalized Gamma–Lindley q-distribution is the Jackson q-analogue of the classical four-parameter generalized Gamma–Lindley family. We show that a single truncated moment of an order statistic characterizes it completely among all distributions carried by the geometric lattice on which it lives. The argument rests on a q-successor kernel, defined as the negative q-derivative of a power of the survival function. This kernel is nonnegative on the lattice, has total Jackson mass one, and gives the conditional distribution of the order statistic of a prescribed rank once the number of observations below a lattice point is fixed. Jackson integration by parts expresses the associated truncated moment through the survival function evaluated one lattice step to the left; the resulting first-order q-difference equation, iterated along the lattice, reconstructs the survival function without a normalizing constant. The characterization is necessary and sufficient, requires no nondegeneracy assumption on the density, reduces to the classical one as $q \to 1^-$, and contains the Gamma, exponential, Lindley, Gamma–Lindley, and weighted Lindley q-distributions as particular cases.

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Introduction

A characterization theorem identifies a distribution from a single structural property, which turns a modeling assumption into one that can be tested. Truncated moments of order statistics are well suited to this purpose, since the conditional distribution of one order statistic given its predecessor involves the parent distribution only through its survival function. Kilany [16] characterized the Lindley distribution in this way, and Laribi, Masmoudi, and Boutouria [17] carried the method over to the four-parameter generalized Gamma–Lindley family, whose density is

$$f_\Theta(x) = \frac{\theta^{\alpha+1} x^{\alpha-1} e^{-\theta x}}{\beta\,\Gamma(\alpha+1)(\gamma+\theta)} \Big((\beta\gamma + \beta\theta - \theta)x + \alpha \Big), \qquad x > 0, \tag{1}$$

with $\alpha, \theta > 0$, $\beta \geq 1$, and $\gamma \geq 0$.

In Jackson's calculus [13–15] the derivative becomes the difference quotient $D_q f(x) = \big(f(qx) - f(x)\big)/\big((q-1)x\big)$ and the integral becomes a weighted sum over a geometric lattice, the fixed parameter $q \in (0,1)$ measuring the deformation. Several distributions have been transported into this calculus [11, 1, 6, 3, 4], and characterizations have followed [3–5, 7]. The term q-distribution is used in the literature for several different constructions: the discrete q-distributions built from q-numbers on the integers [10, 5], the Tsallis q-Gaussian and q-exponential distributions of nonextensive statistical mechanics [18], and the q-Gaussian measures attached to q-Hermite polynomials [8, 9]. Here the term refers exclusively to the distributions obtained by normalizing a nonnegative function against the Jackson integral, as in [11, 1, 3, 4, 6]. Each of them is carried by a geometric lattice, and so is the q-analogue $\mathrm{GGL}_q(\theta, \alpha, \beta, \gamma)$ of (1), which is introduced in Definition 1.4 below. All properties of GGL_q used below are proved in the present paper.

The argument of [17] does not transcribe directly, for four reasons. Classically the conditional density of $X_{(r+1)}$ given $X_{(r)} = x$ is $p\,S(y)^{p-1} f(y)/S(x)^p$ with $p = n - r$, and it has unit mass because $p\,S^{p-1} f = -\frac{d}{dy} S^p$. In Jackson calculus $D_q(S^p)$ is a telescoping sum rather than $p\,S^{p-1} D_q S$, so only the q-derivative itself delivers a kernel of total mass one. Next, integration by parts evaluates one factor at qx, which leaves the shifted power $S(qy)^p$ in the truncated moment, and that shift cannot be removed. Third, a distribution carried by the lattice has atoms, so that conditioning on the event $\{X_{(r)} = x\}$ permits ties, and the event that exactly r observations are at most x must take its place. Finally, the classical inversion formula recovers S only up to a multiplicative constant, whereas its q-analogue is a first-order q-difference equation; since the lattice accumulates at the origin, where S equals one, iterating that equation determines S outright.

The main result, Theorem 3.2, states that a distribution carried by the lattice $\{\omega_{q,\theta} q^j : j \in \mathbb{N}\}$ is $\mathrm{GGL}_q(\theta, \alpha, \beta, \gamma)$ if and only if a single truncated moment of the successor order statistic, conditioned on the number of observations at or

below a lattice point, has an explicit form that involves the parent only through the function Ψ_Θ of (17). The proof introduces the q-successor kernel $-D_q(S^p)$ (Section 2), shows that the truncated moment satisfies a first-order q-difference equation whose iteration returns S with no normalizing constant (Theorem 2.10), and then specializes this general mechanism to the survival function of GGL_q. The characterization is necessary and sufficient and needs no nondegeneracy assumption on the density; it contains the Gamma, exponential, Lindley, Gamma–Lindley, and weighted Lindley q-distributions (Corollary 3.4) and reduces to the characterization of [17] along lattice points converging to a fixed argument as $q \to 1^-$ (Theorem 3.5). Because $\mathrm{GGL}_q(\Theta)$ depends on (β, γ) only through one combination of them, what is characterized is the distribution and not the parameter vector (Remark 1.6). Section 1 collects the preliminaries, including the lattice framework and the properties of GGL_q; Section 2 develops the general theory on the lattice; Section 3 contains the characterization, its particular cases, and the classical limit.

1 Preliminaries and the GGL_q distribution

1.1 Jackson calculus

Throughout, $q \in (0,1)$ is fixed and $\mathbb{N} = \{0, 1, 2, \dots\}$. The q-derivative is

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \qquad x > 0, \tag{2}$$

and $[t]_q = (1-q^t)/(1-q)$ for $t \in \mathbb{R}$, with $[n]_q! = \prod_{i=1}^n [i]_q$ and $[0]_q! = 1$. Each $[t]_q$ depends continuously on q, tends to t as $q \to 1^-$, and stays below $[\infty]_q := 1/(1-q)$ when $t > 0$. For real ν and $x > 0$,

$$D_q x^\nu = \frac{(qx)^\nu - x^\nu}{(q-1)x} = [\nu]_q \, x^{\nu-1}. \tag{3}$$

Splitting $f(qx)g(qx) - f(x)g(x)$ as $f(qx)\big(g(qx) - g(x)\big) + g(x)\big(f(qx) - f(x)\big)$, and then again with f and g interchanged, yields the two product rules

$$D_q(fg)(x) = f(qx)D_q g(x) + g(x)D_q f(x) = f(x)D_q g(x) + g(qx)D_q f(x). \tag{4}$$

Which of the two is used matters in Theorem 2.10.

The q-exponential is $\mathrm{E}_q^x = \prod_{n\geq 0}\big(1+(1-q)xq^n\big)$. Since $\mathrm{E}_q^{cx} = (1+(1-q)cx)\,\mathrm{E}_q^{cqx}$, one has $D_q \mathrm{E}_q^{cx} = c\,\mathrm{E}_q^{cqx}$ for every constant c. Its factor of index $n = 0$ vanishes at $x = -[\infty]_q$, so

$$\mathrm{E}_q^{-[\infty]_q} = \prod_{n\geq 0}(1-q^n) = 0, \tag{5}$$

and $x^\nu \mathrm{E}_q^{-\lambda x}$ vanishes at $x = 1/\big((1-q)\lambda\big)$ for all $\nu \in \mathbb{R}$ and $\lambda > 0$. Every factor of

$$\mathrm{E}_q^{-\lambda q x} = \prod_{n\geq 0}\big(1 - (1-q)\lambda q^{n+1} x\big) \tag{6}$$

lies in $(0,1]$ when $0 \leq x \leq 1/\big((1-q)\lambda\big)$, so $\mathrm{E}_q^{-\lambda q x} > 0$ there.

For $b > 0$ the Jackson integral is

$$\begin{aligned} \int_0^b f(x)\, d_q x &= (1-q)\, b \sum_{j\geq 0} q^j f(q^j b), \\ \int_a^b f(x)\, d_q x &= \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x, \end{aligned} \tag{7}$$

and it converges whenever $|f(x)|x^\delta$ is bounded near the origin for some $\delta < 1$ [15, Ch. 19]. If F is a q-antiderivative of f and is continuous at 0, then [15, Ch. 20]

$$\int_a^b D_q F(x)\, d_q x = F(b) - F(a), \qquad D_q \int_0^x f(t)\, d_q t = f(x), \tag{8}$$

while integration by parts reads

$$\int_a^b f(x)\, d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)\, d_q f(x), \tag{9}$$

with $d_q g(x) = D_q g(x)\, d_q x$. The second integrand is evaluated at qx, and this is the source of every shifted argument below.

The q-Gamma function and its incomplete counterpart are [15, Ch. 21]

$$\begin{aligned} \Gamma_q(t) &= \int_0^{[\infty]_q} x^{t-1} \mathrm{E}_q^{-qx}\, d_q x \quad (t > 0), \\ \Gamma_q(a, z) &= \int_z^{[\infty]_q} u^{a-1} \mathrm{E}_q^{-qu}\, d_q u \quad (a > 0), \end{aligned}$$

for $0 \leq z \leq [\infty]_q$, and they satisfy $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$, $\Gamma_q(1) = 1$, $\Gamma_q(a, 0) = \Gamma_q(a)$, $\Gamma_q(a, [\infty]_q) = 0$, together with $\Gamma_q(t) \to \Gamma(t)$ as $q \to 1^-$.

Lemma 1.1. *Let $\lambda > 0$, $\omega_\lambda = 1/\big((1-q)\lambda\big)$, $a > 0$, and $0 \leq x \leq \omega_\lambda$. Then*

$$\begin{aligned} \int_x^{\omega_\lambda} t^{a-1} \mathrm{E}_q^{-\lambda q t}\, d_q t &= \frac{\Gamma_q(a, \lambda x)}{\lambda^a}, \\ \Gamma_q(a+1, z) &= z^a \mathrm{E}_q^{-z} + [a]_q \Gamma_q(a, z) \quad (0 \leq z \leq [\infty]_q), \end{aligned} \tag{10}$$

and $\Gamma_q(1, z) = \mathrm{E}_q^{-z}$.

Proof. Since $\lambda \omega_\lambda = [\infty]_q$, formula (7) with $b = \omega_\lambda$ gives

$$\int_0^{\omega_\lambda} t^{a-1} \mathrm{E}_q^{-\lambda q t}\, d_q t = \lambda^{-a}(1-q)[\infty]_q \sum_{j\geq 0} q^j \Big(\frac{q^j}{1-q}\Big)^{a-1} \mathrm{E}_q^{-q^{j+1}[\infty]_q} = \frac{\Gamma_q(a)}{\lambda^a},$$

the last sum being the Jackson sum of $\Gamma_q(a)$. Running the same computation on $[0, x]$ and subtracting gives the first identity. For the second, apply (9) on $[z, [\infty]_q]$ with $f(u) = u^a$ and $g(u) = -\mathrm{E}_q^{-u}$, so that $D_q g(u) = \mathrm{E}_q^{-qu}$ and $g(qu) = -\mathrm{E}_q^{-qu}$:

$$\Gamma_q(a+1, z) = \Big[-u^a \mathrm{E}_q^{-u}\Big]_z^{[\infty]_q} + [a]_q \int_z^{[\infty]_q} u^{a-1} \mathrm{E}_q^{-qu}\, d_q u = z^a \mathrm{E}_q^{-z} + [a]_q \Gamma_q(a, z),$$

the boundary term at $[\infty]_q$ vanishing by (5). Lastly, $-\mathrm{E}_q^{-u}$ is a q-antiderivative of E_q^{-qu}, so $\Gamma_q(1, z) = \big[-\mathrm{E}_q^{-u}\big]_z^{[\infty]_q} = \mathrm{E}_q^{-z}$ by (8) and (5). □

The classical limit in Section 3 requires the next two convergences to be uniform on compact sets. We know of no reference covering the incomplete function, so both are proved here.

Lemma 1.2. *Let $a > 0$ and $Z > 0$. As $q \to 1^-$, both $\mathrm{E}_q^{-z} \to e^{-z}$ and $\Gamma_q(a,z) \to \Gamma(a,z)$ uniformly for $z \in [0,Z]$.*

Proof. Suppose $(1-q)Z \le \frac{1}{2}$ and $0 \le z \le Z$. Each factor $u_n = (1-q)zq^n$ of E_q^{-z} then lies in $[0,\frac{1}{2}]$, where $-u-u^2 \le \log(1-u) \le -u$. Since $\sum_{n\ge0} u_n = z$ and $\sum_{n\ge0} u_n^2 = (1-q)z^2/(1+q) \le (1-q)Z^2$,

$$-z-(1-q)Z^2 \le \log \mathrm{E}_q^{-z} \le -z,$$

so $\log \mathrm{E}_q^{-z} \to -z$ uniformly on $[0,Z]$. The first claim follows because exp is uniformly continuous on $[-Z-1,0]$.

For the second, split $\Gamma_q(a,z) = \Gamma_q(a) - \gamma_q(a,z)$ and $\Gamma(a,z) = \Gamma(a) - \gamma(a,z)$, where $\gamma_q(a,z) = \int_0^z u^{a-1}\mathrm{E}_q^{-qu}\,d_qu$ and $\gamma(a,z) = \int_0^z u^{a-1}e^{-u}\,du$. Because $\Gamma_q(a) \to \Gamma(a)$, it is enough that $\gamma_q(a,z) \to \gamma(a,z)$ uniformly on $[0,Z]$. Setting $u_j = zq^j$, so that $u_j - u_{j+1} = (1-q)u_j$, formula (7) reads

$$\gamma_q(a,z) = \sum_{j\ge0} (u_j - u_{j+1})\, u_j^{a-1}\mathrm{E}_q^{-qu_j}.$$

Fix $\delta \in (0,Z]$ and put $J = \min\{j : u_j \le \delta\}$. From $0 < \mathrm{E}_q^{-qu} \le 1$,

$$\sum_{j\ge J}(u_j-u_{j+1})u_j^{a-1}\mathrm{E}_q^{-qu_j} \le (1-q)\sum_{i\ge0}(u_Jq^i)^a = \frac{u_J^a}{[a]_q} \le \frac{\delta^a}{[a]_q},$$

$$\int_0^{u_J} u^{a-1}e^{-u}\,du \le \frac{u_J^a}{a} \le \frac{\delta^a}{a}.$$

The terms with $0 \le j < J$ form a right-endpoint tagged Riemann sum for $u \mapsto u^{a-1}\mathrm{E}_q^{-qu}$ on $[u_J,z]$, of mesh at most $(1-q)Z$. By the first part that integrand tends uniformly to $u^{a-1}e^{-u}$ on $[q\delta,Z]$, and $u_J > q\delta$ by minimality of J, so the partial sum converges to $\int_{u_J}^z u^{a-1}e^{-u}\,du$ uniformly in $z \in [\delta,Z]$. When $z \le \delta$ one has $J = 0$, and the two bounds above already control $\gamma_q(a,z)$ and $\gamma(a,z)$. Since $[a]_q \to a$, picking δ with $4\delta^a/a < \varepsilon$ yields $|\gamma_q(a,z) - \gamma(a,z)| < \varepsilon$ on $[0,Z]$ for q near 1. □

1.2 The lattice and distributions carried by it

Fix $\omega > 0$ and write $y_j = \omega q^j$ for $j \in \mathbb{N}$,

$$\mathcal{L}_\omega = \{y_j : j \in \mathbb{N}\} \subset (0,\omega], \qquad \mathcal{L}_\omega^0 = \mathcal{L}_\omega \cup \{0\}.$$

The lattice accumulates only at the origin. By (7), a Jackson integral over $[0,\omega]$ is a sum over $\mathcal{L}_\omega$. For $x = y_k \in \mathcal{L}_\omega$ the two sums (7) over $[0,\omega]$ and over $[0,x]$ run over nested sublattices, whence

$$\int_x^\omega h(y)\,d_qy = (1-q)\,\omega\sum_{j=0}^{k-1} q^j h(y_j) = \sum_{j=0}^{k-1}(y_j - y_{j+1})\,h(y_j), \tag{11}$$

a finite sum with positive weights; for $x = 0$ the same expression is a convergent series with positive weights. If $x \notin \mathcal{L}_\omega^0$, however, $\int_x^\omega$ is a difference of Jackson sums over two different lattices and need not be a positive combination of values of h. Positivity statements below are therefore made only at points of $\mathcal{L}_\omega^0$, whereas identities proved from (8) and (9) hold for all x.

Three objects must be kept apart. The first is a *Jackson q-density* (a q-density, for short): a function $g \geq 0$ on $(0, \omega]$ with $\int_0^\omega g\, d_q x = 1$. The second is the distribution it defines, namely the distribution of a random variable X with $\mathbb{P}(X \in \mathcal{L}_\omega) = 1$ and masses

$$m_j = \mathbb{P}(X = y_j) = (1-q)\,\omega q^j\, g(y_j), \qquad j \in \mathbb{N}, \qquad \sum_{j \geq 0} m_j = 1; \tag{12}$$

only the values of g on $\mathcal{L}_\omega$ matter for this distribution, and conversely every distribution carried by $\mathcal{L}_\omega$ is given by the density $g(y_j) = m_j/\big((1-q)y_j\big)$ on $\mathcal{L}_\omega$. The third is the survival function. For $x = y_k$, formula (11) gives

$$S(y_k) := \int_{y_k}^\omega g\, d_q t = \sum_{j<k} m_j = \mathbb{P}(X > y_k), \qquad \mathbb{P}(X \leq y_k) = 1 - S(y_k), \tag{13}$$

and $S(0) := 1$. Thus the *probabilistic* survival function $\mathbb{P}(X > x)$ is defined on $\mathcal{L}_\omega^0$; it is nonincreasing, $S(\omega) = 0$, and $S(y_k) \uparrow 1$ as $k \to \infty$, that is, S is continuous at the origin along the lattice. The *analytic* survival function $x \mapsto \int_x^\omega g\, d_q t$ on $[0, \omega]$ agrees with S on $\mathcal{L}_\omega^0$, but at $x \notin \mathcal{L}_\omega^0$ it depends on the values of g outside the lattice and carries no probabilistic meaning. Finally, for $y \in \mathcal{L}_\omega$,

$$D_q S(y_j) = \frac{S(y_{j+1}) - S(y_j)}{(q-1)y_j} = -g(y_j). \tag{14}$$

Lemma 1.3. *Let $F, G : \mathcal{L}_\omega^0 \to \mathbb{R}$ and $x = y_k \in \mathcal{L}_\omega$. Then*

$$\int_x^\omega D_q F(y)\, d_q y = F(\omega) - F(x),$$

$$\int_x^\omega F\, d_q G = F(\omega)G(\omega) - F(x)G(x) - \int_x^\omega G(qy)\, d_q F(y).$$

Proof. By (2), $(y_j - y_{j+1}) D_q F(y_j) = F(y_j) - F(y_{j+1})$, so (11) gives the telescoping sum $\sum_{j<k} \big(F(y_j) - F(y_{j+1})\big) = F(y_0) - F(y_k)$, which is the first identity. The second follows by applying the first to FG and using the second product rule in (4). □

Unlike (8) and (9), these two identities involve finite sums and require no continuity at the origin; this is what allows Section 2 to dispense with any assumption on the behavior of g off the lattice.

1.3 The GGL_q *distribution*

The parameter vector is $\Theta = (\theta, \alpha, \beta, \gamma)$ with $\theta, \alpha > 0$, $\beta \geq 1$, and $\gamma \geq 0$, and we abbreviate

$$\lambda = [\theta]_q, \quad \mu = [\gamma]_q, \quad A_q = \beta(\mu + \lambda) - \lambda, \quad B_q = \beta(\mu + \lambda),$$
$$\omega_{q,\theta} = \frac{1}{(1-q)[\theta]_q} = \frac{[\infty]_q}{[\theta]_q}, \tag{15}$$

so that $A_q \geq \mu \geq 0$, $B_q \geq \beta\lambda > 0$, $A_q + \lambda = B_q$, and $\omega_{q,\theta} \to \infty$ as $q \to 1^-$. Note that $\omega_{q,\theta} = 1/(1-q^\theta)$. From now on, whenever $\omega = \omega_{q,\theta}$ is used, the lattice of Section 1.2 is $\mathcal{L}_{\omega_{q,\theta}}$.

Definition 1.4. $X \sim \mathrm{GGL}_q(\Theta)$ if X is carried by $\mathcal{L}_{\omega_{q,\theta}}$ and has the Jackson q-density

$$f_q(x) = \frac{[\theta]_q^{\alpha+1} x^{\alpha-1} \mathrm{E}_q^{-[\theta]_q q x}}{\beta\,\Gamma_q(\alpha+1)\big([\gamma]_q + [\theta]_q\big)} \Big(\big(\beta[\gamma]_q + \beta[\theta]_q - [\theta]_q\big)x + [\alpha]_q \Big), \qquad 0 < x \leq \omega_{q,\theta}, \tag{16}$$

and $f_q \equiv 0$ outside $(0, \omega_{q,\theta}]$.

In the notation (15) the density (16) reads $f_q(x) = C_\Theta^{-1} \lambda^{\alpha+1} x^{\alpha-1} \big(A_q x + [\alpha]_q\big) \mathrm{E}_q^{-\lambda q x}$ with $C_\Theta = \Gamma_q(\alpha+1) B_q$. It mixes a q-Gamma density of shape $\alpha+1$ with a generalized Lindley q-density, with weights $(\beta-1)/\beta$ and $1/\beta$, exactly as (1) does classically. To check this, put $G_q(x) = \lambda^{\alpha+1} x^\alpha \mathrm{E}_q^{-\lambda q x} / \Gamma_q(\alpha+1)$ and $L_q(x) = \lambda^{\alpha+1} x^{\alpha-1} \big(\mu x + [\alpha]_q\big) \mathrm{E}_q^{-\lambda q x} / \big(\Gamma_q(\alpha+1)(\mu+\lambda)\big)$. Both are nonnegative on $(0, \omega_{q,\theta}]$ and have Jackson integral one over $[0, \omega_{q,\theta}]$, by (10) at $x = 0$ and $[\alpha]_q \Gamma_q(\alpha) = \Gamma_q(\alpha+1)$. Moreover $\frac{\beta-1}{\beta} G_q + \frac{1}{\beta} L_q = f_q$: the coefficient of $x^\alpha \mathrm{E}_q^{-\lambda q x}$ is $\frac{\lambda^{\alpha+1}}{\Gamma_q(\alpha+1)} \cdot \frac{(\beta-1)(\mu+\lambda)+\mu}{\beta(\mu+\lambda)} = \frac{\lambda^{\alpha+1} A_q}{C_\Theta}$, and that of $x^{\alpha-1} \mathrm{E}_q^{-\lambda q x}$ is $\frac{\lambda^{\alpha+1}[\alpha]_q}{\beta\,\Gamma_q(\alpha+1)(\mu+\lambda)} = \frac{\lambda^{\alpha+1}[\alpha]_q}{C_\Theta}$.

Proposition 1.5. *Let Θ be above, $\omega = \omega_{q,\theta}$, and set*

$$\Psi_\Theta(x) = A_q (\lambda x)^\alpha \mathrm{E}_q^{-\lambda x} + [\alpha]_q B_q \Gamma_q(\alpha, \lambda x), \qquad 0 \leq x \leq \omega. \tag{17}$$

Then:

(i) $f_q(x) > 0$ for every $x \in (0, \omega]$, and $\int_0^\omega f_q \, d_q x = 1$, so that f_q is a Jackson q-density;

(ii) the analytic survival function $S_q(x) = \int_x^\omega f_q(t)\, d_q t$ satisfies

$$S_q(x) = \frac{\Psi_\Theta(x)}{C_\Theta}, \qquad 0 \leq x \leq \omega, \tag{18}$$

and S_q is continuous on $[0, \omega]$ with $S_q(0) = 1$, $S_q(\omega) = 0$, and $D_q S_q = -f_q$;

(iii) if $X \sim \mathrm{GGL}_q(\Theta)$, all masses $m_j = (1-q)\omega q^j f_q(\omega q^j)$ are positive, so that $0 < S_q(y) < 1$ for every $y \in \mathcal{L}_\omega \setminus \{\omega\}$ and $\Psi_\Theta(y) > 0$ there.

Proof. Since $A_q x + [\alpha]_q \geq [\alpha]_q > 0$ and $\mathrm{E}_q^{-\lambda q x} > 0$ on $(0, \omega]$ by (6), we have $f_q > 0$ there. For the normalization, (10) at $x = 0$ gives $\int_0^\omega t^{a-1} \mathrm{E}_q^{-\lambda q t}\, d_q t = \Gamma_q(a) \lambda^{-a}$, which we apply with $a = \alpha+1$ and $a = \alpha$:

$$\int_0^\omega f_q \, d_q x = \frac{\lambda^{\alpha+1}}{C_\Theta} \Big(A_q \frac{\Gamma_q(\alpha+1)}{\lambda^{\alpha+1}} + [\alpha]_q \frac{\Gamma_q(\alpha)}{\lambda^\alpha} \Big) = \frac{\Gamma_q(\alpha+1)(A_q+\lambda)}{C_\Theta} = 1,$$

because $[\alpha]_q \Gamma_q(\alpha) = \Gamma_q(\alpha+1)$ and $A_q + \lambda = B_q$. This proves (i). For (ii) write

$$S_q(x) = \frac{\lambda^{\alpha+1}}{C_\Theta} \Big(A_q I_1 + [\alpha]_q I_2 \Big),$$

$$I_1 = \int_x^{\omega} t^{\alpha} \mathrm{E}_q^{-\lambda q t}\, d_q t, \qquad I_2 = \int_x^{\omega} t^{\alpha-1} \mathrm{E}_q^{-\lambda q t}\, d_q t.$$

Apply (9) with $f(t) = t^{\alpha}$ and $g(t) = -\lambda^{-1}\mathrm{E}_q^{-\lambda t}$, so that $D_q g(t) = \mathrm{E}_q^{-\lambda q t}$, $g(qt) = -\lambda^{-1}\mathrm{E}_q^{-\lambda q t}$, and $D_q f(t) = [\alpha]_q t^{\alpha-1}$ by (3). The boundary term at ω vanishes by (5) because $\lambda\omega = [\infty]_q$. With $I_2 = \lambda^{-\alpha}\Gamma_q(\alpha, \lambda x)$ from (10) and $A_q + \lambda = B_q$,

$$\begin{aligned} I_1 &= \Big[-\frac{t^{\alpha}}{\lambda}\mathrm{E}_q^{-\lambda t} \Big]_x^{\omega} + \frac{[\alpha]_q}{\lambda} I_2 = \frac{x^{\alpha}}{\lambda}\mathrm{E}_q^{-\lambda x} + \frac{[\alpha]_q}{\lambda} I_2, \\ S_q(x) &= \frac{\lambda^{\alpha+1}}{C_\Theta}\Big(\frac{A_q}{\lambda} x^{\alpha}\mathrm{E}_q^{-\lambda x} + \frac{A_q + \lambda}{\lambda}[\alpha]_q \frac{\Gamma_q(\alpha, \lambda x)}{\lambda^{\alpha}} \Big) \\ &= \frac{A_q (\lambda x)^{\alpha}\mathrm{E}_q^{-\lambda x} + [\alpha]_q B_q \Gamma_q(\alpha, \lambda x)}{C_\Theta}. \end{aligned}$$

At $x = 0$ the first summand vanishes because $\alpha > 0$, while $[\alpha]_q B_q \Gamma_q(\alpha) = B_q \Gamma_q(\alpha + 1) = C_\Theta$, so $S_q(0) = 1$. At $x = \omega$ both summands vanish by (5) and $\Gamma_q(\alpha, [\infty]_q) = 0$. Continuity of S_q follows from (17), since $z \mapsto \Gamma_q(\alpha, z)$ is continuous on $[0, [\infty]_q]$. Finally $S_q = 1 - \int_0^x f_q\, d_q t$, so $D_q S_q = -f_q$ by (8). For (iii), $m_j > 0$ by (i); hence $S_q(y_k) = \sum_{j<k} m_j \in (0, 1)$ for $k \geq 1$ by (13), and $\Psi_\Theta = C_\Theta S_q > 0$ at those points. □

Remark 1.6. The distribution $\mathrm{GGL}_q(\Theta)$ depends on (β, γ) only through $B_q = \beta([\gamma]_q + [\theta]_q)$: indeed $A_q = B_q - \lambda$ and $C_\Theta = \Gamma_q(\alpha + 1)B_q$, so (16) is a function of (θ, α, B_q). The same holds classically, since (1) depends on (β, γ) only through $\beta(\gamma + \theta)$. Consequently the pair (β, γ) is not identifiable from the distribution (for instance, the Lindley case $\alpha = \beta = \gamma = 1$ is attained along a whole curve of values of (β, γ)), and the theorems below characterize the *distribution* $\mathrm{GGL}_q(\Theta)$, that is, the class of parameter vectors sharing the same (θ, α, B_q). No statement about the identifiability of the parameter vector is made or needed.

The q-moments of $\mathrm{GGL}_q(\Theta)$ enter the proof of Theorem 3.5, and we derive them here from (16).

Lemma 1.7. *Let $X \sim \mathrm{GGL}_q(\Theta)$ and let $m \geq 1$ be an integer. Then*

$$\mathbb{E}_q\big[X^m\big] = \int_0^{\omega_{q,\theta}} x^m f_q(x)\, d_q x = \frac{1}{[\theta]_q^m B_q}\Big(\prod_{i=1}^{m-1} [\alpha + i]_q \Big)\Big([\alpha + m]_q A_q + [\alpha]_q [\theta]_q \Big),$$

an empty product being 1. *The right-hand side depends continuously on $q \in (0, 1)$ and has a finite limit as $q \to 1^-$.*

Proof. By (16), $x^m f_q(x) = C_\Theta^{-1}\lambda^{\alpha+1}\big(A_q x^{\alpha+m} + [\alpha]_q x^{\alpha+m-1}\big)\mathrm{E}_q^{-\lambda q x}$. Taking $x = 0$ in (10) gives $\int_0^{\omega_{q,\theta}} t^{a-1}\mathrm{E}_q^{-\lambda q t}\, d_q t = \Gamma_q(a)\lambda^{-a}$, which we apply with $a = \alpha + m + 1$ and $a = \alpha + m$:

$$\begin{aligned} \mathbb{E}_q\big[X^m\big] &= \frac{\lambda^{\alpha+1}}{C_\Theta}\Big(A_q \frac{\Gamma_q(\alpha + m + 1)}{\lambda^{\alpha+m+1}} + [\alpha]_q \frac{\Gamma_q(\alpha + m)}{\lambda^{\alpha+m}} \Big) \\ &= \frac{A_q \Gamma_q(\alpha + m + 1) + [\alpha]_q \lambda\, \Gamma_q(\alpha + m)}{\lambda^m C_\Theta}. \end{aligned}$$

Table 1. Main symbols used throughout the paper

Symbol	Meaning
D_q, $[t]_q$, E_q^x	q-derivative (2), q-number, q-exponential $\prod_{n\geq 0}(1+(1-q)xq^n)$
$\Gamma_q(t)$, $\Gamma_q(a,z)$	q-Gamma and incomplete q-Gamma functions
$\mathcal{L}_\omega$, $y_j=\omega q^j$	geometric lattice and its points
m_j, S	masses $\mathbb{P}(X=y_j)$ and survival function $\mathbb{P}(X>x)$ on $\mathcal{L}_\omega^0$
λ, μ, A_q, B_q, $\omega_{q,\theta}$	abbreviations (15)
C_Θ, Ψ_Θ	$\Gamma_q(\alpha+1)B_q$, and the function (17), with $S_q=\Psi_\Theta/C_\Theta$
Λ_p, $\kappa_p(\cdot\,|\,x)$	q-minimum density and q-successor kernel (Section 2)
$W_{p,s}$, $J_{p,s}$	truncated q-moment and shifted integral, (23), (24)
$\tau_{k,s}$	the integrals of Definition 3.1

Iterating $\Gamma_q(t+1)=[t]_q\Gamma_q(t)$ down to $\Gamma_q(\alpha+1)$,

$$\Gamma_q(\alpha+m+1)=\Big(\prod_{i=1}^{m}[\alpha+i]_q\Big)\Gamma_q(\alpha+1),$$

$$\Gamma_q(\alpha+m)=\Big(\prod_{i=1}^{m-1}[\alpha+i]_q\Big)\Gamma_q(\alpha+1),$$

so the numerator equals $\big(\prod_{i=1}^{m-1}[\alpha+i]_q\big)\Gamma_q(\alpha+1)\big([\alpha+m]_qA_q+[\alpha]_q\lambda\big)$. Dividing by $\lambda^mC_\Theta=\lambda^m\Gamma_q(\alpha+1)B_q$ gives the stated formula. Each $[t]_q$ is continuous in q and tends to t, while $B_q\geq\beta\lambda>0$, whence the last assertion. □

1.4 Notation

Table 1 collects the main symbols; ω stands for $\omega_{q,\theta}$ whenever the lattice is that of $\mathrm{GGL}_q(\Theta)$.

2 The q-successor kernel, truncated moments, and inversion

Throughout this section $\omega>0$ is fixed, $\mathcal{L}=\mathcal{L}_\omega$, $y_j=\omega q^j$, and X is a random variable with $\mathbb{P}(X\in\mathcal{L})=1$, masses $m_j=\mathbb{P}(X=y_j)$, and survival function S on $\mathcal{L}^0$ as in (13); $p\geq 1$ is an integer. No assumption is made on any extension of S or of the density off the lattice. We use the notation $g(y_j)=m_j/\big((1-q)y_j\big)$ of Section 1.2, so that $D_qS=-g$ on $\mathcal{L}$ by (14).

Classically, with $p=n-r$, the conditional density of $X_{(r+1)}$ given $X_{(r)}=x$ is $p\,S(y)^{p-1}f(y)/S(x)^p$ for $y>x$; see [2, Ch. 2]. Its mass equals one because $p\,S(y)^{p-1}f(y)=-\frac{d}{dy}S(y)^p$, and it is this second expression that survives in Jackson calculus.

Lemma 2.1. *For any function u, any integer $p\geq 1$, and $x>0$,*

$$D_q\big(u^p\big)(x)=\Big(\sum_{i=0}^{p-1}u(qx)^iu(x)^{p-1-i}\Big)D_qu(x). \tag{19}$$

Proof. Put $a=u(qx)$ and $b=u(x)$. Then $a^p-b^p=(a-b)\sum_{i=0}^{p-1}a^ib^{p-1-i}$, and dividing by $(q-1)x$ gives the claim. □

The sum in (19) tends to $p\,u(x)^{p-1}$ as $q \to 1^-$.

Definition 2.2. The *q-minimum density of order p* attached to X is

$$\Lambda_p(y) = -D_q\big(S^p\big)(y) = g(y)\sum_{i=0}^{p-1} S(qy)^i S(y)^{p-1-i}, \qquad y \in \mathcal{L}, \tag{20}$$

the second equality being Lemma 2.1 with $u = S$ together with $D_q S = -g$.

Proposition 2.3. *Let $\nu_j = S(y_{j+1})^p - S(y_j)^p$ for $j \in \mathbb{N}$. Then $\Lambda_p \geq 0$ on $\mathcal{L}$, $(y_j - y_{j+1})\Lambda_p(y_j) = \nu_j$, and*

$$\int_x^{\omega} \Lambda_p(y)\, d_q y = S(x)^p \qquad (x \in \mathcal{L}^0). \tag{21}$$

Hence Λ_p is a Jackson q-density on $[0, \omega]$ with survival function S^p, and for independent copies $X_1, \ldots, X_p$ of X,

$$\nu_j = \mathbb{P}\big(\min(X_1, \ldots, X_p) = y_j\big).$$

Proof. Since $S(y_{j+1}) \geq S(y_j) \geq 0$, both forms in (20) are nonnegative, and by (2) one has $(y_j - y_{j+1})\Lambda_p(y_j) = S(y_{j+1})^p - S(y_j)^p = \nu_j$. For $x = y_k$, Lemma 1.3 with $F = -S^p$ gives $\int_x^{\omega} \Lambda_p\, d_q y = S(x)^p - S(\omega)^p = S(x)^p$. For $x = 0$, (11) gives $\sum_{j\geq 0} \nu_j = \lim_k S(y_k)^p = 1$. Finally, $\mathbb{P}(\min(X_1, \ldots, X_p) > y_k) = S(y_k)^p$ by independence and (13), so the mass of the minimum at y_j is $S(y_{j+1})^p - S(y_j)^p = \nu_j$. □

Remark 2.4. The transcription $[p]_q\, S(y)^{p-1} g(y)$ of the classical conditional density is also nonnegative on $\mathcal{L}$, but its Jackson integral over $[0, \omega]$ is in general not one, and the defect depends on the distribution of X. For $p = 2$, for instance, $\int_0^{\omega} [2]_q S\, g\, d_q y = (1 + q)\sum_j m_j S(y_j) = (1 + q)\big(1 - \sum_j m_j^2\big)/2$, which varies with (m_j); hence no constant depending on p alone can repair it. This is why Definition 2.2 takes the q-derivative itself.

Definition 2.5. Let $x \in \mathcal{L}$ with $x < \omega$ and $S(x) > 0$. The *q-successor kernel of order p* at x is

$$\kappa_p(y \,|\, x) = \frac{\Lambda_p(y)}{S(x)^p}, \qquad y \in \mathcal{L},\ y > x.$$

Dividing (21) by $S(x)^p$ shows $\kappa_p(\cdot \,|\, x)$ to be a Jackson q-density on $[x, \omega]$: it is nonnegative on $\mathcal{L}$ and $\int_x^{\omega} \kappa_p(y \,|\, x)\, d_q y = 1$, the integral being a finite sum of nonnegative terms by (11). Pointwise, it converges to $p\, S(y)^{p-1} f(y)/S(x)^p$ as $q \to 1^-$. The kernel also admits an exact reading in terms of order statistics.

A distribution carried by the lattice has atoms, so that the event $\{X_{(r)} = x\}$ has positive probability and permits the tie $X_{(r+1)} = X_{(r)}$; conditioning on it would not produce the kernel above. In the classical continuous setting, $\{X_{(r)} = x\}$ and the event that exactly r observations are at most x induce the same conditional distribution of $X_{(r+1)}$ and the formula used in [17] results. On the lattice only the second event is the right substitute.

Proposition 2.6. *Let $X_1, \dots, X_n$ be independent copies of X with order statistics $X_{(1)} \le \cdots \le X_{(n)}$, let $0 \le r < n$ and $p = n - r$, and let $x \in \mathcal{L}$ satisfy $x < \omega$ and $0 < S(x)$, with $S(x) < 1$ if $r \ge 1$. Set*

$$A_r(x) = \{\text{exactly } r \text{ of } X_1, \dots, X_n \text{ are } \le x\}.$$

Then $\mathbb{P}(A_r(x)) > 0$, on $A_r(x)$ one has $X_{(r+1)} > x$, and

$$\mathbb{P}\big(X_{(r+1)} > y \,\big|\, A_r(x)\big) = \Big(\frac{S(y)}{S(x)}\Big)^p, \qquad y \in \mathcal{L}^0,\ y \ge x. \tag{22}$$

The conditional distribution of $X_{(r+1)}$ given $A_r(x)$ therefore has Jackson q-density $\kappa_p(\cdot \,|\, x)$ on $(x, \omega]$.

Proof. By (13), $\mathbb{P}(A_r(x)) = \binom{n}{r}\big(1 - S(x)\big)^r S(x)^p > 0$ under the stated conditions. Given $A_r(x)$, exactly p observations exceed x. Independence and exchangeability make them independent with the distribution of X conditioned on $\{X > x\}$, and $X_{(r+1)}$ is their minimum. Hence, for $y \ge x$,

$$\mathbb{P}\big(X_{(r+1)} > y \,\big|\, A_r(x)\big) = \Big(\frac{\mathbb{P}(X > y)}{\mathbb{P}(X > x)}\Big)^p = \Big(\frac{S(y)}{S(x)}\Big)^p = \int_y^\omega \kappa_p(t \,|\, x)\, d_q t,$$

using (13) and Proposition 2.3. A distribution on $\mathcal{L}$ is determined by its survival function at lattice points. □

Definition 2.7. Let $s > 0$ be real and let $x \in \mathcal{L}$ satisfy $x < \omega$ and $S(x) > 0$. Put

$$W_{p,s}(x) = \int_x^\omega y^s \kappa_p(y \,|\, x)\, d_q y = \frac{1}{S(x)^p} \int_x^\omega y^s \Lambda_p(y)\, d_q y. \tag{23}$$

Proposition 2.6 identifies $W_{p,s}(x)$ with $\mathbb{E}\big[X_{(r+1)}^s \,\big|\, A_r(x)\big]$ for $p = n - r$, the q-analogue of the statistic $\mathbb{E}\big[X_{(r+1)}^s \,\big|\, X_{(r)} = x\big]$ of [16, 17].

Theorem 2.8. *With the notation of Definition 2.7,*

$$W_{p,s}(x) = x^s + \frac{[s]_q}{S(x)^p} J_{p,s}(x), \qquad J_{p,s}(x) = \int_x^\omega y^{s-1} S(qy)^p\, d_q y. \tag{24}$$

Proof. Apply Lemma 1.3 (the lattice form of (9)) on $[x, \omega]$ with $F(y) = y^s$ and $G(y) = S(y)^p$. Here $d_q G(y) = -\Lambda_p(y)\, d_q y$ by (20), and $D_q F(y) = [s]_q y^{s-1}$ by (3), so

$$\begin{aligned} -\int_x^\omega y^s \Lambda_p(y)\, d_q y &= \Big[y^s S(y)^p\Big]_x^\omega - [s]_q \int_x^\omega S(qy)^p y^{s-1}\, d_q y \\ &= -x^s S(x)^p - [s]_q J_{p,s}(x), \end{aligned}$$

because $S(\omega) = 0$. Divide by $-S(x)^p$ and use (23). □

The shifted power $S(qy)^p$ originates in the term $g(qx)$ of (9). The classical identity $W_{p,s}(x) = x^s + s\, S(x)^{-p} \int_x^\infty y^{s-1} S(y)^p\, dy$ appears only in the limit $q \to 1^-$ (Theorem 3.5).

Dividing by $W_{p,s}(x) - x^s$ and by $W_{p,s}(qx) - x^s$ in the next step calls for a nondegeneracy condition. Positivity of the mass at the upper endpoint is enough.

Proposition 2.9. *Assume $\mathbb{P}(X=\omega)=m_0>0$, equivalently $g(\omega)>0$, since $m_0=(1-q)\omega g(\omega)$. Then $S(x)>0$ for every $x\in\mathcal{L}\setminus\{\omega\}$, and $W_{p,s}(x)>x^s$ and $W_{p,s}(qx)>x^s$ for every $x\in\mathcal{L}\setminus\{\omega\}$.*

Proof. Let $x=y_k$ with $k\geq 1$. By (13), $S(y_k)=\sum_{j<k}m_j\geq m_0>0$. By (23) and Proposition 2.3, $W_{p,s}(x)=\sum_{j<k}y_j^s\nu_j/S(x)^p$ with weights $\nu_j/S(x)^p$ summing to one over $j<k$ (telescoping), and $y_j>x$ for $j<k$, so $W_{p,s}(x)\geq y_{k-1}^s>x^s$. Likewise $W_{p,s}(qx)=\sum_{j\leq k}y_j^s\nu_j/S(y_{k+1})^p\geq y_k^s=x^s$, and the inequality is strict because the weight at $j=0$ is $\nu_0/S(y_{k+1})^p=m_0^p/S(y_{k+1})^p>0$ and $y_0=\omega>x$ since $k\geq 1$. □

Theorem 2.10. *Let $m_0>0$, let $s>0$, and write $W=W_{p,s}$. Then, for every $x\in\mathcal{L}\setminus\{\omega\}$,*

$$\frac{D_q\big(S^p\big)(x)}{S(qx)^p}=-\frac{D_qW(x)}{W(x)-x^s},\qquad\text{equivalently}\qquad\Big(\frac{S(qx)}{S(x)}\Big)^p=\frac{W(x)-x^s}{W(qx)-x^s},\tag{25}$$

and consequently

$$S(x)^p=\prod_{j=0}^{\infty}\frac{W(q^{j+1}x)-(q^jx)^s}{W(q^jx)-(q^jx)^s},\tag{26}$$

the product being convergent. In particular W determines S on $\mathcal{L}$, hence the distribution of X, with no normalizing constant.

Proof. Set $v=W-x^s$, so that (24) reads $S(x)^pv(x)=[s]_qJ(x)$ with $J=J_{p,s}$. By (11), $J(qx)-J(x)=(x-qx)\,x^{s-1}S(qx)^p$, hence $D_qJ(x)=-x^{s-1}S(qx)^p$ for $x\in\mathcal{L}$. Applying D_q through the first product rule of (4), with S^p evaluated at qx, and using $D_qv=D_qW-[s]_qx^{s-1}$ from (3),

$$\begin{aligned}&S(qx)^pD_qv(x)+v(x)D_q\big(S^p\big)(x)=-[s]_qx^{s-1}S(qx)^p\\ \Longrightarrow\quad&v(x)D_q\big(S^p\big)(x)=-S(qx)^pD_qW(x),\end{aligned}$$

the terms carrying $[s]_qx^{s-1}S(qx)^p$ cancelling. For $x\in\mathcal{L}\setminus\{\omega\}$ Proposition 2.9 gives $v(x)>0$, and $S(qx)>0$ since $qx\in\mathcal{L}\setminus\{\omega\}$, so division yields the first identity of (25). Substituting $D_q(S^p)(x)=\big(S(qx)^p-S(x)^p\big)/\big((q-1)x\big)$ and $D_qW(x)=\big(W(qx)-W(x)\big)/\big((q-1)x\big)$ from (2), then multiplying by $(q-1)x$,

$$\begin{aligned}&1-\Big(\frac{S(x)}{S(qx)}\Big)^p=-\frac{W(qx)-W(x)}{W(x)-x^s}\\ \Longrightarrow\quad&\Big(\frac{S(x)}{S(qx)}\Big)^p=\frac{W(x)-x^s+W(qx)-W(x)}{W(x)-x^s}=\frac{W(qx)-x^s}{W(x)-x^s},\end{aligned}$$

both denominators being positive by Proposition 2.9. This is the second identity of (25).

Now fix $x=\omega q^k$ with $k\geq 1$ and apply (25) at $x,qx,\dots,q^{M-1}x$, all lying in $\mathcal{L}\setminus\{\omega\}$. Multiplying and telescoping,

$$\frac{S(q^Mx)^p}{S(x)^p}=\prod_{j=0}^{M-1}\frac{W(q^jx)-(q^jx)^s}{W(q^{j+1}x)-(q^jx)^s}.$$

By (13), $S(\omega q^{k+M}) = \sum_{j<k+M} m_j \to 1$ as $M \to \infty$, so the left side tends to $S(x)^{-p}$. The product therefore converges to $S(x)^{-p}$, and (26) follows on inverting, all factors being positive. Only W enters the right side of (26), and by (13) the survival function determines the masses. □

Equation (25) is the q-analogue of $\frac{d}{dx}\log S(x)^p = -w'(x)/\big(w(x) - x^s\big)$, which underlies the classical characterization [17]. Its classical integrated form recovers S only up to a multiplicative constant, to be fixed afterwards by a boundary condition. No constant appears in (26), because the lattice accumulates at 0 and the boundary condition $S(0^+) = 1$ enters through the limit. The positivity $m_0 > 0$ is used only to make (25) meaningful at every point of $\mathcal{L} \setminus \{\omega\}$; Theorem 3.2 shows that it need not be assumed in the characterization, because it is forced there.

3 Characterization, particular cases, and the classical limit

Definition 3.1. For $p \geq 1$, $s > 0$, $0 \leq k \leq p$, and $x \in \mathcal{L}_{\omega_{q,\theta}}$, set

$$\tau_{k,s}(x) = \int_x^{\omega_{q,\theta}} y^{s-1+\alpha k} \Big(\mathrm{E}_q^{-[\theta]_q q y}\Big)^k \Big(\Gamma_q\big(\alpha, [\theta]_q q y\big)\Big)^{p-k} \, d_q y,$$

and write $\tau_k = \tau_{k,1}$.

For $x \in \mathcal{L}_{\omega_{q,\theta}}$ these are finite sums with nonnegative terms by (11), since $0 < \mathrm{E}_q^{-[\theta]_q q y} \leq 1$ and $0 \leq \Gamma_q(\alpha, [\theta]_q q y) \leq \Gamma_q(\alpha)$ on $[0, \omega_{q,\theta}]$. No closed form for $\tau_{k,s}$ is required below. The factor $\big(\mathrm{E}_q^{-[\theta]_q q y}\big)^k$ must be kept as a k-th power, since $\mathrm{E}_q^a \mathrm{E}_q^b \neq \mathrm{E}_q^{a+b}$ in general. The classical simplification $(e^{-\theta y})^k = e^{-k\theta y}$ has no analogue here, and the same applies to $\Gamma_q(\alpha, [\theta]_q q y)^{p-k}$.

Theorem 3.2. *Let $q \in (0,1)$, let $\Theta = (\theta, \alpha, \beta, \gamma)$ with $\theta, \alpha > 0$, $\beta \geq 1$, and $\gamma \geq 0$, and let $\omega = \omega_{q,\theta}$ be as in (15). Fix an integer $p \geq 1$ and a real $s > 0$. Let X be a random variable with $\mathbb{P}(X \in \mathcal{L}_\omega) = 1$, and let $S(x) = \mathbb{P}(X > x)$. Then $X \sim \mathrm{GGL}_q(\theta, \alpha, \beta, \gamma)$ if and only if, for every $x \in \mathcal{L}_\omega \setminus \{\omega\}$ with $S(x) > 0$,*

$$W_{p,s}(x) = x^s + \frac{[s]_q}{\Psi_\Theta(x)^p} \int_x^{\omega} y^{s-1} \Psi_\Theta(qy)^p \, d_q y, \tag{27}$$

or, equivalently,

$$W_{p,s}(x) = x^s + [s]_q \, \frac{\displaystyle\sum_{k=0}^{p} \binom{p}{k} \Big(q^\alpha [\theta]_q^\alpha A_q\Big)^k \Big([\alpha]_q B_q\Big)^{p-k} \tau_{k,s}(x)}{\Psi_\Theta(x)^p}. \tag{28}$$

In terms of order statistics, let $s = 1$ and $p = n - r$. By Proposition 2.6, $W_{p,1}(x) = \mathbb{E}\big[X_{(r+1)} \,\big|\, A_r(x)\big]$ at every x with $\mathbb{P}(A_r(x)) > 0$, that is, for $S(x) > 0$ if $r = 0$ and for $0 < S(x) < 1$ if $r \geq 1$. There (28) reads

$$\mathbb{E}\big[X_{(r+1)} \,\big|\, A_r(x)\big] = x + \frac{\displaystyle\sum_{k=0}^{n-r} \binom{n-r}{k} \Big(q^\alpha [\theta]_q^\alpha A_q\Big)^k \Big([\alpha]_q B_q\Big)^{n-r-k} \tau_k(x)}{\Big(A_q \big([\theta]_q x\big)^\alpha \mathrm{E}_q^{-[\theta]_q x} + [\alpha]_q B_q \Gamma_q\big(\alpha, [\theta]_q x\big)\Big)^{n-r}}. \tag{29}$$

For $r = 0$ this is equivalent to (27), since $\mathbb{P}(A_0(x)) = S(x)^n > 0$ exactly when $S(x) > 0$. For $r \geq 1$, (29) holds at every $x \in \mathcal{L}_\omega \setminus \{\omega\}$ when $X \sim \mathrm{GGL}_q(\Theta)$, but it is only a necessary condition: a distribution carried by a single lattice point satisfies it vacuously, because no x has $0 < S(x) < 1$. The characterization itself is (27).

Formula (27) is the generic identity (24) with the survival function S replaced by $S_q = \Psi_\Theta / C_\Theta$, the constant C_Θ cancelling; this is the general q-successor and inversion mechanism of Section 2, and the characterization of GGL_q is its specific application. Formula (28) is the binomial expansion of (27). The theorem does not require $\mathbb{P}(X = \omega) > 0$ or any other nondegeneracy assumption: for $X \sim \mathrm{GGL}_q(\Theta)$ this holds by Proposition 1.5(iii), and in the converse direction it is deduced from (27) itself. The hypothesis that $S(x) > 0$ only restricts the points at which $W_{p,s}$ is defined.

Proof. *Equivalence of (27) and (28), and the form (29).* Replacing x by qy in (17) and using $(\lambda q y)^\alpha = q^\alpha \lambda^\alpha y^\alpha$,

$$\Psi_\Theta(qy) = q^\alpha \lambda^\alpha A_q \, y^\alpha \mathrm{E}_q^{-\lambda q y} + [\alpha]_q B_q \Gamma_q(\alpha, \lambda q y).$$

Raising this finite sum of two nonnegative terms to the power p by the binomial theorem, multiplying by y^{s-1}, and integrating term by term over the lattice interval $[x, \omega]$, which is legitimate for a finite sum of finite sums, produces $\tau_{k,s}(x)$ in the k-th term; this is (28). For $s = 1$ one has $[1]_q = 1$, and (29) is (28) with Ψ_Θ written out.

Necessity. Let $X \sim \mathrm{GGL}_q(\Theta)$. By Proposition 1.5, $S = S_q = \Psi_\Theta / C_\Theta$ on $\mathcal{L}_\omega^0$, $S > 0$ on $\mathcal{L}_\omega \setminus \{\omega\}$, and $m_0 > 0$. Theorem 2.8 applies, and inserting $S = \Psi_\Theta / C_\Theta$ makes C_Θ cancel:

$$\begin{aligned} W_{p,s}(x) &= x^s + \frac{[s]_q C_\Theta^p}{\Psi_\Theta(x)^p} \cdot \frac{1}{C_\Theta^p} \int_x^\omega y^{s-1} \Psi_\Theta(qy)^p \, d_q y \\ &= x^s + \frac{[s]_q}{\Psi_\Theta(x)^p} \int_x^\omega y^{s-1} \Psi_\Theta(qy)^p \, d_q y. \end{aligned}$$

Denote by w the right side of (27); thus $W_{p,s}^{(q)} = w$ on $\mathcal{L}_\omega \setminus \{\omega\}$, where $W_{p,s}^{(q)}$ is the truncated moment of $\mathrm{GGL}_q(\Theta)$.

Sufficiency. Let X be carried by $\mathcal{L}_\omega$, with masses m_j and survival function S, and suppose that $W_{p,s}(x) = w(x)$ at every $x \in \mathcal{L}_\omega \setminus \{\omega\}$ with $S(x) > 0$. Since $S(y_k) \uparrow 1$, such points exist. Let $j_0 = \min\{j : m_j > 0\}$. We first show $j_0 = 0$. Suppose $j_0 \geq 1$ and take $x = y_{j_0+1} \in \mathcal{L}_\omega \setminus \{\omega\}$, for which $S(x) = m_{j_0} > 0$. Then $S(y_j) = 0$ for $j \leq j_0$, so $\nu_j = 0$ for $j < j_0$ and $\nu_{j_0} = S(x)^p$; the conditional distribution of the minimum given that it exceeds x is the point mass at y_{j_0}, and $W_{p,s}(x) = y_{j_0}^s$. On the other hand, for $\mathrm{GGL}_q(\Theta)$ the same quantity is $w(x) = \sum_{j \leq j_0} y_j^s \nu_j^{(q)} / S_q(x)^p$ with weights summing to one, and $\nu_0^{(q)} > 0$, $y_0 > y_{j_0}$; hence $w(x) > y_{j_0}^s$, a contradiction. Therefore $m_0 > 0$.

Now $m_0 > 0$ gives $S > 0$ on $\mathcal{L}_\omega \setminus \{\omega\}$, so (27) holds at every point of $\mathcal{L}_\omega \setminus \{\omega\}$ for both X and a variable with the $\mathrm{GGL}_q(\Theta)$ distribution, whose survival function is S_q and whose truncated moment is w by necessity. Theorem 2.10 applies to S and to

S_q with this common w, so for every $x \in \mathcal{L}_\omega \setminus \{\omega\}$,

$$S(x)^p = \prod_{j=0}^{\infty} \frac{w(q^{j+1}x) - (q^j x)^s}{w(q^j x) - (q^j x)^s} = S_q(x)^p.$$

Both being nonnegative, $S(x) = S_q(x)$, while $S(\omega) = S_q(\omega) = 0$. By (13) the survival function fixes the masses through $m_j = S(y_{j+1}) - S(y_j)$, so $X \sim \mathrm{GGL}_q(\Theta)$. □

Remark 3.3. The condition $m_0 > 0$, that is $f(\omega) > 0$, is the q-analogue of the classical requirement that the parent be supported on all of $]a, b[$ with $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$. It is needed by the inversion mechanism of Theorem 2.10, because (25) divides by $S(qx)$ and by $W(x) - x^s$ and $W(qx) - x^s$; it is natural, since $m_0 = (1-q)\omega f(\omega)$ is the mass of the top lattice point. Within the class of distributions carried by $\mathcal{L}_\omega$, however, the identity (27) cannot hold for a distribution with $m_0 = 0$, as the proof of Theorem 3.2 shows, so the condition does not have to be imposed in the characterization.

Corollary 3.4. *Let X be as in Theorem 3.2, $p \geq 1$, and $s > 0$. Substituting parameter values in (15) and (17), and using $\Gamma_q(1, z) = \mathrm{E}_q^{-z}$ from Lemma 1.1, identity (27) becomes the following.*

(a) q-Gamma, $\beta = 1$, $\gamma = 0$. *Here* $A_q = 0$, $B_q = [\theta]_q$, *and* $\Psi_\Theta(x) = [\alpha]_q[\theta]_q\Gamma_q(\alpha, [\theta]_q x)$, *so only $k = 0$ survives and*

$$W_{p,s}(x) = x^s + \frac{[s]_q}{\Gamma_q\big(\alpha, [\theta]_q x\big)^p} \int_x^\omega y^{s-1} \Big(\Gamma_q\big(\alpha, [\theta]_q qy\big)\Big)^p \, d_q y.$$

(b) q-exponential, $\beta = 1$, $\gamma = 0$, $\alpha = 1$. *Here* $\Psi_\Theta(x) = [\theta]_q \mathrm{E}_q^{-[\theta]_q x}$ *and*

$$W_{p,s}(x) = x^s + \frac{[s]_q}{\big(\mathrm{E}_q^{-[\theta]_q x}\big)^p} \int_x^\omega y^{s-1} \Big(\mathrm{E}_q^{-[\theta]_q qy}\Big)^p \, d_q y.$$

(c) Lindley q-distribution [6], $\alpha = \beta = \gamma = 1$. *Here* $A_q = 1$, $B_q = 1 + [\theta]_q$, *and* $\Psi_\Theta(x) = \big([\theta]_q x + [\theta]_q + 1\big)\mathrm{E}_q^{-[\theta]_q x}$. *Since* $\Gamma_q(1, [\theta]_q qy) = \mathrm{E}_q^{-[\theta]_q qy}$, *the sum over k recombines by the binomial theorem and*

$$W_{p,s}(x) = x^s + \frac{[s]_q \displaystyle\int_x^\omega y^{s-1} \big([\theta]_q qy + [\theta]_q + 1\big)^p \Big(\mathrm{E}_q^{-[\theta]_q qy}\Big)^p \, d_q y}{\Big(\big([\theta]_q x + [\theta]_q + 1\big)\mathrm{E}_q^{-[\theta]_q x}\Big)^p}.$$

(d) Gamma–Lindley q-distribution, $\alpha = \gamma = 1$, $\beta \geq 1$. *Here* $A_q = \beta(1 + [\theta]_q) - [\theta]_q$, $B_q = \beta(1 + [\theta]_q)$, *and* $\Psi_\Theta(x) = \big(A_q[\theta]_q x + B_q\big)\mathrm{E}_q^{-[\theta]_q x}$; *in the same way,*

$$W_{p,s}(x) = x^s + \frac{[s]_q \displaystyle\int_x^\omega y^{s-1} \big(A_q[\theta]_q qy + B_q\big)^p \Big(\mathrm{E}_q^{-[\theta]_q qy}\Big)^p \, d_q y}{\Big(\big(A_q[\theta]_q x + B_q\big)\mathrm{E}_q^{-[\theta]_q x}\Big)^p}.$$

(e) q-analogue of the weighted Lindley distribution [12], $\gamma = \alpha$, $\beta = 1$. *Here* $A_q = [\alpha]_q$, $B_q = [\alpha]_q + [\theta]_q$, *and* $\Psi_\Theta(z) = [\alpha]_q\, \Xi(z)$, *where*

$$\Xi(z) = ([\theta]_q z)^\alpha \mathrm{E}_q^{-[\theta]_q z} + ([\alpha]_q + [\theta]_q)\Gamma_q\big(\alpha, [\theta]_q z\big).$$

The common factor $[\alpha]_q^p$ *cancels in* (27), *which becomes*

$$W_{p,s}(x) = x^s + [s]_q \frac{\displaystyle\int_x^\omega y^{s-1}\, \Xi(qy)^p \, d_q y}{\Xi(x)^p}.$$

In each case Theorem 3.2 characterizes the corresponding distribution among the distributions carried by $\mathcal{L}_\omega$.

The truncated q-moment lives on $\mathcal{L}_\omega^0$, and $\omega_{q,\theta}$ varies with q, so a fixed $x > 0$ will not generally lie on $\mathcal{L}_\omega^0$. We therefore evaluate along lattice points approaching x. For $0 < a < \omega_{q,\theta}$ let

$$\pi_q(a) = \omega_{q,\theta}\, q^{n(a)}, \qquad n(a) = \min\{n \in \mathbb{N} : \omega_{q,\theta} q^n \le a\}$$

denote the largest lattice point not exceeding a. Then $\pi_q(a) \in \mathcal{L}_{\omega_{q,\theta}}$ and

$$qa < \pi_q(a) \le a, \qquad \text{so} \qquad \pi_q(a) \to a \ \text{ as } q \to 1^-. \tag{30}$$

The proof of the next theorem rests on four facts: $\omega_{q,\theta} \to \infty$; the lattices become asymptotically dense in $(0, \infty)$; Jackson sums over them converge to ordinary integrals; and the q-exponential and the incomplete q-Gamma function converge to their classical counterparts (Lemma 1.2). Because the lattice has unbounded extent in the limit, the integral in (24) is split into a bounded part, treated as a Riemann sum, and a tail, controlled uniformly in q by a moment bound.

Theorem 3.5. *Fix* Θ, *an integer* $p \ge 1$, *a real* $s > 0$, *and a point* $x > 0$. *For* q *close enough to* 1 *set* $x_q = \pi_q(x)$, *let* $Y_q \sim \mathrm{GGL}_q(\Theta)$ *have survival function* S_q *and truncated* q*-moment* $W_{p,s}^{(q)}$, *and let* $X \sim \mathrm{GGL}(\Theta)$ *of* (1) *have survival function* S *and truncated moment* $W_{p,s}$. *Then*

$$\lim_{q\to 1^-} W_{p,s}^{(q)}(x_q) = W_{p,s}(x) = x^s + \frac{s}{S(x)^p} \int_x^\infty y^{s-1} S(y)^p \, dy. \tag{31}$$

Consequently (28) *converges to the identity characterizing* GGL(Θ) *in [17], and* (29) *to its order-statistic form.*

Proof. Throughout this proof we write $\omega = \omega_{q,\theta}$. Since $\omega \to \infty$, for q near 1 we have $x < \omega$ and $x_q \in \mathcal{L}_\omega \setminus \{\omega\}$, with $x_q \to x$ by (30). As $q \to 1^-$, $[t]_q \to t$ for real t, while $\mathrm{E}_q^{-z} \to e^{-z}$ and $\Gamma_q(a, z) \to \Gamma(a, z)$ uniformly on compact subsets of $[0, \infty)$ by Lemma 1.2. Hence $S_q \to S$ uniformly on compacts by (17) and (18), and $S_q(x_q) \to S(x) > 0$ by continuity of S. In view of (24) it remains to prove

$$\lim_{q\to 1^-} \int_{x_q}^\omega y^{s-1} S_q(qy)^p \, d_q y = \int_x^\infty y^{s-1} S(y)^p \, dy. \tag{32}$$

A uniform tail bound. Fix an integer $m > s$. At lattice points z one has $S_q(z) = \mathbb{P}(Y_q > z)$ by (13), so Markov's inequality gives $S_q(z) \le \mathbb{E}_q\left[Y_q^m\right] z^{-m}$. By Lemma 1.7 the moment $\mathbb{E}_q\left[Y_q^m\right]$ is continuous in q with a finite limit as $q \to 1^-$, so there are $q_0 < 1$ and $K < \infty$ with $\mathbb{E}_q\left[Y_q^m\right] \le K$ for all $q \in [q_0, 1)$. Since $S_q \le 1$ and $p \ge 1$, and since qy is again a lattice point when y is,

$$y^{s-1} S_q(qy)^p \le y^{s-1} S_q(qy) \le \frac{K}{q_0^m}\, y^{s-1-m} \qquad \text{for every } y \in \mathcal{L}_\omega,\ q \in [q_0, 1). \tag{33}$$

A lattice-aligned splitting. Let $\varepsilon > 0$ and choose $M > x$ with

$$\frac{2K}{q_0^m} \cdot \frac{M^{s-m}}{m-s} < \varepsilon \qquad \text{and} \qquad \int_M^\infty y^{s-1} S(y)^p \, dy < \varepsilon.$$

For q near 1 put $M_q = \pi_q(M)$, so that $M_q \to M$ and $x_q < M_q < \omega$. Both $\int_{x_q}^{M_q}$ and $\int_{M_q}^{\omega}$ are then finite sums over the same lattice with positive weights, by (11), so (33) may be summed termwise. Writing $M_q = \omega q^n$ and $\nu = s - m < 0$, and using (11) with $q^{n\nu} = (M_q/\omega)^\nu$,

$$\begin{aligned} 0 \le \int_{M_q}^{\omega} y^{s-1} S_q(qy)^p \, d_q y &\le \frac{K}{q_0^m} \int_{M_q}^{\omega} y^{\nu-1} \, d_q y = \frac{K}{q_0^m} (1-q)\, \omega \sum_{j=0}^{n-1} q^j \left(\omega q^j\right)^{\nu-1} \\ &= \frac{K}{q_0^m} (1-q)\, \omega^\nu \sum_{j=0}^{n-1} q^{j\nu} = \frac{K}{q_0^m} (1-q)\, \omega^\nu \, \frac{1-q^{n\nu}}{1-q^\nu} \\ &= \frac{K}{q_0^m} \cdot \frac{1-q}{q^\nu - 1} \left(M_q^\nu - \omega^\nu\right). \end{aligned}$$

Because $\nu < 0$ we have $q^\nu > 1$ and $M_q^\nu > \omega^\nu$, so the bound is positive. As $q \to 1^-$, $(1-q)/(q^\nu - 1) \to -1/\nu = 1/(m-s)$, $M_q^\nu \to M^\nu$, and $\omega^\nu \to 0$, so the bound tends to $K M^{s-m} / \left(q_0^m (m-s)\right)$. The tail is therefore at most ε for q near 1.

The head. Put $g_q(y) = y^{s-1} S_q(qy)^p$ and $g(y) = y^{s-1} S(y)^p$. On $[x/2, M]$ the convergence $g_q \to g$ is uniform, since $S_q \to S$ uniformly on compacts, S is uniformly continuous there, and $qy \to y$ uniformly. Because $\int_{x_q}^{M_q} 1 \, d_q y = M_q - x_q$ by (8),

$$\left| \int_{x_q}^{M_q} g_q \, d_q y - \int_{x_q}^{M_q} g \, d_q y \right| \le (M_q - x_q)\, \|g_q - g\|_{\infty, [x/2, M]} \longrightarrow 0.$$

Writing $x_q = \omega q^k$ and $M_q = \omega q^n$ with $n < k$, formula (11) gives

$$\int_{x_q}^{M_q} g \, d_q y = \sum_{j=n}^{k-1} \left(\omega q^j - \omega q^{j+1}\right) g\left(\omega q^j\right),$$

a Riemann sum for g on $[x_q, M_q]$ over the partition $\{\omega q^j\}_{j=n}^{k}$, tagged at right endpoints, of mesh $\max_j (1-q)\omega q^j \le (1-q)M \to 0$. As g is continuous on $[x/2, M]$ and $x_q \to x$, $M_q \to M$, this sum converges to $\int_x^M g(y)\, dy$.

Combining the three estimates and letting $\varepsilon \to 0$ gives (32), hence (31). The classical identity in (31) is the integration-by-parts formula for $\mathbb{E}\big[X^s_{(r+1)} \,\big|\, X_{(r)} = x\big]$ used in [17].

Finally, in (29) we have $[\theta]_q \to \theta$, $q^\alpha \to 1$, $[\alpha]_q \to \alpha$, $A_q \to \beta\gamma + \beta\theta - \theta$, $B_q \to \beta(\gamma+\theta)$, $\mathrm{E}_q^{-[\theta]_q qy} \to e^{-\theta y}$, and $\Gamma_q(\alpha, [\theta]_q qy) \to \Gamma(\alpha, \theta y)$. The numerator therefore tends to

$$\sum_{k=0}^{n-r} \binom{n-r}{k} (\beta\theta + \beta\gamma - \theta)^k \big(\alpha\beta(\gamma+\theta)\big)^{n-r-k} \int_x^\infty (\theta y)^{\alpha k} e^{-\theta y k}\, \Gamma(\alpha, \theta y)^{n-r-k}\, dy$$

and the denominator to $\big((\beta\theta+\beta\gamma-\theta)(\theta x)^\alpha e^{-\theta x} + \alpha\beta(\gamma+\theta)\Gamma(\alpha,\theta x)\big)^{n-r}$, which is the classical expression of [17]. □

Conclusion

Among all distributions carried by the lattice $\{\omega_{q,\theta} q^j : j \in \mathbb{N}\}$, the generalized Gamma–Lindley q-distribution is characterized by the truncated q-moment (27) of the successor order statistic, conditioned on the number of observations at or below a lattice point, with no nondegeneracy assumption on the density. The methodological contribution is the general q-successor and inversion mechanism of Section 2, which holds for every distribution on the lattice: the conditional density of the successor order statistic must be taken as $-D_q(S^p)$, whose total Jackson mass is one by construction, and not as $[p]_q S^{p-1} f$; integration by parts leaves the shifted power $S(qy)^p$ in (24), which is what makes (25) exact; and, because the lattice accumulates at the origin, the inversion formula (26) returns the survival function with no normalizing constant, so that uniqueness follows from the recursion itself. The application to the GGL_q family consists in inserting the explicit survival function Ψ_Θ / C_Θ, which yields (27) and its expanded form, and Corollary 3.4 covers the Gamma, exponential, Lindley, Gamma–Lindley, and weighted Lindley q-distributions. The classical characterization is recovered along lattice points converging to a fixed argument (Theorem 3.5). Since the distribution depends on (β, γ) only through one combination, the result characterizes the distribution and not the parameter vector.

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