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Scaling limits for the discretization error of a Brownian stochastic integral

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Abstract In this note, we derive a large-deviation-type scaling limit for the discretization error of a Brownian stochastic integral. Somewhat surprisingly, and to the best of our knowledge, this result has not been previously obtained in the literature.

Keywords Brownian motion, discretization of stochastic integrals, scaling limits
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1 Introduction and the main result

Let $W = (W_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and endowed with its augmented natural filtration $(\mathcal{F}_t)_{t\geq 0}$.

The well-known Martingale Representation Theorem (see [17, Section 4]) states that, for a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(W_1) \in L_2$, we have

$$f(W_1) = u(0,0) + \int_0^1 u_x(t, W_t)\, dW_t,$$

where $u = u(t,x)$ is the unique solution, under suitable growth conditions, of the heat equation

$$u_t + \frac{1}{2}u_{xx} = 0, \qquad u(1,x) = f(x),$$

and is given by

$$u(t,x) := \mathbb{E}\big[f(x + W_{1-t})\big] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f\big(x + \sqrt{1-t}\, y\big) e^{-y^2/2}\, dy.$$

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From the point of view of mathematical finance, the Martingale Representation Theorem can be viewed as a perfect replication result, and the process

$$\big(u_x(t,W_t)\big)_{t\in[0,1)}$$

is the corresponding delta-hedging strategy. In real market conditions, however, portfolio rebalancing can occur only at discrete times. Roughly speaking, as the frequency of rebalancing increases and the discrete hedging strategy approaches continuous rebalancing, the corresponding portfolio values converge to those of the continuously rebalanced portfolio.

Before formulating our main result, we need some preparation. Introduce the function

$$\Psi(c) := \frac{-c-\log(1-c)}{2}, \qquad c<1.$$

Observe that, for a standard normal random variable $Z\sim N(0,1)$,

$$\mathbb{E}\left[\exp\left(\frac{c}{2}\big(Z^2-1\big)\right)\right] = e^{\Psi(c)}, \qquad \forall c<1. \tag{1}$$

We now arrive at our main result, which will be proved in the next section.

Theorem 1. *Let $f\in C^3(\mathbb{R})$ be such that*

$$-\infty < \inf_{x\in\mathbb{R}} f''(x) \le \sup_{x\in\mathbb{R}} f''(x) < 1,$$

and suppose that f''' is Lipschitz continuous.

Set

$$S_n := f(W_1) - u(0,0) - \sum_{k=0}^{n-1} u_x\left(\frac{k}{n}, W_{\frac{k}{n}}\right)\big(W_{\frac{k+1}{n}} - W_{\frac{k}{n}}\big), \qquad n\in\mathbb{N},$$

and define $g:[0,1]\to\mathbb{R}_+$ by

$$g(t) := \sup_{x\in\mathbb{R}} \Psi\big(u_{xx}(t,x)\big).$$

Then g is a continuous function and

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\big[e^{nS_n}\big] = \int_0^1 g(t)\,dt.$$

Although discrete-time hedging in the Brownian setting has been studied extensively, the existing literature has focused primarily on the approximation of hedging strategies and stochastic integrals. The convergence of discrete-time delta hedging on equidistant grids was analyzed in [1, 15, 16], while [9, 12, 14, 11] considered more general deterministic discretization schemes. The connection between fractional smoothness of the payoff and L_p-approximation of stochastic integrals, including approximation along equidistant, adapted, and certain random time nets, was studied in [13]. The papers [10, 7, 8, 3] further investigated discretization schemes based on random stopping times.

These works are concerned with discretization errors under a fixed preference specification. By contrast, our analysis is carried out in a large-deviations regime induced by an exponential scaling of the risk-aversion parameter and is a natural continuation of the papers [4, 5], which investigated the following question: if an investor is allowed to rebalance only on an equidistant grid of n trading times, what is the limiting behavior, as $n \to \infty$, under exponential utility with risk-aversion parameter n? The corresponding asymptotically optimal rebalancing strategy is characterized by a nonlinear PDE that does not admit an explicit solution.

In the present paper, we analyze the same scaling regime but replace the asymptotically optimal rebalancing strategy with the simpler strategy obtained by directly discretizing the continuous-time delta hedge.

The assumption

$$\sup_{x\in\mathbb{R}} f''(x) < 1$$

is natural for ensuring that the scaling limit is finite, and it was also imposed in [4, 5]. To gain some intuition, consider

$$f(x) = \frac{1}{2}x^2.$$

Then

$$S_n = \frac{1}{2}\left(\sum_{k=0}^{n-1}(W_{\frac{k+1}{n}} - W_{\frac{k}{n}})^2 - 1\right),$$

and therefore

$$\mathbb{E}\left[e^{nS_n}\right] = \infty$$

for all n.

2 Proof of Theorem 1

Given a function $H : [0,1]\times\mathbb{R} \to \mathbb{R}$, we say that H is Lipschitz continuous if there exists $L < \infty$ such that

$$\left|H(t,x) - H(s,y)\right| \le L\left(|t-s| + |x-y|\right), \qquad (t,x),(s,y) \in [0,1]\times\mathbb{R}.$$

Proof. We start with some preparations. For $n \in \mathbb{N}$ and $j = 0, 1, \dots, n-1$, let

$$I_j^n = u\left(\frac{j+1}{n}, W_{\frac{j+1}{n}}\right) - u\left(\frac{j}{n}, W_{\frac{j}{n}}\right) - u_x\left(\frac{j}{n}, W_{\frac{j}{n}}\right)(W_{\frac{j+1}{n}} - W_{\frac{j}{n}})$$

and

$$Z_j^n := \sqrt{n}(W_{\frac{j+1}{n}} - W_{\frac{j}{n}}).$$

Set

$$a := \sup_{x\in\mathbb{R}} f''(x) < 1, \qquad b := \frac{1}{2}\sup_{x\in\mathbb{R}}\left|f''(x)\right|.$$

Then

$$u_{xx}(t,x) \le a, \qquad \left|u_t(t,x)\right| = \frac{1}{2}\left|u_{xx}(t,x)\right| \le b, \qquad (t,x) \in [0,1]\times\mathbb{R}.$$

Clearly, for all $n \in \mathbb{N}$ and $j = 0, 1, ..., n-1$ we have

$$I_j^n = u\left(\frac{j+1}{n}, W_{\frac{j+1}{n}}\right) - u\left(\frac{j}{n}, W_{\frac{j+1}{n}}\right) + u\left(\frac{j}{n}, W_{\frac{j}{n}} + \frac{Z_j^n}{\sqrt{n}}\right) - u\left(\frac{j}{n}, W_{\frac{j}{n}}\right) - u_x\left(\frac{j}{n}, W_{\frac{j}{n}}\right)\frac{Z_j^n}{\sqrt{n}}.$$

By the fundamental theorem of calculus, for the first difference we have

$$u\left(\frac{j+1}{n}, W_{\frac{j+1}{n}}\right) - u\left(\frac{j}{n}, W_{\frac{j+1}{n}}\right) = \int_0^{1/n} u_t\left(\frac{j}{n} + s, W_{\frac{j+1}{n}}\right) ds. \tag{2}$$

For the second difference, the one-dimensional Taylor formula with integral remainder,

$$F(x+h) - F(x) - F'(x)h = h^2 \int_0^1 (1-r)F''(x+rh)\, dr,$$

applied to $F(x) := u(\frac{j}{n}, x)$ and $h := \frac{Z_j^n}{\sqrt{n}}$, gives

$$u\left(\frac{j}{n}, W_{\frac{j}{n}} + \frac{Z_j^n}{\sqrt{n}}\right) - u\left(\frac{j}{n}, W_{\frac{j}{n}}\right) - u_x\left(\frac{j}{n}, W_{\frac{j}{n}}\right)\frac{Z_j^n}{\sqrt{n}} = \frac{|Z_j^n|^2}{n} \int_0^1 (1-r) u_{xx}\left(\frac{j}{n}, W_{\frac{j}{n}} + r\frac{Z_j^n}{\sqrt{n}}\right) dr. \tag{3}$$

Since, by Lemma 1, u_t and u_{xx} are Lipschitz continuous on $[0,1] \times \mathbb{R}$, combining (2) and (3) we obtain that there exists a constant $\hat{L} > 0$ such that, for all $n \geq 2$ and $j = 0, 1, \ldots, n-1$,

$$\left| nI_j^n - u_t\left(\frac{j}{n}, W_{\frac{j}{n}}\right) - \frac{1}{2} u_{xx}\left(\frac{j}{n}, W_{\frac{j}{n}}\right)|Z_j^n|^2 \right| \leq \hat{L}\left(\frac{1}{n} + \frac{|Z_j^n|}{\sqrt{n}} + \frac{|Z_j^n|^3}{\sqrt{n}}\right).$$

In particular, using $u_t = -u_{xx}/2$ and increasing $\hat{L}$ if necessary, for all $n \geq 2$,

$$\left| nI_j^n - \frac{1}{2} u_{xx}\left(\frac{j}{n}, W_{\frac{j}{n}}\right)\left(|Z_j^n|^2 - 1\right) \right| \leq \frac{\hat{L}}{\log n} \quad \text{on the event } |Z_j^n| \leq \left(\frac{\sqrt{n}}{\log n}\right)^{1/3}. \tag{4}$$

Also we observe that by applying (2), (3) and the bounds $|u_t| \leq b$ and $u_{xx} \leq a$ we have

$$nI_j^n \leq b + \frac{a}{2}|Z_j^n|^2, \quad n \in \mathbb{N}. \tag{5}$$

We finish the preparations by noting that since $u_{xx}(t,x) \leq a < 1$ for all $(t,x) \in [0,1] \times \mathbb{R}$, and Ψ is Lipschitz continuous on $(-\infty, a]$, Lemma 1 implies that $\Psi(u_{xx}(t,x))$ is Lipschitz continuous on $[0,1] \times \mathbb{R}$. Hence, g is also Lipschitz continuous.

We now divide the remainder of the proof of Theorem 1 into two steps.

Step I. In this step, we show the upper bound

$$\limsup_{n\to\infty} \frac{1}{n} \log\big(\mathbb{E}\big[e^{nS_n}\big]\big) \le \int_0^1 g(t)\,dt.$$

Choose $n \in \mathbb{N}$. Clearly, $S_n = \sum_{j=0}^{n-1} I_j^n$.

Thus, to complete the proof of the upper bound, it suffices to show that there exists a constant $L > 0$ such that, for all $n \ge 2$ and $j = 0, 1, \dots, n-1$,

$$\mathbb{E}\big[e^{nI_j^n} \,\big|\, \mathcal{F}_{j/n}\big] \le e^{\frac{L}{\log n} + g(\frac{j}{n})} + \frac{2e^b (\log n)^{1/3}}{\sqrt{2\pi}(1-a)n^{1/6}} \exp\left(-\frac{(1-a)n^{1/3}}{2(\log n)^{2/3}}\right). \tag{6}$$

Indeed, (6) implies that for every $\epsilon > 0$ and all sufficiently large n,

$$\mathbb{E}\big[e^{nI_j^n} \,\big|\, \mathcal{F}_{j/n}\big] \le e^{\epsilon + g(\frac{j}{n})}, \qquad j = 0, 1, \dots, n-1.$$

Hence, by the tower property of conditional expectation,

$$\mathbb{E}\big[e^{n\sum_{j=0}^{n-1} I_j^n}\big] \le e^{n\epsilon + \sum_{j=0}^{n-1} g(\frac{j}{n})}.$$

Since g is continuous,

$$\limsup_{n\to\infty} \frac{1}{n} \log\big(\mathbb{E}\big[e^{n\sum_{j=0}^{n-1} I_j^n}\big]\big) \le \epsilon + \int_0^1 g(t)\,dt.$$

Since $\epsilon > 0$ was arbitrary, we obtain the desired upper bound.

It remains to prove (6). Fix j and define the event

$$A_j^n = \left\{ \big|Z_j^n\big| \le \left(\frac{\sqrt{n}}{\log n}\right)^{1/3} \right\}.$$

We split the exponential according to the disjoint events A_j^n and $(A_j^n)^c$. Applying (4) on A_j^n and (5) on its complement gives

$$\begin{aligned} e^{nI_j^n} &= e^{nI_j^n} \mathbf{1}_{A_j^n} + e^{nI_j^n} \mathbf{1}_{(A_j^n)^c} \\ &\le \exp\left(\frac{1}{2} u_{xx}\left(\frac{j}{n}, W_{j/n}\right)\big(|Z_j^n|^2 - 1\big) + \frac{\hat{L}}{\log n}\right)\mathbf{1}_{A_j^n} \\ &\quad + \exp\left(b + \frac{a}{2}|Z_j^n|^2\right)\mathbf{1}_{(A_j^n)^c}. \end{aligned}$$

We now take conditional expectations and bound the first term by dropping its indicator.

Since $Z_j^n \sim N(0,1)$ is independent of $\mathcal{F}_{j/n}$, we obtain

$$\begin{aligned} \mathbb{E}\big[e^{nI_j^n} \,\big|\, \mathcal{F}_{j/n}\big] &\le \exp\left(\frac{\hat{L}}{\log n}\right)\mathbb{E}\left[\exp\left(\frac{1}{2} u_{xx}\left(\frac{j}{n}, W_{\frac{j}{n}}\right)\big(|Z_j^n|^2 - 1\big)\right) \,\middle|\, \mathcal{F}_{j/n}\right] \\ &\quad + \mathbb{E}\left[\exp\left(b + \frac{a}{2}|Z_j^n|^2\right)\mathbf{1}_{\{|Z_j^n| > (\frac{\sqrt{n}}{\log n})^{1/3}\}}\right] \end{aligned}$$

$$= \exp\left(\frac{\hat{L}}{\log n} + \Psi\left(u_{xx}\left(\frac{j}{n}, W_{\frac{j}{n}}\right)\right)\right)$$
$$+ \mathbb{E}\left[\exp\left(b + \frac{a}{2}\left|Z_j^n\right|^2\right)\mathbf{1}_{\{|Z_j^n|>(\frac{\sqrt{n}}{\log n})^{1/3}\}}\right].$$

Finally, by a direct estimate for the standard normal variable Z_j^n,

$$\mathbb{E}\left[\exp\left(b + \frac{a}{2}\left|Z_j^n\right|^2\right)\mathbf{1}_{\{|Z_j^n|>(\frac{\sqrt{n}}{\log n})^{1/3}\}}\right]$$
$$= \frac{2e^b}{\sqrt{2\pi}} \int_{(\frac{\sqrt{n}}{\log n})^{1/3}}^{\infty} \exp\left(-\frac{1-a}{2}z^2\right) dz$$
$$\leq \frac{2e^b (\log n)^{1/3}}{\sqrt{2\pi}(1-a)n^{1/6}} \exp\left(-\frac{(1-a)n^{1/3}}{2(\log n)^{2/3}}\right).$$

This yields (6) and completes the first step.

Step II. Fix $\epsilon > 0$. By Lemma 2, choose a continuous, piecewise-linear function $h : [0,1] \to \mathbb{R}$, with $h(0) = 0$, such that

$$\int_0^1 \Psi\big(u_{xx}\big(t, h(t)\big)\big)\, dt \geq \int_0^1 g(t)\, dt - \epsilon.$$

Choose $\delta > 0$ such that, for all $t \in [0,1]$ and $x, y \in \mathbb{R}$,

$$|x - y| < \delta \quad \Longrightarrow \quad \big|\Psi\big(u_{xx}(t,x)\big) - \Psi\big(u_{xx}(t,y)\big)\big| < \epsilon.$$

For $n \geq 2$, retain the notation I_j^n, Z_j^n from the preparations and set

$$r_n = \left(\frac{\sqrt{n}}{\log n}\right)^{1/3}, \qquad F_n = \Big\{\max_{0\leq j<n}\left|Z_j^n\right| \leq r_n\Big\},$$
$$E_n = \Big\{\max_{0\leq j\leq n}\big|W_{j/n} - h(j/n)\big| < \delta\Big\}.$$

The local remainder estimate (4) holds on F_n. Consequently,

$$\mathbb{E}\big[e^{nS_n}\big] \geq e^{-\hat{L}n/\log n}\, \mathbb{E}\left[\exp\left(\sum_{j=0}^{n-1} \tfrac{1}{2} u_{xx}(j/n, W_{j/n})\big(\big(Z_j^n\big)^2 - 1\big)\right)\mathbf{1}_{F_n}\right].$$

Define

$$c_j^n = u_{xx}(j/n, W_{j/n}), \qquad D_j^n = \exp\big(\tfrac{1}{2}c_j^n\big(\big(Z_j^n\big)^2 - 1\big) - \Psi\big(c_j^n\big)\big).$$

Then set

$$\frac{d\mathbb{Q}_n}{d\mathbb{P}} = \prod_{j=0}^{n-1} D_j^n.$$

By (1), $\mathbb{E}[D_j^n \mid \mathcal{F}_{j/n}] = 1$; successive conditioning therefore shows that this density has expectation one. Changing measure and then restricting to E_n gives

$$\mathbb{E}\big[e^{nS_n}\big] \geq e^{-\hat{L}n/\log n}\mathbb{E}_{\mathbb{Q}_n}\big[e^{\sum_{j=0}^{n-1}\Psi(c_j^n)}\mathbf{1}_{F_n}\big]$$

$$\geq \exp\left(\sum_{j=0}^{n-1} \Psi\big(u_{xx}\big(j/n, h(j/n)\big)\big) - n\epsilon - \frac{\hat{L}n}{\log n}\right) \mathbb{Q}_n(E_n \cap F_n).$$

We prove that $\liminf_n \mathbb{Q}_n(E_n \cap F_n) > 0$.

Write $m = \inf_{x\in\mathbb{R}} f''(x) > -\infty$ and retain $a = \sup_{x\in\mathbb{R}} f''(x) < 1$. Since $u_{xx}(t,x) = \mathbb{E}[f''(x + W_{1-t})]$, we have $m \leq u_{xx}(t,x) \leq a$. Define

$$\sigma(t,x) = \big(1 - u_{xx}(t,x)\big)^{-1/2}, \qquad \underline{\sigma} = (1-m)^{-1/2}, \qquad \overline{\sigma} = (1-a)^{-1/2}.$$

Then $0 < \underline{\sigma} \leq \sigma(t,x) \leq \overline{\sigma} < \infty$. If L_q is a Lipschitz constant for u_{xx} with respect to the Euclidean metric, the mean value theorem gives

$$\big|\sigma(t,x) - \sigma(s,y)\big|^2 \leq K_\sigma^2\big(|t-s|^2 + |x-y|^2\big), \qquad K_\sigma = \frac{L_q}{2(1-a)^{3/2}}. \tag{7}$$

Successive conditioning of the factors D_k^n with $k > j$ shows that the conditional density of Z_j^n under $\mathbb{Q}_n$, given $\mathcal{F}_{j/n}$, is

$$z \longmapsto \frac{\sqrt{1-c_j^n}}{\sqrt{2\pi}} \exp\left(-\frac{1-c_j^n}{2} z^2\right).$$

Indeed, this density is the standard normal density multiplied by $\exp(c_j^n(z^2-1)/2 - \Psi(c_j^n))$. Thus, conditionally on $\mathcal{F}_{j/n}$, Z_j^n is centered normal with variance $\sigma^2(j/n, W_{j/n})$. The Gaussian tail bound and the union bound yield

$$\mathbb{Q}_n\big(F_n^c\big) \leq 2n \exp\left(-\frac{r_n^2}{2\overline{\sigma}^2}\right) = 2n \exp\left(-\frac{(1-a)n^{1/3}}{2(\log n)^{2/3}}\right) \longrightarrow 0.$$

To apply Corollary 7.4.2 of [6], we include time in the state. Extend σ to $\mathbb{R}^2$ by $\widetilde{\sigma}(s,x) = \sigma(\max(0, \min(s,1)), x)$. The extension is bounded and Lipschitz. For $z = (s,x)$, let $\mu_n(z,\cdot)$ be the law of $z + H_n(z)$, where

$$H_n(z) = \left(\frac{1}{n}, \frac{\widetilde{\sigma}(z)}{\sqrt{n}} Z\right), \qquad Z \sim N(0,1), \qquad \widetilde{\sigma}(z) = \widetilde{\sigma}(s,x).$$

Under $\mathbb{Q}_n$, the chain $(j/n, W_{j/n})_{j=0}^n$ has this time-homogeneous transition kernel and starts at $(0,0)$. Extend it past time 1 with the same kernel if necessary.

The limiting drift and covariance are $\boldsymbol{b} = (1,0)^\top$ and $\boldsymbol{a}(z) = \operatorname{diag}(0, \widetilde{\sigma}^2(z))$. Expectations in the following estimates are over Z. Uniformly in z,

$$n\mathbb{E}H_n(z) = \boldsymbol{b}, \qquad n\mathbb{E}\big[H_n(z)H_n(z)^\top\big] = \operatorname{diag}\big(1/n, \widetilde{\sigma}^2(z)\big),$$
$$\mathbb{E}\big|H_n(z)\big|^4 = n^{-4} + 2\widetilde{\sigma}^2(z)n^{-3} + 3\widetilde{\sigma}^4(z)n^{-2} \leq C_0 n^{-2},$$

where $|\cdot|$ is the Euclidean norm, $\top$ denotes transpose, and $C_0 = 1 + 2\overline{\sigma}^2 + 3\overline{\sigma}^4$. For the truncated coefficients

$$\boldsymbol{b}_n = n\mathbb{E}[H_n \mathbf{1}_{\{|H_n|\leq 1\}}], \qquad \boldsymbol{a}_n = n\mathbb{E}\big[H_n H_n^\top \mathbf{1}_{\{|H_n|\leq 1\}}\big],$$

with their dependence on z suppressed, the fourth-moment bound gives

$$\sup_z \left|\boldsymbol{b}_n(z) - \boldsymbol{b}\right| \le C_0/n, \qquad \sup_z \left\|\boldsymbol{a}_n(z) - \boldsymbol{a}(z)\right\| \le (1 + C_0)/n,$$

where $\|\cdot\|$ is the matrix operator norm. Moreover, for every $r > 0$,

$$\sup_z n\mu_n\left(z, \left\{z' : \left|z' - z\right| \ge r\right\}\right) \le \frac{C_0}{nr^4} \longrightarrow 0.$$

Thus the coefficient limits and large-jump condition of the corollary hold. For smooth, compactly supported test functions $\varphi : \mathbb{R}^2 \to \mathbb{R}$, the limiting generator is

$$\mathcal{A}\varphi(s,x) = \partial_s\varphi(s,x) + \tfrac{1}{2}\widetilde{\sigma}^2(s,x)\partial_{xx}\varphi(s,x).$$

Its martingale problem is well-posed for every initial distribution: the stochastic differential equation (SDE) $dT_t = dt$, $dY_t = \widetilde{\sigma}(T_t, Y_t)\, dB_t$, where B is a standard one-dimensional Brownian motion, has bounded Lipschitz coefficients and a unique nonexplosive strong solution for every initial point. Also, $\boldsymbol{a}$ is continuous, symmetric and nonnegative definite, as required; nondegeneracy is not assumed in the corollary.

Write $D([0,1])$ for the space of real-valued right-continuous paths with left limits and $C([0,1])$ for its continuous-path subspace; $\|v\|_\infty = \sup_{0\le t\le 1}|v(t)|$. Corollary 7.4.2, applied with initial law the point mass at $(0,0)$, and projection onto the second coordinate therefore yield

$$V_t^n := W_{\lfloor nt\rfloor/n} \Longrightarrow X \quad \text{in } D\left([0,1]\right) \text{ with the Skorokhod } J_1 \text{ topology},$$

where $dX_t = \sigma(t, X_t)\, dB_t$, $X_0 = 0$. Here $T_t = t$ for the limiting process, and restriction to $[0,1]$ is valid because its paths are continuous. Convergence on the left refers to the laws under $\mathbb{Q}_n$.

Define the linear interpolation by

$$W_t^n = W_{j/n} + (nt - j)(W_{(j+1)/n} - W_{j/n}), \qquad j/n \le t \le (j+1)/n,$$

for $j = 0, \dots, n-1$. Then $\|W^n - V^n\|_\infty \le \max_{j<n}|W_{(j+1)/n} - W_{j/n}|$, and, for every $r > 0$,

$$\mathbb{Q}_n\left(\left\|W^n - V^n\right\|_\infty > r\right) \le 2n\exp\left(-\frac{nr^2}{2\overline{\sigma}^2}\right) \longrightarrow 0.$$

The perturbation theorem and continuity of X now give $W^n \Longrightarrow X$ in $C([0,1])$ with the uniform topology (see [2, Sections 3 and 12]).

Denote the probability governing X by $\mathbf{P}$.

We also verify directly that

$$p := \mathbf{P}\left(\sup_{0\le t\le 1}\left|X_t - h(t)\right| < \delta/2\right) > 0. \tag{8}$$

Since h is continuous and piecewise linear, it is absolutely continuous with bounded derivative, defined almost everywhere. Set $\theta_t = h'(t)/\sigma(t, X_t)$. The lower bound on

σ implies $|\theta_t| \le \|h'\|_\infty/\underline{\sigma}$, so Novikov's condition holds. Girsanov's theorem gives a probability $\mathbf{R}$, equivalent to $\mathbf{P}$ on the time interval $[0,1]$, with

$$\frac{d\mathbf{R}}{d\mathbf{P}} = \exp\left(\int_0^1 \theta_s\, dB_s - \frac{1}{2}\int_0^1 \theta_s^2\, ds\right), \qquad B_t^{\mathbf{R}} = B_t - \int_0^t \theta_s\, ds$$

a Brownian motion under $\mathbf{R}$. Hence

$$M_t := X_t - h(t) = \int_0^t \sigma(s, X_s)\, dB_s^{\mathbf{R}},$$

$$\langle M\rangle_t = \int_0^t \sigma^2(s, X_s)\, ds \le \overline{\sigma}^2, \qquad 0 \le t \le 1.$$

By the Dambis–Dubins–Schwarz theorem, on an extension if necessary, $M_t = \beta_{\langle M\rangle_t}$ for a Brownian motion β. One may extend M after time 1 by an independent Brownian motion so that its quadratic variation tends to infinity. Therefore

$$\mathbf{R}\Big(\sup_{0\le t\le 1}\big|X_t - h(t)\big| < \delta/2\Big) \ge \mathbf{R}\Big(\sup_{0\le v\le \overline{\sigma}^2} |\beta_v| < \delta/2\Big) > 0.$$

Here the final inequality is the standard positive probability that a Brownian motion remains in an open interval for a finite time. Equivalence of $\mathbf{R}$ and $\mathbf{P}$ proves (8). For these standard stochastic-calculus results, see [17, 18].

Finally the set $G = \{x \in C([0,1]) : \|x - h\|_\infty < \delta\}$ is open, and $\{W^n \in G\} \subseteq E_n$. The Portmanteau theorem gives

$$\liminf_n \mathbb{Q}_n(E_n) \ge \liminf_n \mathbb{Q}_n\big(W^n \in G\big) \ge \mathbf{P}(X \in G) \ge p > 0.$$

It follows that

$$\liminf_{n\to\infty} \mathbb{Q}_n(E_n \cap F_n) \ge \liminf_{n\to\infty}\big(\mathbb{Q}_n(E_n) - \mathbb{Q}_n\big(F_n^c\big)\big) \ge p > 0.$$

Returning to the lower bound for $\mathbb{E}[e^{nS_n}]$, taking logarithms, dividing by n and using the Riemann-sum convergence, we obtain

$$\liminf_{n\to\infty} \frac{1}{n}\log \mathbb{E}\big[e^{nS_n}\big] \ge \int_0^1 \Psi\big(u_{xx}\big(t, h(t)\big)\big)\, dt - \epsilon \ge \int_0^1 g(t)\, dt - 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, this completes the proof. □

Remark 1. *It is natural to ask whether Theorem 1 can be extended to a multidimensional Brownian motion or to more general diffusion processes. Since the aim of this note is to focus on the one-dimensional case, we do not study these extensions here. A full treatment would require further analysis and is left for future research.*

We also note that the financial interpretation of the result is limited to the Bachelier model, where the asset price can take negative values. Extending the present exponential utility approach to geometric Brownian motion is not straightforward. Indeed, the lognormal distribution does not have finite exponential moments for positive arguments. For this reason, the criterion used in this paper is not well suited to the geometric Brownian motion model. Such an extension would require a different functional and a different analytical approach.

3 Auxiliary lemmas

In this section, we prove two auxiliary results that were used in the proof of Theorem 1. We start with the following regularity result.

Lemma 1. *Let f satisfy the assumptions of Theorem 1. Then u_{xx} and $u_t = -u_{xx}/2$ are Lipschitz continuous on $[0,1] \times \mathbb{R}$ with respect to the Euclidean metric.*

Proof. Set $A = \|f''\|_\infty < \infty$ and let $K \geq 0$ be a Lipschitz constant for f'''. Taylor's formula gives

$$f''(x+1) - f''(x) - f'''(x) = \int_0^1 \big(f'''(x+r) - f'''(x)\big)\, dr.$$

Consequently,

$$\big|f'''(x)\big| \leq 2A + K/2 =: B_0, \qquad x \in \mathbb{R}.$$

This argument also covers $A = 0$ and $K = 0$, without division by either constant. The representations

$$q(t,x) := u_{xx}(t,x) = \mathbb{E}\big[f''(x + W_{1-t})\big], \qquad q_x(t,x) = \mathbb{E}\big[f'''(x + W_{1-t})\big]$$

show that $|q(t,x) - q(t,y)| \leq B_0|x-y|$ and that $q_x(t,\cdot)$ is K-Lipschitz. Hence, for every $z \in \mathbb{R}$,

$$\big|q(t,x+z) - q(t,x) - q_x(t,x)z\big| \leq \tfrac{K}{2} z^2.$$

If $0 \leq s < t \leq 1$, the heat-semigroup identity and a standard normal variable Z give

$$q(s,x) = \mathbb{E}\big[q(t, x + \sqrt{t-s}\, Z)\big].$$

Using the remainder bound and $\mathbb{E}Z = 0$, $\mathbb{E}Z^2 = 1$, we obtain

$$\big|q(s,x) - q(t,x)\big| \leq \tfrac{K}{2}(t-s).$$

Thus, for arbitrary (t,x), (s,y),

$$\begin{aligned}\big|q(t,x) - q(s,y)\big| &\leq B_0|x-y| + \tfrac{K}{2}|t-s| \\ &\leq \sqrt{B_0^2 + K^2/4}\sqrt{|t-s|^2 + |x-y|^2}.\end{aligned}$$

The assertion for u_t follows from the heat equation. □

Next, we prove the following.

Lemma 2. *Let f satisfy the assumptions of Theorem 1. For every $\epsilon > 0$, there exists a continuous, piecewise-linear function $h : [0,1] \to \mathbb{R}$, which depends on ϵ, such that $h(0) = 0$ and*

$$\int_0^1 \Psi\big(u_{xx}\big(t, h(t)\big)\big)\, dt \geq \int_0^1 g(t)\, dt - \epsilon.$$

Proof. Again, we notice that for the constant $a := \sup_{x\in\mathbb{R}} f''(x) < 1$, we have $u_{xx}(t,x) \le a$ for all $(t,x) \in [0,1]\times\mathbb{R}$. Since the function Ψ is Lipschitz continuous on $(-\infty, a]$, Lemma 1 implies that $\Psi(u_{xx}(t,x))$ is Lipschitz continuous on $[0,1]\times\mathbb{R}$. Hence, g is also Lipschitz continuous.

Fix $\epsilon > 0$. By the uniform continuity of $\Psi(u_{xx}(t,x))$ and g, we may choose a partition $0 = t_0 < t_1 < \cdots < t_N = 1$ and points $x_0, \dots, x_{N-1} \in \mathbb{R}$ such that the piecewise-constant function

$$\widetilde{h}(t) := x_i, \qquad t \in [t_i, t_{i+1}), \qquad \widetilde{h}(1) = x_{N-1}.$$

satisfies

$$\int_0^1 \Psi\big(u_{xx}\big(t, \widetilde{h}(t)\big)\big)\, dt \ge \int_0^1 g(t)\, dt - \frac{\epsilon}{2}. \tag{9}$$

Notice that for the constant $b := \frac{1}{2}\sup_{x\in\mathbb{R}} |f''(x)|$, we have $|u_{xx}(t,x)| \le 2b$ for all $(t,x) \in [0,1]\times\mathbb{R}$. Together with the relation $u_{xx}(t,x) \le a$ for all $(t,x) \in [0,1]\times\mathbb{R}$, this yields $M := \sup_{(t,x)\in[0,1]\times\mathbb{R}} \Psi(u_{xx}(t,x)) < \infty$. Next, replace each jump of $\widetilde{h}$ by a linear interpolation on sufficiently short disjoint intervals whose total length is less than $\frac{\epsilon}{8(1+M)}$ and, on an initial interval of length less than $\frac{\epsilon}{8(1+M)}$, interpolate linearly from 0 to x_0. In this way, we obtain a continuous, piecewise-linear function h satisfying $h(0) = 0$, which differs from $\widetilde{h}$ only on a set $A \subset [0,1]$ with $|A| < \frac{\epsilon}{4(1+M)}$. Set $h(1) = x_{N-1}$. Therefore,

$$\left| \int_0^1 \Psi\big(u_{xx}\big(t, h(t)\big)\big)\, dt - \int_0^1 \Psi\big(u_{xx}\big(t, \widetilde{h}(t)\big)\big)\, dt \right| \le 2M|A| < \frac{\epsilon}{2}.$$

This together with (9) gives

$$\int_0^1 \Psi\big(u_{xx}\big(t, h(t)\big)\big)\, dt \ge \int_0^1 g(t)\, dt - \epsilon. \qquad \square$$

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