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Lévy bases on manifolds

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Abstract This paper studies Lévy bases on Riemannian manifolds and related processes built from them. We argue that the canonical choice of intensity measure is the Riemannian measure on the manifold, and show how this allows for transporting properties of Lévy bases on $\mathbb{R}^d$ to the manifold setting via metatimes and subordination constructions. We show how this links to time-changed Lévy processes and how to naturally extend the setting to manifolds with (possibly stochastically) evolving metrics via generalized time-changes. As applications, we introduce so-called M-trawl processes, which generalize the classical trawl processes by letting the trawl operation be induced by a geometric flow, and we discuss the interpretation of Lévy bases on manifolds as Lévy white noise for stochastic partial differential equations on manifolds with Lévy-type forcing.

Keywords Lévy bases, infinite divisibility, random fields, time-change
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1 Introduction

Lévy bases are the extension of processes with independent increments to random measures. They have been studied in detail in [36, 32, 29] and are intimately connected to infinite divisibility and stochastic integrals. Lévy bases can also be seen as a natural generalization of classical Brownian noise, which provides connections and applications to stochastic partial differential equations and random fields, as discussed in [8]. In recent years, these connections have been explored quite extensively in the setting of ambit stochastics, where stochastic integrals against Lévy bases are used to construct a large and flexible class of random fields known as ambit fields. We refer to the monograph [10] for a comprehensive overview of ambit fields and

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ambit stochastics. A Lévy basis Λ can be abstractly defined on a δ-ring of a Lusin space S, but most of the literature is focused on the case of $S = \mathbb{R}^d$, which arises in many applications. In this setting, Λ exhibits many properties analogous to that of Lévy processes, and is characterized by a characteristic quadruplet, which is a usual characteristic triplet for Lévy processes, plus a so-called intensity measure. The intensity measure is tied to stationarity properties of Λ, and in a temporo-spatial setting on $\mathbb{R}^d$, the only choice yielding a stationary and homogeneous random measure is that of the Lebesgue measure. It is therefore not suprising that this choice is more or less taken for granted in most applied contexts. In this paper, we study the case where $S = M$ is a Riemannian manifold, which is a natural generalization of the Euclidean setting that has recently been motivated in an applied context in [23]. In this setting, the choice of the Lebesgue measure as intensity measure is no longer appropriate, but we argue that the canonical intensity measure is that of the Riemannian measure on M. The rich structure of manifolds lead to a more nuanced treatment of the properties of Lévy bases themselves, where the notions of homogeneity and stationarity are in particular more flexible. The generalization of known fields and processes built from Lévy bases on $\mathbb{R}^d$ therefore leads to interesting and non-trivial results in the manifold setting, once the canonical choice of intensity measure is pinned down. We relate Lévy bases on a k-manifold M, to Lévy bases on $\mathbb{R}^k$ via a subordination concept based on so-called *metatimes* cf. [7]. This allows for transferring several results from the Euclidean setting to the manifold setting, such as a Lévy-Itô decomposition in the spirit of [29]. We also briefly consider processes arising from evaluating Lévy bases along subsets of the manifold, characterizing some cases where this leads to Lévy processes and Lévy sheets, and connect this to conventional time-changes of Lévy processes. For example, when M is a regular C^1 curve embedded in a manifold N and parametrized as $\varphi(t)$, we show that the process $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process if and only if φ has constant speed (Proposition 3.5), and that in general, the process has the law of a Lévy process with the same characteristics, but time-changed by $\int_0^t \|\dot{\varphi}(s)\| ds$ (Proposition 3.7).

The analysis is extended to the case where the manifold on which Λ is defined evolves stochastically over time, and we connect this to extended subordination of Lévy bases, as studied in [5]. This offers an interesting viewpoint, as stochastic partial differential equations on evolving manifolds driven by Gaussian noise have been studied previously, e.g. in [18]. Lévy bases on such (possibly stochastically) evolving manifolds may be used to define non-Gaussian noise for the equations via stochastic integration, where the connections to extended subordination provide some tractable modelling tools. To this end, we also rigorously introduce appropriate notions of Lévy noise on manifolds, and show that Lévy bases on manifolds may be considered as classical types of Lévy white noise, as studied in [22, 8]. Here we note that the classical case of Lévy white noise is restricted, as these are usually defined on the space of Schwartz functions, which is not easily generalized to the manifold setting, and we are therefore only able to treat this type of noise on smaller classes of manifolds. Finally, we also study a generalization of trawl processes as introduced in [6], which are traditionally defined as processes of the form $\Lambda(A_t)$, where A_t is the time translation (trawling) of a fixed set A. We generalize this construction to so-called M-trawl processes, where the trawling is done via a general geometric flow, and we characterize their fundamental distributional properties which parallels the traditional setting.

Throughout the paper, we rely on many results which are standard to readers familiar with Lévy bases and/or basic geometry. Some basic geometric concepts are recapped in Appendix A, while some results on Lévy bases are recapped in Section 2, where we also extend some standard definitions and results to better accommodate the non-Euclidean setting. In Section 3, we argue that the natural choice of intensity measure is that of the Riemannian measure, and we establish connections to subordination of Lévy bases on Euclidean space, along with a Lévy-Itô decomposition for Lévy bases with the Riemannian measure as intensity measure. We also consider various processes of the form $t \mapsto \Lambda(M_t)$ that arise from this choice of intensity measure. In Section 4, we study the extension to Lévy bases defined on stochastically evolving manifolds and Section 5 extends the study of Lévy bases on evolving manifolds via the class of M-trawl processes and studies their distributional properties. Finally, in Section 6, we build upon the analysis of [8] to define Lévy type noise on manifolds, where the noise is interpreted as a suitable derivative of the Lévy basis.

2 Lévy bases and their properties

In this section, we introduce Lévy bases and the related properties that we shall need. Most fundamental results can be found in [32, 29], with the various associated integration theories also developed in [36, 32, 14, 37]. A detailed comparison of these integration theories may be found in [10]. We shall spend some effort reviewing the definition and properties of Lévy bases to set up the machinery for the following sections, but some novel results are also given. For example, in Section 2.2 we provide a characterization of the equivalence of homogeneity and stationarity of Lévy bases on general spaces.

2.1 Lévy bases

In all of the following, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Lusin space S, i.e., S is Borel isomorphic to a subset of $[0,1]$. We note that all Polish spaces, which are separable and completely metrizable topological spaces, are Lusin spaces. We let $\mathcal{S}$ be a σ-algebra on S, but Lévy bases are typically defined on δ-rings, which are non-empty sets that are closed under union, relative complementation, and countable intersection. We typically work on the δ-ring $\mathcal{B}_b(S)$ of totally bounded Borel sets in S, and for a general σ-algebra $\mathcal{S}$ we denote by $\mathcal{S}_b$ the δ-ring of all $\mathcal{S}$-measurable sets which are totally bounded in S. If $A \in \mathcal{S}$, we denote by $\mathcal{S} \restriction_A$ the restriction of $\mathcal{S}$ to A, and similarly for measures on $(S, \mathcal{S})$. We now recall the definition of a Lévy basis as an independently scattered and infinitely divisible random measure, along with their Lévy-Khintchine representation (derived in [32]), which leads to the so-called characteristic quadruplet that uniquely identifies the Lévy basis in law. It is straightforward to extend these results to $\mathbb{R}^d$-valued Lévy bases, which are studied in, e.g., [29].

Definition 2.1 (Lèvy basis)**.** A Lévy basis Λ on $(S, \mathcal{S})$ is a collection of random variables $\{\Lambda(A) \mid A \in \mathcal{S}_b\}$ with the following properties

1. For any sequence $A_1, A_2, \ldots$ of disjoint sets in $\mathcal{S}_b$ with $\bigcup_{j=1}^{\infty} A_j \in \mathcal{S}_b$ we have $\Lambda(\bigcup_{j=1}^{\infty} A_j) \overset{a.s.}{=} \sum_{j=1}^{\infty} \Lambda(A_j)$,.

2. Λ is *independently scattered*: For disjoint elements $A_1, \dots, A_n$ in $\mathcal{S}_b$ the random variables $\Lambda(A_1), \dots, \Lambda(A_n)$ are independent.

3. Λ is *infinitely divisible*: The law of the random variable $\Lambda(A)$ is infinitely divisible for any $A \in \mathcal{S}_b$.

Proposition 2.1 (Lévy-Khintchine formula for Lévy bases)**.** *Let Λ be a Lévy basis on $(S, \mathcal{S})$ and $A \in \mathcal{S}_b$. Then Λ satisfies the following Lévy-Khintchine formula.*

$$\begin{aligned} C\big(u; \Lambda(A)\big) &= \log\big(\mathbb{E}\big[\exp\big(\mathrm{i}u\Lambda(A)\big)\big]\big) \\ &= \mathrm{i}u\zeta^*(A) - \frac{1}{2}u^2 a^*(A) + \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbf{1}_{[-1,1]}(x)\big)\nu^*(dx, A), \quad (1) \end{aligned}$$

where ζ^* is a signed measure on $\mathcal{S}_b$, a^* is a measure on $\mathcal{S}_b$, and $\nu^*(\cdot, \cdot)$ is the so-called generalized Lévy measure such that $\nu^*(dx, A)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{S}_b$ and a measure on $\mathcal{S}_b$ for fixed dx.

Definition 2.2 (Control measure)**.** Let Λ be a Lévy basis with Lévy-Khintchine representation (1). Define the measure c by

$$c(A) = |\zeta^*|(A) + a^*(A) + \int_{\mathbb{R}} \min\{1, x^2\}\nu^*(dx, A), \quad A \in \mathcal{S}_b,$$

where $|\cdot|$ denotes the total variation. Then the extension of c to a σ-finite measure on $(S, \mathcal{S})$ is called the control measure of Λ.

Definition 2.3 (Characteristic quadruplet (CQ))**.** Let $(\zeta, a, \nu(dx, \cdot), c)$ be a quadruplet satisfying the following conditions

1. ζ and a are measurable functions on S and a is restricted to be non-negative.

2. For fixed z, $\nu(dx, z)$ is a Lévy measure on $\mathbb{R}$ and for fixed dx it is a measurable function on S.

3. c is a measure on $(S, \mathcal{S})$ such that $\int_A \zeta(z)c(dz)$ is a signed measure on $(S, \mathcal{S})$, $\int_A a(z)c(dz)$ is a measure on $(S, \mathcal{S})$ and $\int_A \nu(dx, z)c(dz)$ is a Lévy measure on $\mathbb{R}$ for fixed $A \in \mathcal{S}$.

Then $(\zeta, a, \nu(dx, \cdot), c)$ is called the characteristic quadruplet (henceforth CQ) of the Lévy basis with Lévy-Khintchine representation (1) which satisfies

$$\zeta(z) = \frac{d\zeta^*}{dc}(z), \quad a(z) = \frac{da^*}{dc}(z), \quad \nu(dx, z) = \frac{\nu^*(dx, \cdot)}{dc}(z).$$

where ζ^*, a^*, ν^* are defined in (2.2).

The CQ uniquely characterizes a Lévy basis in the sense that every Lévy basis determines a CQ of the form $(\zeta, a, \nu, c) = (\zeta(z), a(z), \nu(dx, z), c(dz))_{z \in S}$, and each such CQ determines, in law, a Lévy basis on $(S, \mathcal{S})$. In the converse case, however, the measure c is no longer the control measure in the sense of [32], and it is therefore in general termed the *intensity measure*. With the CQ at hand we may write the Lévy-Khintchine representation (1) more succinctly as

$$C\big(u; \Lambda(A)\big) = \int_A C\big(u; \Lambda'(z)\big)c(dz), \quad (2)$$

where Λ' denotes the so-called *Lévy seed*, which for each z is an infinitely divisible random variable having Lévy-Khintchine representation

$$C\big(u;\Lambda'(z)\big) = \mathrm{i}u\zeta(z) - \frac{1}{2}u^2 a(z) + \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbf{1}_{[-1,1]}(x)\big)\nu(dx,z).$$

In the following, whenever we speak of a Lévy basis with some CQ, we shall implicitly assume that there exists a u such that $C(u;\Lambda') \neq 0$, i.e., that the Lévy basis is not identically equal to zero.

Lemma 2.1. *Let Λ be an $\mathbb{R}^n$-valued Lévy basis on $(S,\mathcal{S})$ with CQ (ζ,a,ν,c) and suppose that there exists an increasing sequence $(S_k)_{k=1}^\infty \subseteq \mathcal{S}_b$ such that $\bigcup_{k=1}^\infty S_k = S$. Define the function $Q(z)$ by*

$$Q(z) = \big|\zeta(z)\big| + a(z) + \int_{\mathbb{R}^n} \big(1 \wedge |y|^2\big)\nu(dy,z), \tag{3}$$

and suppose that $\int_A Q(z)c(dz) < \infty$ whenever $c(A) < \infty$. Then Λ admits a unique extension (up to modification) to the δ-ring $\mathcal{R}_c = \{A \in \mathcal{S} \mid c(A) < \infty\}$ and this extension is unique (up to modification) and given by $\overline{\Lambda}$ with

$$\overline{\Lambda}(A) = \mathbb{P}\text{-}\lim_{k\to\infty} \Lambda(A \cap S_k), \quad A \in \mathcal{R}_c, \tag{4}$$

and has CQ (ζ,a,ν,c) on $(S,\mathcal{R}_c)$.

Proof. Let $A \in \mathcal{R}_c$. Since Λ is additive at the sets S_k increasing in k, it holds for $m > k$ that

$$\Lambda(A \cap S_m) - \Lambda(A \cap S_k) = \Lambda\big(A \cap (S_m \setminus S_k)\big). \tag{5}$$

Furthermore, it holds by assumption that $\int_A Q(z)c(dz) < \infty$ and $A \setminus S_k \downarrow \emptyset$, so

$$\int_{A\setminus S_k} Q(z)c(dz) \xrightarrow[k\to\infty]{} 0.$$

Note then that since $Q(z) \geq 0$, we have

$$c\big(A \cap (S_m \setminus S_k)\big) = \int_{A\cap(S_m\setminus S_k)} Q(z)c(dz) \leq \int_{A\setminus S_k} Q(z)c(dz) \xrightarrow[k\to\infty]{} 0.$$

It therefore follows from Proposition 2.1 in [32] that (5) tends to 0 in probability. Thus the sequence $(\Lambda(A \cap S_m))_{m=1}^\infty$ is Cauchy in probability and there exists a limit which is independent of the localizing sequence. The probability limit $\overline{\Lambda}$ inherits finite additivity and it is simple to show that $\overline{\Lambda}$ is independently scattered, since

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^N \big\langle u_j, \overline{\Lambda}(A_j)\big\rangle\right)\right] &= \lim_{k\to\infty}\prod_{j=1}^N \mathbb{E}\big[\exp\big(\mathrm{i}\big\langle u_j, \Lambda(A_j \cap S_k)\big\rangle\big)\big] \\ &= \prod_{j=1}^N \mathbb{E}\big[\exp\big(\mathrm{i}\big\langle u_j, \overline{\Lambda}(A_j)\big\rangle\big)\big],\end{aligned}$$

for pairwise disjoint $A_1, \ldots, A_N$. The limit $\overline{\Lambda}$ also inherits the same Lévy triplet, since $\Lambda(A \cap S_k)$ converges in probability, hence in distribution. It remains to show that $\overline{\Lambda}$ is also countably additive, in which case it is a Lévy basis on $\mathcal{R}_c$ with the same CQ as Λ. To this end, let $(A_j)_{j=1}^{\infty}$ be disjoint with $A = \bigcup_j A_j \in \mathcal{R}_c$. Then

$$\overline{\Lambda}(A) - \sum_{j=1}^{N} \overline{\Lambda}(A_j) = \overline{\Lambda}\left(A \setminus \bigcup_{j=1}^{N} A_j \right) \xrightarrow[N\to\infty]{} 0,$$

in probability. Since the $\overline{\Lambda}(A_j)$ are independent, it follows from Lèvy's equivalence theorem that the series $\sum_{j=1}^{\infty} \overline{\Lambda}(A_j)$ also converges almost surely. It is simple to show that any two extensions defined via (4) are modifications, since they agree on $\mathcal{S}_b$ and the probability limit is unique. □

2.2 *Stationarity and homogeneity*

Stationarity of a random measure on $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is typically defined as law invariance under translation, while law invariance under rotation is termed isotropy. Given that we wish to work on more general spaces, it is natural to consider stationarity as invariance under different operations. We shall therefore consider a group of operations on S, which we denote G, and where function composition is the group action. When needed, we shall use the notation $(S, \mathcal{S}, G)$ to represent the space on which we define the Lévy basis, along with the group of operations under consideration.

Definition 2.4 (Stationary Lévy basis)**.** Let Λ be a Lévy basis on $(S, \mathcal{S}, G)$. Let also $A_1, A_2, \ldots, A_n$ be a finite collection of elements of $\mathcal{S}_b$, such that $g(A_1), g(A_2), \ldots, g(A_n)$ belong to $\mathcal{S}_b$ for $g \in G$. We say that Λ is G-stationary, if it holds that

$$\begin{pmatrix} \Lambda(A_1) & \Lambda(A_2) & \cdots & \Lambda(A_n) \end{pmatrix} \overset{d}{=} \begin{pmatrix} \Lambda(g(A_1)) & \Lambda(g(A_2)) & \cdots & \Lambda(g(A_n)) \end{pmatrix}.$$

Definition 2.5 (Homogeneous Lévy basis)**.** Let Λ be a Lévy basis on $(S, \mathcal{S}, G)$ with CQ $(\zeta, a, \nu(dx, \cdot), c)$. We say that Λ is

1. *Factorisable* if the quantities $\zeta(z)$, $a(z)$, $\nu(dx, z)$ do not depend on z.
2. *G-invariant* if the measure $c(dz)$ is invariant on $\mathcal{S}$ with respect to any element of G, i.e. $c(g(A)) = c(A)$ for any $g \in G$ and $A \in \mathcal{S}$.
3. *G-Homogeneous* if Λ is factorisable and G-invariant.

Remark 2.1. The typical definition of homogenity as given in [10] corresponds to a factorisable Lévy basis on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), G)$, where $G = \mathbb{T}$ is the group of translations on $\mathbb{R}^d$. It is then simple to deduce that for such a $\mathbb{T}$-homogeneous Lévy basis, the intensity measure c must be a multiple of the Lebesgue measure λ^d on $\mathbb{R}^d$.

In applications, it is often most convenient to model Λ by specifying its characteristic quadruplet, much like it is simpler to specify the triplet of a Lévy process rather than the distribution of its increments. One is in principle free to specify the intensity measure c, as long as it satisfies the required regularity conditions of Definition 2.3. This provides a great deal of modelling flexibility, but stationarity properties are not guaranteed. In Proposition 2.2 and Corollary 2.1, we extend Proposition 32 of [10]

to characterize when homogeneity and stationarity is equivalent. In particular, Corollary 2.1 recovers the fact that the notions of G-homogeneity and G-stationarity are equivalent on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{T})$, but we note that in the non-Euclidean setting, homogeneity is generally stronger than stationarity.

Proposition 2.2. *Let Λ be a Lévy basis on $(S, \mathcal{S}, G)$. Then Λ is G-stationary if it is G-homogeneous. Conversely, if Λ is G-stationary and in addition factorisable, then Λ is G-homogeneous.*

Proof. Suppose first that Λ is G-homogeneous. We need to show that

$$\big(\Lambda(A_1), \Lambda(A_2), \ldots, \Lambda(A_n)\big) \overset{d}{=} \big(\Lambda\big(g(A_1)\big), \Lambda\big(g(A_2)\big), \ldots, \Lambda\big(g(A_n)\big)\big),$$

for any finite collection $A_1, A_2, \ldots, A_n$ of elements of $\mathcal{S}_b$, such that $g(A_1), g(A_2), \ldots, g(A_n)$ are in $\mathcal{S}_b$. Following the proof of Proposition 32 in [10], there exists a partition of say $N_n \in \mathbb{N}$ mutually disjoint sets $g(B_l)$ such that

$$\bigcup_{j=1}^{n} g(A_j) = \bigcup_{l=1}^{N_n} g(B_j).$$

Define the indices $I_j \subseteq \{1, \ldots, N_n\}$ for $j = 1, \ldots, n$ such that

$$g(A_j) = \bigcup_{k \in I_j} g(B_l).$$

As the $g(B_l)$'s are disjoint, we have that

$$\Lambda\big(g(A_j)\big) = \sum_{l \in I_j} \Lambda\big(g(B_l)\big).$$

The characteristic function of $(\Lambda(g(A_1)), \ldots, \Lambda(g(A_n)))$ for $u \in \mathbb{R}^n$ is thus

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{n} u_j \Lambda\big(g(A_j)\big)\right)\right] &= \prod_{l=1}^{N_n} \mathbb{E}\left[\exp\left(\mathrm{i}\cdot\Lambda\big(g(B_l)\big)\sum_{j=1}^{k} u_j \mathbf{1}_{I_j}(l)\right)\right] \\ &= \prod_{l=1}^{N_n} \exp\left(C\left(\sum_{j=1}^{k} u_j \mathbf{1}_{I_j}(l)\ ;\ \Lambda\big(g(B_l)\big)\right)\right),\end{aligned}$$

Using (1) and the factorisability property of Λ, we find that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{n} u_j \Lambda\big(g(A_j)\big)\right)\right] &= \prod_{l=1}^{N_n} \exp\left(c\big(g(B_l)\big) C\left(\sum_{j=1}^{k} u_j \mathbf{1}_{I_j}(l); \Lambda'\right)\right) \\ &= \prod_{l=1}^{N_n} \exp\left(c(B_l) C\left(\sum_{j=1}^{k} u_j \mathbf{1}_{I_j}(l); \Lambda'\right)\right) \\ &= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{k} u_j \Lambda(A_j)\right)\right],\end{aligned}$$

which shows that Λ is stationary with respect to G. On the other hand, if Λ is factorisable and stationary with respect to G, it holds that

$$C\big(u;\Lambda\big(g(A)\big)\big) = C\big(u;\Lambda(A)\big), \quad A \in \mathcal{S}_b, \tag{6}$$

which in turn implies

$$C(u;\Lambda\big(g(A)\big) = c\big(g(A)\big)C\big(u;\Lambda'\big) = C\big(u;\Lambda(A)\big) = c(A)C\big(u;\Lambda'\big).$$

As this holds for arbitrary $g \in G$ and $A \in \mathcal{S}_b$, it follows that $c(A) = c(g(A))$. □

Corollary 2.1. *Let Λ be a Lévy basis on $(S, \mathcal{S}, G)$ and let F denote the set*

$$F = \left\{ f \in L^1_{\mathrm{loc}}(S) : \int_A \big|f\big(g(z)\big) - f(z)\big|c(dz) = 0,\ g \in G, A \in \mathcal{S}_b \right\}.$$

Suppose that Λ is G-stationary and G-invariant. Suppose also that every function in F is constant c-almost everywhere. Then Λ is G-homogeneous.

Proof. Suppose that Λ is G-stationary and G-invariant and that F coincides with the set of c-a.e. constant functions. Then by (1) it follows that, for $u \in \mathbb{Q}$,

$$\int_{g(A)} C\big(u;\Lambda'(z)\big)c(dz) = \int_A C\big(u;\Lambda'(z)\big)c(dz).$$

It is then simple to deduce that

$$C\big(u;\Lambda'\big(g(z)\big)\big)) = C\big(u;\Lambda'(z)\big), \quad c\text{-a.e.}$$

This means that $C(u;\Lambda'(z)) \in F$ and therefore, by assumption, independent of z. Using that $C(u;\Lambda'(z))$ is continuous in u, we obtain a common c-null set outside which $C(u;\Lambda'(z))$ is independent of z. Hence Λ is factorisable and thus G-homogeneous. □

Besides controlling stationarity properties, the intensity measure is also related to path properties of the Lévy basis. To this end, we introduce the notion of dispersiveness, which is also a condition on the intensity measure. Proposition 2.3 gives a Lévy-Itô decomposition for dispersive Lévy bases on subsets of $\mathbb{R}^d$ and is essentially proven in [29, Proposition 4.5]. We also note that in the temporo-spatial setting where $S = \mathbb{R}\times S'$ with S' a general Lusin space, a similar result is proven in [16] (see their Remark 4.4). Finally, in Lemma 2.2 we show how the intensity measure transforms when the space on which Λ is defined is transformed.

Definition 2.6 (Dispersive Lévy basis)**.** A Lévy basis Λ on $(S, \mathcal{S})$ is said to be dispersive if

$$\Lambda\big(\{s\}\big) = 0, \quad \forall s \in S. \tag{7}$$

Proposition 2.3. *Let $S \in \mathcal{B}(\mathbb{R}^d)$ and let Λ be a dispersive $\mathbb{R}^n$-valued Lévy basis on $(S, \mathcal{S})$ with CQ (ζ, a, ν, c). Then there exists a modification Λ^* of Λ with the same CQ, which admits the following Lévy-Itô decomposition*

$$\Lambda^*(A) = \int_A \zeta(z)c(dz) + W(A) + \int_{\mathbb{R}^n} y\mathbf{1}_D(y)\big(N^d - \widetilde{N}\big)(dy, A) + \int_{\mathbb{R}^n} y\mathbf{1}_{D^c}(y)N(dy, A), \tag{8}$$

where W is an $\mathbb{R}^n$-valued Gaussian Lévy basis on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with CQ $(0, a, 0, c)$ and N is an $\mathbb{R}^n$-valued Poisson random measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with compensator $\widetilde{N}(dy, dz) = \nu(dy, z)\lambda_M(dz)$.

Proof. Set $\widetilde{S} = \mathbb{R}^d \setminus S$ and let $\widetilde{\Lambda}$ be a Lévy basis on $(\widetilde{S}, \widetilde{\mathcal{S}})$ with CQ (ζ, a, ν, c). Define Λ_R as the random measure

$$\Lambda_R(A) = \Lambda(A \cap S) + \widetilde{\Lambda}(A \cap \widetilde{S}), \quad A \in \mathcal{B}_b\big(\mathbb{R}^d\big),$$

which, due to the independently scattered property of Λ and $\widetilde{\Lambda}$, is a Lévy basis on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with CQ (ζ, a, ν, c). From Proposition 4.5 in [29], there exists a modification Λ_R^* of Λ_R with Lévy-Itô decomposition (8). Defining Λ^* by the restriction of Λ_R^* to $(S, \mathcal{S})$, we then obtain a modification of Λ with Lévy-Itô decomposition (8). □

Lemma 2.2. *Let Λ be a Lévy basis on $(S, \mathcal{S})$ with CQ (ζ, a, ν, c). Suppose that the conditions of Lemma 2.1 hold and let $\overline{\Lambda}$ denote the extension of Λ onto the δ-ring $\mathcal{R}_c = \{A \in \mathcal{S} : c(A) < \infty\}$ per Lemma 2.1. For $U \in \mathcal{S}$, let $\phi : (U, \mathcal{S} \restriction_U) \longrightarrow (S', \mathcal{S}')$ be a measurable function and suppose that the pushforward measure $c_\phi := \phi_*(c \restriction_U)$ is σ-finite. Then*

$$\Lambda_\phi(A) = \overline{\Lambda}\big(\phi^{-1}(A)\big), \quad A \in \mathcal{R}_{c_\phi},$$

defines a Lévy basis on the δ-ring $\mathcal{R}_{c_\phi}$. If the relevant standard-Borel assumptions for disintegration hold, then there exists a c_ϕ-almost everywhere unique Markov kernel κ from S' to U such that

$$c\big(B \cap \phi^{-1}(A)\big) = \int_A \kappa\big(B \mid z'\big)\, c_\phi\big(dz'\big),$$

for $A \in \mathcal{S}'$ and $B \in \mathcal{S} \restriction_U$. The CQ of Λ_ϕ is $(\zeta_\phi, a_\phi, \nu_\phi, c_\phi)$, where

$$\begin{aligned}
\zeta_\phi\big(z'\big) &= \int_U \zeta(z)\, \kappa\big(dz \mid z'\big), \\
a_\phi\big(z'\big) &= \int_U a(z)\, \kappa\big(dz \mid z'\big), \\
\nu_\phi\big(B, z'\big) &= \int_U \nu(B, z)\, \kappa\big(dz \mid z'\big), \quad B \in \mathcal{B}(\mathbb{R}).
\end{aligned}$$

If, in addition, it holds that $\mathcal{S}'_b \subseteq \mathcal{R}_{c_\phi}$, then the restriction of Λ_ϕ to $\mathcal{S}'_b$ is a Lévy basis on $(S', \mathcal{S}')$ in the usual sense.

Proof. For $A \in \mathcal{R}_{c_\phi}$, we have

$$c\big(\phi^{-1}(A)\big) = c_\phi(A) < \infty.$$

Hence $\phi^{-1}(A) \in \mathcal{R}_c$, so $\overline{\Lambda}(\phi^{-1}(A))$ is well-defined. Since inverse images preserve disjointness and countable unions, the independent scattering and countable additivity

properties of Λ_ϕ follow immediately from the corresponding properties of $\overline{\Lambda}$. It therefore holds that

$$\begin{aligned} C\big(u;\Lambda_\phi(A)\big) &= \int_{\phi^{-1}(A)} C\big(u;\Lambda'(z)\big)c(dz) \\ &= \int_A \int_U C\big(u;\Lambda'(z)\big)\kappa\big(dz \mid z'\big)c_\phi\big(dz'\big). \end{aligned}$$

The inner integral is the Lévy exponent corresponding to $(\zeta_\phi(z'), a_\phi(z'), \nu_\phi(\,\cdot\,, z'))$, leading to the claimed CQ for Λ_ϕ. □

2.3 *Stochastic integrals*

Lévy bases lend themselves well to defining random fields via stochastic integration. In this paper, we shall mainly rely on the integration theory of [32], which we briefly summarize. We write the stochastic integral with respect to a Lévy basis Λ on $(S, \mathcal{S})$ as $\int_A f(z)\Lambda(dz)$, for $A \in \mathcal{S}$ and $f : S \to \mathbb{R}$ a deterministic function. The integral is defined for simple functions $f_n(z) = \sum_{j=1}^n \mathbf{1}_{A_j}(z)x_j$ as

$$\int_A f_n(z)\Lambda(dz) = \sum_{j=1}^n x_j \Lambda(A \cap A_j),$$

which can be extended to general measurable and sufficiently integrable functions $f(z) = \lim_{n\to\infty} f_n(z)$ as the probability limit of simple functions

$$\int_A f(z)\Lambda(dz) = \mathbb{P}\text{-}\lim_{n\to\infty} \int_A f_n(z)\Lambda(dz).$$

The class of Λ-integrable functions is also characterized in [32, Proposition 2.4], and we recall this result in Proposition 2.4. Furthermore, we state Proposition 2.6 of [32] in Proposition 2.5, which characterizes the cumulant function of the stochastic integral.

Proposition 2.4. *Let $f : S \to \mathbb{R}$ be a measurable function, and let Λ be a Lévy basis on $(S, \mathcal{S})$ with CQ (ζ, a, ν, c). Then f is integrable with respect to Λ if and only if the following conditions hold*

$$\begin{aligned} \int_S \big|V_1\big(f(z), z\big)\big|c(dz) &< \infty, \\ \int_S \big|f(z)\big|^2 a(z)c(dz) &< \infty, \\ \int_S \big|V_2\big(f(z), z\big)\big|c(dz) &< \infty \end{aligned}$$

where

$$\begin{aligned} V_1(x, z) &= x\zeta(z) + \int_{\mathbb{R}} \big(h(zx) - zh(x)\big)\nu(dx, z), \\ V_2(x, z) &= \int_{\mathbb{R}} \min\big\{1, |xz|^2\big\}\nu(dx, z), \end{aligned}$$

with $h(y) = y\mathbf{1}_{\{|y|\le 1\}}(y)$.

Proposition 2.5. *Let Λ be a Lévy basis on $(S, \mathcal{S})$ with CQ (ζ, a, ν, c) and let Λ' be the associated Lévy seed. Let $f : S \to \mathbb{R}$ be a measurable and Λ-integrable function. Then $\int_S |C(uf(z); \Lambda'(z))| c(dz) < \infty$, and*

$$C\left(u; \int_S f(z)\Lambda(dz)\right) = \int_S C\big(uf(z); \Lambda'(z)\big)c(dz), \quad u \in \mathbb{R}.$$

It is often of interest to integrate stochastic functions, and in particular other random fields. In the theory of [32], we may integrate random fields X, provided that these are independent of the driving Lévy basis Λ. Indeed, the independent random field X can be realized on a product probability space and the stochastic integral defined pathwise, for fixed realizations of X. This approach is outlined and utilized in [10, 9]. In particular, this allows for obtaining the cumulant function in Proposition 2.5 conditionally on X (see [10, Proposition 40] or [9, Proposition 1]).

In many settings, it is also the case that the space S has a temporo-spatial structure of the form $S = \mathbb{I} \times S'$, where $\mathbb{I}$ is an interval in $\mathbb{R}$. In this case, Lévy bases on S exhibit important dynamical properties, due to their relation to martingale measures. In particular, square integrable Lévy bases are orthogonal martingale measures in the sense of [37] in their natural filtration $\mathcal{F}_t^\Lambda$ defined as

$$\mathcal{F}_t^\Lambda = \bigcap_{n=1}^{\infty} \sigma(\Lambda\big(\big(s, t + \tfrac{1}{n}\big] \times A\big) : s \le t + \tfrac{1}{n},\ A \in \mathcal{B}\big(S'\big)\big). \tag{9}$$

This allows for a stronger integration theory, in which we may integrate fields X that are stochastically dependent on Λ, provided that X is $\mathcal{F}_t^\Lambda$-predictable, in the sense that it is $\mathcal{P}$-measurable, where $\mathcal{P}$ is the σ-algebra generated by all simple random fields Y of the form

$$Y(s, z) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_j Y_i(\omega) \mathbf{1}_{(a_i, a_{i+1}]}(s) \mathbf{1}_{A_j}(z), \tag{10}$$

where each Y_i is $\mathcal{F}_{a_i}^\Lambda$-measurable. To this end, the integrability criteria of Proposition 2.4 have been extended to general predictable random fields in [16].

Definition 2.7 (Ambit field)**.** Let Λ be a Lévy basis on $(S, \mathcal{S})$ where $S = \mathbb{R} \times S'$ has a temporo-spatial structure. Let $f : \mathbb{R} \times \mathbb{R} \times S' \times S' \to \mathbb{R}$ be a measurable function and let $\sigma_t(z)$ be a predictable random field. Let also $(A_t(z))_{(t,z)\in S}$ be a family of measurable sets in $\mathcal{S}$. Then an ambit field $X_t(z)$ is a random field of the form

$$X_t(z) = \int_{A_t(z)} f(t, s, z, y)\sigma_s(y)\Lambda(ds, dy). \tag{11}$$

2.4 Subordination of Lévy bases

Just as for Lévy processes, it is possible to define a concept of subordination of Lévy bases, and for stable Lévy bases, the volatility modulation of an ambit field can equivalently be cast as subordination of the driving Lévy basis. This concept relies on so-called *metatimes*, as introduced to study subordination of Lévy bases in [7] and independently studied in [13].

Definition 2.8 (Metatimes and chronometers)**.** Let $(S_1, \mathcal{S}_1)$, $(S_2, \mathcal{S}_2)$ be measure spaces. A mapping $T : \mathcal{S}_1 \times \Omega \to \mathcal{S}_2$ is called a metatime, if it satisfies the following criteria.

1. $T(A;\omega)$ and $T(B;\omega)$ are disjoint whenever $A, B \in \mathcal{S}_1$ are disjoint.

2. $T(\bigcup_{j=1}^{\infty} A_j;\omega) = \bigcup_{j=1}^{\infty} T(A_j;\omega)$ whenever $A_1, A_2, \ldots \in \mathcal{S}_1$ are disjoint.

Furthermore, if the mapping T is injective, i.e., $T(A;\omega) \neq T(B;\omega)$ for any $A \neq B$, then we say that T is a chronometer.

The dependence of a metatime T on ω makes it a random element, and introduces some subtleties related to measurability. For notational convenience we shall sometimes suppress the explicit dependence on ω and it is tacitly understood that any non-deterministic metatimes under consideration are sufficiently regular and refer to Section 2 of [13] for details.

Definition 2.9 (Subordination of Lévy basis)**.** Let Λ be a Lévy basis on $(S_2, \mathcal{S}_2)$ and $T : \mathcal{S}_1 \to \mathcal{S}_2$ a metatime. Then we define the subordiation of Λ to T as the random measure $\Lambda|_T$ on $(S_1, \mathcal{S}_1)$ given as

$$\Lambda|_T(A) = \Lambda\big(T(A)\big), \quad A \in (\mathcal{S}_1)_b.$$

From Definition 2.8 it is clear that any measurable mapping $\phi : S_2 \to S_1$ defines a metatime via $T(A) = \phi^{-1}(A)$, and thus that metatimes are very general objects. The subordination of a Lévy basis to a metatime does therefore not necessarily guarantee any good properties, but in Proposition 2.6 we extend Theorem 15 of [10] from the Euclidean setting to the general setting, which provides an analogue between subordination of Lévy bases to the classical concept of subordination of Lévy processes. The result hinges on the random measure $c \circ T$ being independently scattered and infinitely divisible, where c is the control measure of the Lévy basis under consideration, and we refer to Example 3.1 for an example of an infinitely divisible metatime.

Proposition 2.6. *Let Λ be a factorisable Lévy basis on $(S_2, \mathcal{S}_2)$ with CQ (ζ, a, l, c), and let $T : \mathcal{S}_1 \to \mathcal{S}_2$ be a metatime. Suppose that T satisfies the following*

- *T is independent of Λ.*

- *The random measure $c \circ T$ is independently scattered, i.e., $c(T(A))$ is independent of $c(T(B))$ whenever A and B are disjoint.*

- *The law of $c(T(A))$ is infinitely divisible for any $A \in (\mathcal{S}_1)_b$.*

Then $\Lambda|_T(A)$, the subordination of Λ to T cf. Definition 2.9, is a Lévy basis on $(S_1, \mathcal{S}_1)$.

Proof. Conditionally on T, (1) yields that for any $u \in \mathbb{R}$

$$C\big(u; \Lambda|_T(A) \mid T\big) = C\big(u; \Lambda'\big) c\big(T(A)\big). \tag{12}$$

For a disjoint sequence $(A_i)_{i=1}^n$ of sets in $(\mathcal{S}_1)_b$, we therefore find using the independent scattered property of $c \circ T$

$$\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{i=1}^n u_i \Lambda|_T(A_i)\right)\right] = \mathbb{E}\left[\exp\left(\sum_{i=1}^n C\big(u_i;\Lambda'\big)c\big(T(A_i)\big)\right)\right]$$
$$= \prod_{i=1}^n \mathbb{E}\big[\exp\big(C\big(u_i;\Lambda'\big)c\big(T(A_i)\big)\big)\big],$$

showing that the random measure $\Lambda|_T$ is also independently scattered. We then need to prove that Λ_T is infinitely divisible. As $\Lambda|_T$ is independently scattered, it is enough to prove that $\Lambda|_T(A)$ is infinitely divisible for an arbitrary set $A \in (\mathcal{S}_1)_b$, as per Remark 19 in [10]. Since c is a positive measure, it holds that $c(T(A)) \geq 0$ for all $A \in (\mathcal{S}_1)_b$ and hence $c(T(A))$ necessarily has a Lévy-Khintchine representation of subordinator type such that

$$C(u; c\big(T(A)\big) = \mathrm{i}u\gamma(A) + \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbf{1}_{[-1,1]}(x)\big)\nu^c(dx, A),$$

where γ is a signed measure of finite variation and ν^c a generalized Lévy measure cf. Proposition 2.1. For fixed $A \in (\mathcal{S}_1)_b$, it thus holds that $C(u; c(T(A)))$ is also of Lévy-Khintchine type. From (12), we find that

$$C\big(u;\Lambda|_T(A)\big) = \gamma(A)C\big(u;\Lambda'\big) + \int_{\mathbb{R}} \big(e^{xC(u;\Lambda')} - 1 - C\big(u;\Lambda'\big)x\mathbf{1}_{[-1,1]}(x)\big)\nu^c(dx, A), \tag{13}$$

which is a Lévy-Khintchine exponent in u. We conclude that $\Lambda|_T$ is infinitely divisible and hence a Lévy basis. □

Corollary 2.2. *Let the setting be as in Proposition 2.6 and assume that $c \circ T$ is itself a factorisable positive Lévy basis on $(S_1, \mathcal{S}_1)$ with CQ $(\zeta^c, 0, \nu^c, c^c)$. Then $\Lambda|_T$ is a factorisable Lévy basis with CQ $(\zeta_T, a_T, \nu_T, c_T)$, where*

$$a_T = \zeta^c a, \quad c_T = c^c, \quad \nu_T(A) = \zeta^c \nu(A) + \int_{\mathbb{R}_+} \mathbb{P}(L_r \in A)\nu^c(dr),$$
$$\zeta_T = \zeta^c\zeta + \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} \mathbf{1}_{[-1,1]}(x)\mathbb{P}(L_r \in dx)\right)\nu^c(dr),$$

where $(L_r)_{r\in\mathbb{R}_+}$ is a Lévy process with $L_1 \stackrel{d}{=} \Lambda'$.

Proof. From Definition 2.3, (2), and (13), we find that

$$C\big(u;\Lambda|_T(A)\big) = \left[C\big(u;\Lambda'\big)b + \int_{\mathbb{R}} \big(e^{xC(u;\Lambda')} - 1\big)\nu^c(dx)\right]c^c(A),$$

where $b = \zeta^c + \int_{-1}^1 x\nu^c(dx)$. It is then plain to see that $\Lambda|_T$ has control measure c^c and the concrete form of the local characteristics ζ_T, a_T, ν_T follow by standard Bochner subordination principles (see, e.g., Chapter 30 of [33]). □

Remark 2.2. In Theorem 15 of [10], the authors prove a version of Proposition 2.6 where $S = \mathbb{R}^d$ and T is assumed to be chronometer. If we impose that T is a chronometer, then T^{-1} is well-defined and we obtain reversibility in the sense that $(\Lambda|_T)|_{T^{-1}}(A) = \Lambda(A)$ for sets A within the range of T.

3 Lévy bases on Riemannian manifolds

In this section, we begin the study of the case where $S = M$ is a differentiable Riemannian manifold. Most of the required geometric concepts are recapped in Appendix A, but we note that it is completely standard and covered in detail in, e.g., [25, 26]. A comprehensive introduction to geometry is also provided in [2], which in addition studies C^2 Gaussian random fields indexed by manifolds. Since Lévy bases lend themselves well to defining random fields via stochastic integrals, it is natural to study Lévy bases on manifolds and thereby define (not necessarily C^2) infinitely divisible random fields indexed by manifolds, as well as their extensions to ambit fields indexed by manifolds. The recent work of [23] presents such an application, where an ambit field indexed by a cylinder surface parametrized in $\mathbb{R}^3$ is used to model day-ahead electricity prices (see also Example 3.4). Riemannian manifolds are Polish spaces, and it is therefore trivial to abstractly define a Lévy basis on M as detailed in Section 2, but some natural intricacies arise in applications. Notably, the canonical choice of intensity measure $c(dz) = \lambda^d(dz)$ where λ^d is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is no longer appropriate, as M is usually a null-set with respect to λ^d when it is embedded in $\mathbb{R}^d$. Instead we will argue, perhaps unsurprisingly, that the natural choice of intensity measure is the Riemannian measure λ_M on M, induced by the Riemannian metric on M. This allows us to transfer essentially all of the intuition from Lévy bases on $\mathbb{R}^d$ to Lévy bases on manifolds, but gives rise to much richer structures depending on the Riemannian manifold under consideration.

In the rest of the paper, we let $\langle \cdot, \cdot \rangle_d$ denote the inner product on $\mathbb{R}^d$ and $\|\cdot\|_d$ the associated norm. We shall always use the notation (M, ρ) for a Riemannian manifold with metric ρ and countable atlas $(\phi_\alpha, U_\alpha)_{\alpha \in I}$. We denote by $\mathcal{M}$ the Borel σ-algebra on M and by $\mathcal{M}_b$ the corresponding δ-ring of totally bounded Borel sets in M. Unless explicitly stated otherwise, we will always assume that the Riemannian manifold under consideration is C^1, connected, oriented and without boundary.

3.1 The choice of intensity measure

The goal of this subsection is to motivate the choice of the *Riemannian measure* as the canonical intensity measure for Lévy bases defined on manifolds, and to illustrate some intrinsic properties of such Lévy bases. The Riemannian measure on the k-dimensional manifold (M, ρ) is denoted λ_M, and we recall that it is such that

$$\int_A f(z)\lambda_M(dz) = \int_A f(z) \det\big(\rho(z)\big)^{\frac{1}{2}} \lambda^k(dz) = \int_A f d\mathrm{Vol}_\rho,$$

where Vol_ρ denotes the volume element. On $\mathbb{R}^d$, there are two key properties of a homogeneous Lévy basis Λ, which are natural to require in more general settings.

1. **Volume-scaling:** The cumulant function of a $\Lambda(A)$ is the cumulant function of the Lévy seed, scaled by the volume of A:

$$\begin{aligned} C\big(u; \Lambda(A)\big) &= \int_A C\big(u; \Lambda'(z)\big)\lambda_M(dz) \\ &= C\big(u; \Lambda'\big)\lambda_M(A). \end{aligned} \tag{14}$$

2. **Isometry-invariance:** Λ is G-homogeneous in the sense of Definition 2.5 with respect to the isometry group $G = \mathrm{Isom}(M)$.

Proposition 3.1 below asserts that the only way to preserve both of these properties on a Riemannian manifold M, is to let Λ have intensity measure proportional to λ_M. We note that the choice of Riemannian measure as intensity measure generalizes the Euclidean setting with the Lebesgue measure as intensity measure, since $\lambda_M = \lambda^d$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and λ^d is invariant with respect to $\mathrm{Isom}(\mathbb{R}^d)$. This prompts the question: Which properties from the Euclidean setting generalize to the manifold setting, and how do they generalize? In Proposition 3.2, we show how to directly relate Lévy bases on $\mathbb{R}^k$ to Lévy bases on a k-manifold, essentially by transformation of the intensity measure. In particular, we show that a factorisable Lévy basis on a k-manifold M, with intensity measure λ_M, can be viewed as a subordinated Lévy basis, $\Lambda|_T$, where Λ is defined on $\mathbb{R}^k$ and T is a chronometer induced by the geometry of M. This relation means that Lévy bases on M inherit several properties from the Euclidean setting, such as the Lévy-Itô decomposition (8). The Lévy-Itô decomposition of Lévy bases on manifolds is established in Proposition 3.3. Finally, in Proposition 3.4, we generalize Lemma 3.1 of [7] to the manifold setting, which relates random measures on M to metatimes, and thus by extension to subordination of Lévy bases (see also Example 3.1). In the following sections, we shall see how several other objects related to Lévy bases on manifolds admit interpretations in terms of subordination and generalized time-changes.

Proposition 3.1. *Let Λ be a factorisable and dispersive Lévy basis on a Riemannian manifold $(M, \mathcal{M})$ with σ-finite intensity measure c. Then the only choice of c that ensures volume-scaling and isometry invariance of Λ is, up to scaling, the Riemannian measure $c = \lambda_M$.*

Proof. If we require

$$C\big(u; \Lambda(A)\big) = \mathrm{Vol}_\rho(A) C\big(u; \Lambda'\big), \quad A \in \mathcal{M},$$

then we must have $c = \lambda_M$. Invariance of the Riemannian measure under isometries follow by definition of the Riemannian measure, and uniqueness up to scaling follows from the same properties of the volume form, as discussed in Chapter 1 of [31]. □

Proposition 3.2. *Let (M, ρ) be a k-dimensional Riemannian manifold with (countable) atlas $(U_\alpha, \phi_\alpha)_{\alpha \in I}$ and introduce the following* factorisable *CQ's*

$$q_1 = \big(\zeta, a, \nu, \lambda^k\big), \quad q_2 = (\zeta, a, \nu, \lambda_M).$$

Define $s_M = \lambda_M(M)^{\frac{1}{k}}$ and the set $R_M \subseteq \mathbb{R}^k$ as

$$R_M = \begin{cases} \mathbb{R}^k, & \lambda_M(M) = \infty, \\ [0, s_M)^k, & \lambda_M(M) < \infty. \end{cases}$$

and let Λ be a Lévy basis on $(R_M, \mathcal{B}(R_M))$, with q_1 as CQ. Then there exists a non-random chronometer $T : \mathcal{M} \to \mathcal{B}(R_M)$, independent of ζ, a, ν, such that the subordinated Lévy basis $\Lambda|_T$ is a Lévy basis on $(M, \mathcal{M})$ with q_2 as CQ. Furthermore,

T is bijective up to λ_M-null sets and if Λ_M is a Lévy basis on $(M, \mathcal{M})$ with q_2 as CQ, then the essential inverse $T^{-1} : \mathcal{B}(R_M) \to \mathcal{M}$ is a non-random chronometer such that the subordinated Lévy basis $(\Lambda_M)|_{T^{-1}}$ is a Lévy basis on $(R_M, \mathcal{B}(R_M))$ with q_1 as CQ.

Proof. From Proposition 1.40 in [25], it follows that we can take the atlas (ϕ_α, U_α) such that each U_α is precompact. Define the pairwise disjoint Borel partition $(W_\alpha)_{\alpha \in I}$ of M by setting $W_1 = U_1$ and $W_\alpha = U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta$ for $\alpha \geq 2$. Let $V_\alpha = \phi_\alpha(W_\alpha)$ and define the density w_α as

$$w_\alpha(z) = \sqrt{\det \rho^\alpha(z)}, \tag{15}$$

where ρ^α is defined via the metric tensor in local coordinates as $\rho^\alpha_{ij} = \rho(\frac{\partial}{\partial z^{(i)}}, \frac{\partial}{\partial z^{(j)}})|_{\phi_\alpha^{-1}(z)}$. Define then the measures

$$\lambda_\alpha(dz) = w_\alpha(z)\lambda^k(dz),$$

and the corresponding volumes

$$v_\alpha = \int_{V_\alpha} w_\alpha(z)\lambda^k(dz) = \lambda_M(W_\alpha) < \infty.$$

We may, without loss of generality, discard indices for which $v_\alpha = 0$. Now, because R_M is constructed to have total volume $\lambda^k(R_M) = \lambda_M(M) = \sum_{\alpha \in I} v_\alpha$, we can choose pairwise disjoint sets $(B_\alpha)_{\alpha \in I} \subseteq \mathcal{B}(R_M)$ such that for $\alpha \in I$ we have

$$\lambda^k(B_\alpha) = v_\alpha, \quad \lambda^k\left(R_M \setminus \bigcup_{\alpha \in I} B_\alpha\right) = 0.$$

For example, when $\lambda_M(M) < \infty$, such sets can be constructed as $B_{\alpha_n} = [t_{n-1}, t_n) \times [0, s_M)^{k-1}$ where $t_0 = 0$ and $t_n = t_{n-1} + \frac{v_{\alpha_n}}{s_M^{k-1}}$ where α_n enumerates I. Defining the probability measures $\mathbb{P}_\alpha = \frac{1}{v_\alpha}\lambda_\alpha \restriction_{V_\alpha}$ and $\mathbb{Q}_\alpha = \frac{1}{v_\alpha}\lambda^k \restriction_{B_\alpha}$, we obtain two atomless standard probability spaces $(V_\alpha, \mathcal{B}(V_\alpha), \mathbb{P}_\alpha)$, $(B_\alpha, \mathcal{B}(B_\alpha), \mathbb{Q}_\alpha)$. Hence by the isomorphism theorem for atomless standard probability spaces (see, e.g., [15, Theorem 9.5.1]) there exists conull Borel sets $V_\alpha^0 \subseteq V_\alpha$ and $B_\alpha^0 \subseteq B_\alpha$ and a measure-preserving Borel isomorphism $h_\alpha : V_\alpha^0 \to B_\alpha^0$ such that

$$(h_\alpha)_*(\mathbb{P}_\alpha \restriction_{V_\alpha^0}) = \mathbb{Q}_\alpha \restriction_{B_\alpha^0}, \tag{16}$$

where h_α^{-1} is also Borel. Introduce the conull subsets $W_\alpha^0 = \phi_\alpha^{-1}(V_\alpha^0) \subseteq W_\alpha$ and set

$$M_0 = \bigcup_{\alpha \in I} W_\alpha^0 \subseteq M, \quad R_0 = \bigcup_{\alpha \in I} B_\alpha^0 \subseteq R_M.$$

Then we can define a Borel bijection with Borel inverse $f : M_0 \to R_0$ by

$$f = \bigcup_{\alpha \in I} (h_\alpha \circ \phi_\alpha) \restriction_{W_\alpha^0}.$$

From this, we define the set map T as

$$T(A) = f(A \cap M_0), \quad A \in \mathcal{M}, \tag{17}$$

with inverse $T^{-1}(C) = f^{-1}(C \cap R_0)$ for $C \in \mathcal{B}(R_M)$. As T, T^{-1} are induced by a bijection, they are deterministic chronometers and they satisfy the following pushforward identity for $A \in \mathcal{M}_b$

$$\begin{aligned}\lambda^k\big(T(A)\big) &= \sum_{\alpha\in I} \lambda^k\big(f\big(A \cap W_\alpha^0\big)\big) \\ &= \sum_{\alpha\in I} \lambda_\alpha\big(\phi_\alpha\big(A \cap W_\alpha^0\big)\big) \\ &= \sum_{\alpha\in I} \lambda_M\big(A \cap W_\alpha^0\big) \\ &= \lambda_M(A).\end{aligned}$$

By considering the subordinated Lévy basis $\Lambda|_T$ as the pushforward $(T^{-1})_*\Lambda$, we want to appeal to Lemma 2.2. This requires existence of an increasing and exhausting sequence of $\mathcal{M}_b$, which is satisfied for any Riemannian manifold as these are σ-compact. Furthermore, we require that $\int_A Q(z)\lambda_M(dz) < \infty$ whenever $\lambda_M(A) < \infty$ with Q defined in (3). This holds trivially since ζ, a, ν are constant, due to the assumption of factorisability. Lemma 2.2 then yields that $\Lambda|_T$ is a Lévy basis on $(M_0, \mathcal{M} \restriction_{M_0})$ with CQ $(\zeta, a, \nu, \lambda_M)$, where ζ, a, ν are preserved as they are simply constant. Conversely, starting from a Lévy basis Λ_M on $(M, \mathcal{M})$ with CQ q_2, we find that $T_*\Lambda_M = \Lambda_M|_{T^{-1}}$ is a Lévy basis on $(R_M, \mathcal{B}(R_M))$ with CQ q_1. The full statement of the Proposition follows by noting that, for Borel sets $A \subseteq M \setminus M_0$ we have $T(A) = f(\emptyset) = \emptyset$ and hence $\Lambda|_T$ vanishes on the λ_M-null part, and similarly $(\lambda_M)|_{T^{-1}}$ vanishes on $R_M \setminus R_0$. □

Remark 3.1. In Proposition 3.2, the Lévy basis is assumed to be factorisable cf. Definition 2.5, which means that the chronometer construction (17) is independent of the choice of measure-preserving Borel isomoprhisms. More generally, we could consider non-factorisable CQ's such that the condition of $\int_A Q(z)\lambda_M(dz) < \infty$ whenever $\lambda_M(A) < \infty$ holds with Q defined in (3). In this case, $f_*\Lambda(A) = \overline{\Lambda}(f^{-1}(A))$ is well-defined on $\mathcal{R}_{\lambda^k}$ and with CQ $(\zeta \circ f^{-1}, a \circ f^{-1}, \nu(\cdot, f^{-1}(\cdot)))$ (note that this CQ is still valid cf. Definition 2.3). Conversely, the pullback of a Lévy basis on R_0 has CQ obtained via composition with f. Hence the CQ will depend on the choice of measure space isomorphism h_α in (16).

Proposition 3.3. *Let Λ be an $\mathbb{R}^n$-valued Lévy basis on a Riemannian k-manifold $(M, \mathcal{M})$ with CQ $q = (\zeta, a, \nu, \lambda_M)$. Then Λ admits a modification Λ^* with Lévy-Itô decomposition of the form*

$$\begin{aligned}\Lambda^*(A) = &\int_A \zeta(z)\lambda_M(dz) + W(A) + \int_{\mathbb{R}^n} y\mathbf{1}_D(y)\big(N^d - n\big)(dy, A) \\ &+ \int_{\mathbb{R}^n} y\mathbf{1}_{D^c}(y)N(dy, A),\end{aligned} \tag{18}$$

where W is a Gaussian Lévy basis with CQ $(0, a, 0, \lambda_M)$ and N is a Poisson random measure with compensator $\widetilde{N}(dy, dz) = \nu(dy, z)\lambda_M(dz)$.

Proof. Define the function $r(z) = 1 + Q(z)$ with Q as in (3) and the measure $\widetilde{c}(dz) = r(z)\lambda_M(dz)$. Note that $r(z) \geq 1$ and define the modified characteristics

$$\widetilde{\zeta}(z) = \frac{\zeta(z)}{r(z)}, \quad \widetilde{a}(z) = \frac{a(z)}{r(z)}, \quad \widetilde{\nu}(dy, z) = \frac{\nu(dy, z)}{r(z)}.$$

Then $\widetilde{q} = (\widetilde{\zeta}, \widetilde{a}, \widetilde{\nu}, \widetilde{c})$ is another representation of q and it holds that $Q(z)/r(z) \leq 1$. Lemma 2.1 therefore applies automatically to the CQ $\widetilde{q}$ and we may extend Λ to the δ-ring $\mathcal{R}_{\widetilde{c}}$. The measure $\widetilde{c}$ is atomless and σ-finite, so by the measure-space isomorphism used in (17), there exist conull Borel sets $M_0 \subseteq M$ and $R_0 \subseteq \mathbb{R}^k$ along with a measure-preserving Borel isomorphism $f : M_0 \longrightarrow R_0$, from $\widetilde{c} \restriction_{M_0}$ to $\lambda^k \restriction_{R_0}$. Let T be the corresponding chronometer induced by this as in the proof of Proposition 3.2 and define $\widehat{\Lambda} = \Lambda|_{T^{-1}}$. By Lemma 2.2, $\widehat{\Lambda}$ is an extended Lévy basis on $\mathcal{R}_{\lambda^k|_{R_0}}$ whose restriction to $\mathcal{B}_b(R_0)$ is a Lévy basis on $(R_0, \mathcal{B}(R_0))$ with CQ $\widehat{q}$ given by

$$\widehat{q} = \big(\widetilde{\zeta} \circ f^{-1}, \widetilde{a} \circ f^{-1}, \widetilde{\nu}\big(\,\cdot\,, f^{-1}(\,\cdot\,)\big), \lambda^k \restriction_{R_0}\big).$$

Proposition 2.3 gives a modification $\widehat{\Lambda}^*$ of $\widehat{\Lambda}$, initially on the bounded Borel sets, with the Euclidean Lévy-Itô decomposition (8). Another application of Lemma 2.1 extends both $\widehat{\Lambda}^*$ and its decomposition to $\mathcal{R}_{\lambda^k|_{R_0}}$. By uniqueness of this extension, $\widehat{\Lambda}^*$ remains a modification of $\widehat{\Lambda}$ on that δ-ring. For $A \in \mathcal{M}_b$, let $\Lambda^*(A) = \widehat{\Lambda}^*(T(A))$, which is well-defined as $\lambda^k(T(A)) = \widetilde{c}(A \cap M_0) < \infty$. Pulling the Gaussian and Poisson random measures in the Euclidean Lévy-Itô decomposition back through f shows that Λ^* has the decomposition (18). Finally, for each $A \in \mathcal{M}_b$, note that

$$\begin{aligned}\Lambda^*(A) &= \widehat{\Lambda}^*\big(T(A)\big) \\ &\overset{a.s.}{=} \widehat{\Lambda}\big(T(A)\big) \\ &\overset{a.s.}{=} \Lambda(A \cap M_0) \\ &\overset{a.s.}{=} \Lambda(A),\end{aligned}$$

where the last equality follows because $\widetilde{c}(M \setminus M_0) = 0$, and hence $\Lambda(A \setminus M_0) = 0$ almost surely. Thus Λ^* is a modification of Λ. □

Lemma 3.1. *Let $(E, \mathcal{E}, \mu)$ be a σ-finite and non-atomic measure space such that $(E, \mathcal{E})$ is standard Borel and such that $\mu(E) = \infty$. Then there exists measurable sets $E_0 \subseteq E$ and $R_0 \subseteq \mathbb{R}_+$ with $\mu(E \setminus E_0) = 0$ and $\lambda^1(\mathbb{R}_+ \setminus R_0) = 0$ and a Borel isomorphism $\chi : E_0 \to R_0$ such that*

$$\chi_*(\mu \restriction_{E_0}) = \lambda^1 \restriction_{R_0}.$$

Proof. Let $(K_n)_{n=1}^{\infty}$ be a disjoint Borel partition of E with $0 < \mu(K_n) < \infty$ for all n. Fix $n \in \mathbb{N}$ and consider the probability space

$$(K_n, \mathcal{E} \restriction_{K_n}, \mathbb{Q}_n), \qquad \mathbb{Q}_n(A) = \frac{\mu(A)}{\mu(K_n)},\ A \in \mathcal{E} \restriction_{K_n}.$$

Since $(E, \mathcal{E})$ is standard Borel, so is $(K_n, \mathcal{E} \restriction_{K_n})$ and hence $(K_n, \mathcal{E} \restriction_{K_n}, \mathbb{Q}_n)$ is a standard probability space and non-atomic since μ is non-atomic. Therefore, by the

isomorphism theorem for atomless standard probability spaces [15, Theorem 9.5.1], there exist conull Borel sets $K_n^0 \subseteq K_n$ and $I_n^0 \subseteq (0,1)$ and a measure-preserving Borel isomorphism

$$g_n : K_n^0 \to I_n^0$$

(with g_n^{-1} Borel) such that

$$(g_n)_*(\mathbb{Q}_n \restriction_{K_n^0}) = \lambda^1 \restriction_{I_n^0}.$$

We can then define

$$\psi_n(x) = \mu(K_n) g_n(x), \quad x \in K_n^0,$$

such that $\psi_n : K_n^0 \to \mu(K_n) I_n^0 \subseteq (0, \mu(K_n))$ remains a Borel bijection with Borel inverse. It follows that for Borel $A \subseteq \mu(K_n) I_n^0$

$$\mu \restriction_{K_n^0} \left(\psi_n^{-1}(A)\right) = \mu(K_n)\mathbb{Q}_n\left(g_n^{-1}\left(A/\mu(K_n)\right)\right) = \mu(K_n)\lambda^1\left(A/\mu(K_n)\right) = \lambda^1(A).$$

Thus we conclude that

$$(\psi_n)_*(\mu \restriction_{K_n^0}) = \lambda \restriction_{\mu(K_n) I_n^0}.$$

Set $s_n = \sum_{\ell \leq n} \mu(K_\ell)$ and note that $s_n \to \infty$ due to the assumption of $\mu(E) = \infty$. Hence we have that $\mathbb{R}_+ \setminus \bigcup_{n \in \mathbb{N}} (s_{n-1}, s_n) \subseteq \{s_n : n \geq 0\}$. Define then

$$R_n^0 = s_{n-1} + \mu(K_n) I_n^0 \subseteq (s_{n-1}, s_n),$$

and $R_0 = \bigcup_{n \in \mathbb{N}} R_n^0$ for which it holds that

$$\lambda^1(\mathbb{R}_+ \setminus R_0) \leq \lambda^1\left(\mathbb{R}_+ \setminus \bigcup_{n \in \mathbb{N}} (s_{n-1}, s_n)\right) + \sum_{n \in \mathbb{N}} \lambda^1\left((s_{n-1}, s_n) \setminus R_n^0\right) = 0.$$

Define also $E_0 = \bigcup_n K_n^0$ and note that $\mu(E \setminus E_0) = \sum_{n \in \mathbb{N}} \mu(K_n \setminus K_n^0) = 0$. Then the map $\chi : E_0 \to R_0$ given by

$$\chi(x) = s_{n-1} + \psi_n(x), \quad x \in K_n^0,$$

is bijective with inverse $\chi^{-1}(y) = \psi_n^{-1}(y - s_{n-1})$ for $y \in R_n^0$ and it holds that

$$\chi_*(\mu \restriction_{E_0}) = \lambda \restriction_{R_0}.$$

The maps χ is Borel, since each restriction $\chi \restriction_{K_n^0} = s_{n-1} + \psi_n(x)$ is Borel and $(K_n^0)_{n \in \mathbb{N}}$ is a countable Borel partition of E_0. Likewise, χ^{-1} is Borel since $\chi^{-1} \restriction_{R_n^0}$ is Borel and $(R_n^0)_{n \in \mathbb{N}}$ is a countable Borel partition of R_0. □

Proposition 3.4. *Let M be a Riemannian k-manifold such that $\lambda_M(M) = \infty$. Let $\mu : \mathcal{M} \times \Omega \to \mathbb{R}_+$ be such that for almost all $\omega \in \Omega$, $A \mapsto \mu(A;\omega)$ is a σ-finite measure on $(M, \mathcal{M})$, and for each A, $\omega \mapsto \mu(A;\omega)$ is $\mathcal{F}$-measurable. Then there exists a $\mathcal{M} \otimes \mathcal{F}$-measurable metatime $T : \mathcal{M} \times \Omega \to \mathcal{M}$ such that, for almost all $\omega \in \Omega$, it holds that*

$$\mu(A;\omega) = \lambda_M\left(T(A;\omega)\right), \quad A \in \mathcal{M}_b. \tag{19}$$

Proof. Let $K_1 \subseteq K_2 \subseteq \cdots$ be a sequence of compact Borel sets in M such that $M = \bigcup_{n\in\mathbb{N}} K_n$. Set $L_n = K_n \setminus K_{n-1} \in \mathcal{M}_b$ (with $K_0 = \emptyset$) and note that $\mu(L_n;\omega) < \infty$ for almost all $\omega \in \Omega$. The aim is to construct a metatime T_n on each L_n and patch these together into the final metatime T. For the construction of T_n, we require sequences $(C_n)_{n\in\mathbb{N}} \subseteq \mathcal{M}$ of pairwise disjoint sets of infinite volume, i.e., $\lambda_M(C_n) = \infty$ for all $n \in \mathbb{N}$. Such sets exist due to M being σ-finite and of infinite volume. By σ-finiteness of λ_M, Lemma 3.1 then yields that there are Borel isomorphisms $\chi_n : C_n^0 \to R_n^0 \subseteq \mathbb{R}_+$ such that

$$\big((\chi_n)_*\lambda_M \restriction_{C_n^0}\big)(A) = \lambda^1(A), \quad A \in \mathcal{B}\big(R_n^0\big), \tag{20}$$

where C_n^0, R_n^0 are Borel sets such that $\lambda_M(C_n \setminus C_n^0) = 0$ and $\lambda^1(\mathbb{R}_+ \setminus R_n^0) = 0$. Let $m_n(\omega) = \mu(L_n;\omega)$ be the random μ-volume of L_n and note that $\omega \mapsto m_n(\omega)$ is $\mathcal{F}$-measurable. Since each $(L_n, \mathcal{M} \restriction_{L_n})$ is a standard Borel space and uncountable, there exists a Borel isomorphism $\psi_n : L_n \to (0,1)$ with Borel inverse. This allows for defining the (random) distribution functions

$$F_n(t;\omega) = \mathbf{1}_{\{m_n(\omega)>0\}}(\omega)\frac{1}{m_n(\omega)}\mu(\psi_n^{-1}\big((0,t]\big);\,\omega\big), \quad t \in [0,1].$$

Then for each fixed t, the map $\omega \mapsto F_n(t;\omega)$ is $\mathcal{F}$-measurable by assumption and, for each ω, the map $t \mapsto F(t;\omega)$ is non-decreasing and right-continuous. It then follows that $(t,\omega) \mapsto F_n(t;\omega)$ is $\mathcal{B}([0,1]) \otimes \mathcal{F}$-measurable.

Define the corresponding quantiles as

$$Q_n(u;\omega) = \inf\big\{t \in [0,1] : F_n(t;\omega) \geq u\big\}, \quad u \in [0,1].$$

Then Q_n is $\mathcal{B}([0,1]) \otimes \mathcal{F}$-measurable since for every $x \in \mathbb{R}$,

$$\big\{(u,\omega) : Q_n(u;\omega) < x\big\} = \bigcup_{q\in\mathbb{Q},\ q<x} \big\{(u,\omega) : F_n(q;\omega) > u\big\}.$$

We can then use Q_n to define a transport map H_n as

$$H_n(r;\omega) = \begin{cases} \psi_n^{-1}(Q_n(r/m_n(\omega);\,\omega)), & 0 < r < m_n(\omega), \\ x_n, & \text{otherwise}, \end{cases}$$

for $r \geq 0$ and $x_n \in L_n$ fixed and arbitrary. Since it is a composition of jointly measurable maps, H_n is also $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$-measurable. A standard monotone class argument can be used to verify that for almost all $\omega \in \Omega$

$$\lambda^1\big(\big\{0 < r < m_n(\omega) : H_n(r;\omega) \in A\big\}\big) = \mu(A;\omega), \quad A \in \mathcal{M} \restriction_{L_n}. \tag{21}$$

Now, for $A \in \mathcal{M}$, define the candidate metatime T_n as

$$T_n(A;\omega) = \chi_n^{-1}\big(\big\{0 < r < m_n(\omega) : H_n(r;\omega) \in A \cap L_n\big\} \cap R_n^0\big) \subseteq C_n^0.$$

It follows from the properties of preimages that $T_n(\emptyset;\omega) = \emptyset$ and that $T_n(\cdot;\omega)$ preserves set differences and countable unions of disjoint sets and hence that T_n is a metatime on $\mathcal{M}$. Using (20) and (21), we obtain the volume identity

$$\lambda_M\big(T_n(A;\omega)\big) = \mu(A \cap L_n;\omega).$$

Additionally, we find that T_n is $\mathcal{M} \otimes \mathcal{F}$-measurable since

$$\begin{aligned}&\big\{(x,\omega) : x \in T_n(A;\omega)\big\}\\ &\quad = \big\{(x,\omega) : x \in C_n^0,\, 0 < \chi_n(x) < m_n(\omega),\, H_n\big(\chi_n(x);\omega\big) \in A \cap L_n\big\},\end{aligned}$$

where the right hand side is an element in $\mathcal{M} \otimes \mathcal{F}$. We can then define the final metatime T as the countable disjoint union

$$T(A;\omega) = \bigcup_{n\in\mathbb{N}} T_n(A;\omega).$$

Since each T_n is $\mathcal{M} \otimes \mathcal{F}$-measurable and the union is countable, T is also $\mathcal{M} \otimes \mathcal{F}$-measurable. Furthermore, for $A \in \mathcal{M}_b$, it holds that

$$\lambda_M\big(T(A;\omega)\big) = \sum_{n\in\mathbb{N}} \lambda_M\big(T_n(A;\omega)\big) = \sum_{n\in\mathbb{N}} \mu(A \cap L_n;\omega) = \mu(A;\omega). \qquad \square$$

Remark 3.2. The condition that $\lambda_M(M) = \infty$ in Proposition 3.4 rules out many interesting cases such as that of compact manifolds. However, in applications it is often natural to have a space-time structure of the form $M = \mathbb{R} \times M'$ for a Riemannian manifold M'. In this case, M' can be compact while the conditions of Proposition 3.4 are still satisfied.

Example 3.1. Suppose that μ is a positive random measure on the Riemannian manifold $M = \mathbb{R}_+ \times M'$ (for which $\lambda_M(M) = \infty$ per Remark 3.2) and let T be a metatime on $\mathcal{M}$ such that (19) holds. Let Λ be a factorisable Lévy basis on $(M, \mathcal{M})$ which is independent of T and μ. Then, conditionally on T, the cumulant function of $\Lambda|_T$ is

$$C\big(u;\Lambda|_T(A) \mid T\big) = \lambda_M\big(T(A)\big) C\big(u;\Lambda'\big).$$

An infinitely divisible random measure μ can be constructed via

$$\mu(A) = \int_A \tau(t,x)\big(\lambda^1 \otimes \lambda_{M'}\big)(dt,dx),$$

where τ is of the form

$$\tau(t,x) = \int_0^t \int_{M'} g(t,x;y,s)\Lambda^\perp(ds,dy), \tag{22}$$

for a suitably regular deterministic kernel function g and positive Lévy basis $\Lambda^\perp$ on $(M, \mathcal{M})$, independent of Λ. Letting T be a metatime such that $\mu(A) = \lambda_M(T(A))$ (which exists cf. Proposition 3.4), we find that

$$\begin{aligned}\lambda_M\big(T(A)\big) &= \int_A \int_0^t \int_{M'} g(t,x;y,s)\Lambda^\perp(ds,dy)\big(\lambda^1 \otimes \lambda_{M'}\big)(dt,dx)\\ &= \int_0^t \int_{M'} f(y,s \mid A)\Lambda^\perp(dy,ds),\end{aligned}$$

where $f(s,y \mid A) = \int_A \mathbf{1}_{\{s\le t\}} g(t,x;y,s)(\lambda^1 \otimes \lambda_{M'})(dt,dx)$. The stochastic integral against deterministic functions is again infinitely divisible cf. [32], which shows that for each $A \in \mathcal{M}_b$, $\lambda_M(T(A))$ is infinitely divisible. This can be extended to the measure $\lambda_M(T(\cdot))$ on $\mathcal{M}$. This construction can be extended to more general random measures $\mu(\cdot) = \lambda_M(T(\cdot))$, for example by incorporating a multiplicative noise component in the integral (22), or by probability mixing of the coefficients of the kernel function g.

3.2 Lévy bases on curves

In Remark 3.5 of [29], it is shown how the process $t \mapsto \Lambda([0,t])$ is a Lévy process when Λ is a translation invariant Lévy basis on $\mathbb{R}$. A possible generalization of this property is to study Λ defined on curves embedded in some manifold M with metric ρ. This situation is thus slightly different than in the previous section, where Λ was defined on M; here, Λ is defined on the image of the curve $\varphi(\mathbb{R})$, taking values in M. In general, $\varphi(\mathbb{R})$ is only an immersion, so in all of the following, we impose that φ is injective and a homeomorphism onto its image, making it a proper embedding. This implies that φ is C^1 and regular, in the sense that

$$\left\|\dot{\varphi}(t)\right\|_\rho = \sqrt{\rho_{\varphi(t)}\big(\varphi(t),\varphi(t)\big)} > 0, \quad t \in \mathbb{R}.$$

If φ is parametrized in $\mathbb{R}^d$, this simply corresponds to the condition that

$$\big\langle \varphi(t), \varphi(t) \big\rangle_d > 0, \quad t \in \mathbb{R}.$$

Given the Riemannian measure λ_M on M, the associated Riemannian measure on the curve φ is that arising from the arclength of φ as

$$\lambda_\varphi(A) = \int_{\varphi^{-1}(A)} \left\|\dot{\varphi}(t)\right\|_\rho dt.$$

In Proposition 3.5, we then give necessary and sufficient conditions on the parametrization of φ, such that the process defined by $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process. The crucial property of φ is that it must have constant speed, which severely restricts the parametrizations that lead to Lévy processes, but at the same time implies that any curve reparametrized by its arclength yields a Lévy process.

Proposition 3.5. *Let Λ be a factorisable Lévy basis on $(\varphi(\mathbb{R}), \mathcal{B}(\varphi(\mathbb{R})))$ with CQ $(\zeta, a, \nu, \lambda_\varphi)$. Let φ be a regular parametrized curve on M, parametrized as a regular C^1 curve $\varphi : \mathbb{R} \to M$. Then the following statements are equivalent.*

1. *The process $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process (up to modification).*
2. *The curve φ has constant speed in the sense that $\|\dot{\varphi}(t)\|_\rho = \alpha$, for all $t \in \mathbb{R}$ and some $\alpha > 0$.*

Proof. Set $Y_t = \Lambda(\varphi([0,t]))$, $\ell(t) := \int_0^t \|\dot{\varphi}(s)\|_\rho ds$, and note that φ is injective by assumption of being an embedding, such that for $0 \le s < t$,

$$Y_t - Y_s = \Lambda(\varphi\big((s,t]\big)).$$

Hence the factorisability property of Λ yields that

$$C(u; Y_t - Y_s) = \big(\ell(t) - \ell(s)\big) C\big(u; \Lambda'\big).$$

Suppose first that Y is a Lévy process. Then its one-dimensional distributions are such that $C(u; Y_t) = tC(u; Y_1)$. On the other hand, we know that

$$C(u; Y_t) = \ell(t) C\big(u; \Lambda'\big), \quad C(u; Y_1) = \ell(1) C\big(u; \Lambda'\big).$$

Thus we must have $\ell(t)C(u;\Lambda') = t\ell(1)C(u;\Lambda')$ and it follows that $\ell(t) = t\ell(1)$ for $t \geq 0$. Differentiating and using the continuity of $\dot{\varphi}$ gives

$$\left\|\dot{\varphi}(t)\right\|_\rho = \ell(1) = \alpha > 0, \quad t \geq 0,$$

where $\alpha > 0$ is implied by regularity of φ. Conversely, suppose that $\|\dot{\varphi}(t)\|_\rho = \alpha$. Then, for $0 \leq s < t$,

$$C(u;Y_t - Y_s) = \alpha(t-s)C\left(u;\Lambda'\right),$$

so the distribution of $Y_t - Y_s$ depends only on $t-s$. Moreover, injectivity of φ ensures that the images of disjoint parameter intervals are disjoint, and the independently scattered property of Λ therefore gives independent increments of Y_t. Since $\lambda_\varphi(\{\varphi(0)\}) = 0$, we have $Y_0 = \Lambda(\{\varphi(0)\}) = 0$ almost surely. This shows that Y_t is a Lévy process in law in the sense of [33], but such a process always admits a modification that is continuous in probability and thus a Lévy process. □

Remark 3.3. Proposition 3.5 implies that Lévy bases defined on geodesics on M are Lévy processes. Geodesics are generalizations of the notion of straight lines in $\mathbb{R}^d$ to manifolds, in the sense that these curves locally provide the shortest path between points, and they can be shown to always exhibit constant speed. In particular, we recover the fact that $t \mapsto \Lambda([0,t])$ is a Lévy process when Λ is a translation invariant Lévy basis defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Corollary 3.1. *Let $\varphi : \mathbb{R} \to M$ be a regular C^1 curve, and let Λ be a factorisable Lévy basis on $(\varphi(\mathbb{R}), \mathcal{B}(\varphi(\mathbb{R})))$ with control measure λ_φ. Define $\chi(t) = \int_0^t \|\dot{\varphi}(s)\| ds$ and $\gamma(t) = \varphi(\chi^{-1}(t))$. Then $t \mapsto \Lambda(\gamma([0,t]))$ is a Lévy process in t with $t \in [0, \int_0^\infty \|\dot{\varphi}(s)\|_\rho ds)$.*

Proof. Since φ is regular, it holds that $\chi'(t) = \|\dot{\varphi}(t)\|_\rho > 0$, meaning that χ is strictly increasing and admits a C^1 inverse on its range. Additionally, we note that

$$\left(\chi^{-1}\right)'(t) = \frac{1}{\chi'(\chi^{-1}(t))} = \frac{1}{\|\dot{\varphi}(\chi^{-1}(t))\|_\rho}.$$

The chain rule then implies that

$$\dot{\gamma}(t) = \dot{\varphi}\left(\chi^{-1}(t)\right)\left(\chi^{-1}\right)'(t),$$

and thus that

$$\left\|\dot{\gamma}(t)\right\| = \frac{\|\dot{\varphi}(\chi^{-1}(t))\|}{\|\dot{\varphi}(\chi^{-1}(t))\|} = 1.$$

Hence γ has constant speed and $t \mapsto \Lambda(\gamma([0,t]))$ is therefore a Lévy process by Proposition 3.5. □

Proposition 3.5 essentially states that the curvature of φ must be sufficiently homogeneous to obtain a process with independent and stationary increments, and Corollary 3.1 states that any curve reparametrized by its arclength is exactly sufficiently homogeneous. It is then natural to wonder, if there are some curves φ that are not compatible with Lévy processes at all, without reparametrization. The answer to this question is negative, as Proposition 3.6 shows that it is always possible to obtain any Lévy process on any given curve, by sufficiently distorting a factorisable Lévy basis.

Proposition 3.6. *Let (M, ρ) be a Riemmanian manifold, and let $\varphi : \mathbb{R} \to M$ be a regular C^1 curve on M. Then it is always possible to choose a Lévy basis Λ on $(\varphi(\mathbb{R}), \mathcal{B}(\varphi(\mathbb{R})))$ such that $t \mapsto \Lambda(\varphi[0,t])$ is a Lévy process with characteristic triplet (ζ, a, ν).*

Proof. As φ is regular, it holds by definition that $\|\dot{\varphi}(t)\|_\rho > 0$ for all t. We can then specify Λ to have CQ $(\tilde{\zeta}, \tilde{a}, \tilde{\nu}, \lambda_M)$, where $(\tilde{\zeta}, \tilde{a}, \tilde{\nu}) = \frac{1}{\|\dot{\varphi}(t)\|_\rho} \cdot (\zeta, a, \nu)$. It follows by the same computations as in the proof of Proposition 3.5 that

$$C(u; \Lambda\big(\varphi\big([0,t]\big)\big) = C\big(u; \Lambda'\big)t,$$

which means $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process in law with characteristic triplet (ζ, a, ν). □

Remark 3.4. The converse case to Proposition 3.6 where one fixes a general Lévy basis Λ and seeks a curve φ for which $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process is in general not possible, as the geometry of the curve necessarily applies evenly to the elements ζ, a, ν. As such, if these have differing inhomogeneities, we cannot recover a Lévy process on any φ.

Proposition 3.7. *Let $(M, \mathcal{M})$ be a Riemannian manifold and $\varphi : \mathbb{R} \to M$ a regular C^1 curve. Let Λ be a factorisable Lévy basis on $(\varphi(\mathbb{R}), \mathcal{B}(\varphi(\mathbb{R})))$ with CQ $(\zeta, a, \nu, \lambda_\varphi)$. Then the process $Y_t = \Lambda(\varphi([0,t]))$ has the same finite-dimensional marginal distribution as $\Lambda'(\int_0^t \|\dot{\varphi}(s)\|_\rho ds)$, where $\Lambda'(t)$ is a Lévy process with the same triplet as the Lévy seed of Λ, (ζ, a, ν).*

Proof. Set $\ell(t) = \int_0^t \|\dot{\varphi}(s)\|_\rho \, ds$ and note that $Y_t - Y_s = \Lambda(\varphi((s,t])))$, since $\lambda_\varphi(\{s\}) = 0$. We find that

$$C(u; Y_t - Y_s) = \lambda_\varphi(\varphi\big((s,t]\big))C\big(u; \Lambda'\big) = \big(\ell(t) - \ell(s)\big)C\big(u; \Lambda'\big).$$

This is the cumulant function of $\Lambda'(\ell(t)) - \Lambda'(\ell(s))$. Now let $0 = t_0 < t_1 < \cdots < t_n$ and note that since φ is an embedding and thus injective, the sets $\varphi((t_{k-1}, t_k])$ are pairwise disjoint for $k = 1, \ldots, n$. The independently scattered property of Λ therefore implies that the increments $Y_{t_k} - Y_{t_{k-1}}$ are independent and, furthermore, they have the same distributions as

$$\Lambda'\big(\ell(t_k)\big) - \Lambda'\big(\ell(t_{k-1})\big), \quad k = 1, \ldots, n,$$

which also has independent increments. From this we see that

$$(Y_{t_1}, \ldots, Y_{t_n}) \overset{d}{=} \big(\Lambda'\big(\ell(t_1)\big), \ldots, \Lambda'\big(\ell(t_n)\big)\big),$$

and thus Y_t and $\Lambda'(\ell(t))$ have the same finite-dimensional marginal distributions. □

Proposition 3.7 formalizes the notion that when Λ is homogeneous, the process $t \mapsto \Lambda(\varphi([0,t]))$ is simply that of a Lévy process time-changed by the increasing deterministic process $\int_0^t \|\dot{\varphi}(s)\|_\rho ds$, which has a rather well understood theory. From Proposition 3.7, it also follows that the curve φ together with the Lévy basis Λ cannot uniquely determine the distribution of the resulting process $\Lambda(\varphi([0,t]))$, as C^1 curves are generally not identified by their arclength.

Example 3.2. Consider the choice of a helix parametrized in $\mathbb{R}^3$

$$\varphi(t) = \big(\cos(t), \sin(t), t\big).$$

The metric inherited from $\mathbb{R}^3$ is simply the pullback of the Euclidean metric $\rho = dx^2 + dy^2 + dz^2$ given as $\varphi^*\rho = 2dt^2$ and the induced arclength measure is therefore $\lambda_\varphi(A) = \sqrt{2}\lambda^1(\varphi^{-1}(A))$. It is then plain to see that $\varphi(t)$ has constant speed $\|\dot\varphi(t)\|_\rho = \sqrt{2}$ and Proposition 3.5 yields that $t \mapsto \Lambda(\varphi([0,t]))$ is a Lévy process when Λ is a factorisable Lévy basis with CQ $(\zeta, a, \nu, \lambda_\varphi)$.

Example 3.3. Consider the setting of Example 3.2, but with a non-uniformly parametrized helix in $\mathbb{R}^3$ of the form

$$\varphi(t) = \big(\cos\big(t^2\big), \sin\big(t^2\big), t^2\big).$$

This is only injective and regular on $(0,\infty)$, so we restrict to this region with the convention $\Lambda(\{0\}) = 0$. Then the pullback metric becomes $\varphi^*\rho = 8t^2dt^2$, such that $\lambda_\varphi(A) = 2\sqrt{2}\int_{\varphi^{-1}(A)} t\lambda^1(dt)$ and $\|\dot\varphi(t)\|_\rho = 2\sqrt{2}|t|$. Hence φ does not have constant speed and Proposition 3.5 yields that $t \mapsto \Lambda(\varphi((0,t]))$ is not a Lévy process. However, re-parametrizing φ by $\sqrt{\frac{t}{\sqrt{2}}}$ yields a constant speed curve with unit speed and it follows from Corollary 3.1 that $t \mapsto \Lambda(\gamma((0,t]))$ is a Lévy process where

$$\gamma(t) = \varphi\Big(\sqrt{\tfrac{t}{\sqrt{2}}}\Big).$$

3.3 Lévy bases on manifolds

In addition to Lévy processes, it is also illustrated in [29] how Lévy bases on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ can be considered as Lévy sheets, when evaluated along increasing sets in $\mathbb{R}^d_+$. In the manifold setting, we can similarly consider an increasing family of sets M_t indexed by $\mathbb{R}^d_+$. Under some regularity on the limiting space $M_\infty = \bigcup_t M_t$, we may then define the Lévy basis across all submanifolds by defining Λ on some manifold, M, containing M_∞, and then let the intensity measure be $c = \lambda_M$. this corresponds to the setting of [23], where the family M_t is a continuously extending cylinder surface in $\mathbb{R}^3$ with a fixed Riemannian metric (see Example 3.4). This view is opposed to that of the Lévy basis being defined on a general family of Riemannian manifolds (M_t, ρ_t), where it is less clear how to define and interpret the Lévy basis Λ consistently across all submanifolds. Note that we implicitly assume that each M_t is totally bounded, such that the Lévy basis will be almost surely finite when evaluated along M_t.

In Proposition 3.8 below, we generalize the results of [29] in a possible direction, by characterizing a situation in which the process $t \mapsto \Lambda(M_t)$ yields a homogeneous Lévy sheet, of which we recall the definition, following [7]. To this end, we introduce the increment $\Delta_s^t F$ of a random field $X(t)$ indexed by $t \in \mathbb{R}^k$ as

$$\Delta_s^t X = \sum_{e_1=0}^{1} \cdots \sum_{e_k=0}^{1} (-1)^{k-(e_1+\cdots+e_k)} X\big(b^1(e_1), \ldots, b^k(e_k)\big),$$

where $b^j(1) = t^{(j)}$ and $b^j(0) = s^{(j)}$.

Definition 3.1 (Homogeneous Lévy sheet)**.** Let $(X_t)_{t\in\mathbb{R}^k_+}$ be an $\mathbb{R}$-valued random field indexed by $\mathbb{R}^k_+$. We say that X_t is a homogeneous Lévy sheet if the following holds

- $\Delta^{t_1}_{s_1}X, \ldots, \Delta^{t_n}_{s_n}X$ are independent when $n \geq 2$ and $(s_1, t_1], \ldots, (s_n, t_n]$ are disjoint.
- $\Delta^{t+b}_{t+a}X \overset{d}{=} \Delta^b_a X$ for all $a, b, t \in \mathbb{R}^k_+$.
- $t \mapsto X_t$ is continuous in probability.
- Almost all sample paths of X_t have limits along monotone paths in the sense of [1].

Proposition 3.8. *Let Λ be a factorisable Lévy basis on $(M, \mathcal{M})$ with CQ $(\zeta, a, \nu, \lambda_M)$, where (M, ρ) is a Riemannian manifold. Let $(M_t)_{t\in\mathbb{R}^k}$ be a sequence of sets in M such that $M_t \subseteq M_{t'}$ whenever $t \leq t'$ with the inequality interpreted coordinate-wise, and such that each M_t is totally bounded in M and define $M_\infty = \bigcup_{t\in\mathbb{R}^k_+} M_t$. Define the random field $X_t = \Lambda(M_t)$. Then the following holds.*

1. *If X_t is a homogeneous Lévy sheet, then it holds that*

$$\lambda_M(M_t) = \lambda_M(M_{\mathbf{1}}) \prod_{j=1}^{k} t^{(j)}, \quad \mathbf{1} = (1, \ldots, 1).$$

2. *If there exists an $\alpha > 0$ and a measurable map $\pi : M_\infty \to \mathbb{R}^k_+$ such that, for every $t \in \mathbb{R}^k_+$, the following holds*

$$\lambda_M\left(M_t \triangle \pi^{-1}\left([0, t]\right)\right) = 0,$$

where $\triangle$ denotes the symmetric set difference and

$$\pi_*(\lambda_M \restriction_{M_\infty}) = \alpha\lambda^k.$$

Then X_t admits a modification that is a homogeneous Lévy sheet.

Proof. Suppose that X_t is a homogeneous Lévy sheet. By [29], it follows that there exists a Lévy seed Λ'' such that

$$C(u; X_t) = C\left(u; \Lambda''\right)\lambda^k\left([0, t_1] \times [0, t_2] \times \cdots \times [0, t_k]\right) = C\left(u; \Lambda''\right) \prod_{j=1}^{k} t^{(j)}. \quad (23)$$

Letting Λ' denote the Lévy seed of Λ, it therefore holds that

$$C\left(u; \Lambda'\right)\lambda_M(M_t) = C\left(u; \Lambda''\right) \prod_{j=1}^{k} t^{(j)}. \quad (24)$$

Evaluating this expression at $t = \mathbf{1}$, we find that $C(u; \Lambda'') = \lambda_M(M_{\mathbf{1}})C(u; \Lambda')$ and substituting this into (24) we find that

$$\lambda_M(M_t)C\left(u; \Lambda'\right) = \lambda_M(M_{\mathbf{1}})C\left(u; \Lambda'\right) \prod_{j=1}^{k} t^{(j)},$$

and hence that $\lambda_M(M_t) = \lambda_M(M_{\mathbf{1}}) \prod_{j=1}^k t^{(j)}$. Now suppose that the condition (2) holds. We first construct a Lévy basis on $\mathbb{R}_+^k$ as follows. Let $B \subseteq \mathbb{R}_+^k$ be a bounded Borel set and choose $r \in \mathbb{R}_+^k$ such that $B \subseteq [0, r]$. Note that $M_r \cap \pi^{-1}(B) \subseteq M_r$, so we can define the random measure

$$\widetilde{\Lambda}(B) = \Lambda\big(M_r \cap \pi^{-1}(B)\big).$$

This definition of $\widetilde{\Lambda}$ does (almost surely) not depend on the choice of r. Indeed, if $B \subseteq [0, r] \cap [0, q]$, then

$$\begin{aligned}\big(M_r \cap \pi^{-1}(B)\big) \triangle \big(M_q \cap \pi^{-1}(B)\big) \\ \subseteq \big(M_r \triangle \pi^{-1}\big([0, r]\big)\big) \cup \big(M_q \triangle \pi^{-1}\big([0, q]\big)\big).\end{aligned}$$

The set on the right has λ_M-measure zero and since the intensity measure of Λ is λ_M, it follows that Λ is almost surely equal to zero on this set. Now let $(B_j)_{j=1}^\infty$ be pairwise disjoint bounded Borel sets with bounded union and let r be such that $\bigcup_{j=1}^\infty B_j \subseteq [0, r]$. The sets $M_r \cap \pi^{-1}(B_j)$ are then also pairwise disjoint. Since Λ is countably additive and independently scattered, this is inherited by $\widetilde{\Lambda}$ and the latter is thus a Lévy basis on $\mathbb{R}_+^k$ with intensity measure $\alpha\lambda^k$. Additionally, for $B \in \mathcal{B}_b(\mathbb{R}_+^k)$, we have

$$\begin{aligned}C\big(u; \widetilde{\Lambda}(B)\big) &= \lambda_M\big(M_r \cap \pi^{-1}(B)\big) C\big(u; \Lambda'\big) \\ &= \lambda_M\big(\pi^{-1}(B)\big) C\big(u; \Lambda'\big) \\ &= \alpha\lambda^k(B) C\big(u; \Lambda'\big).\end{aligned}$$

$\widetilde{\Lambda}$ is therefore factorisable and hence homogeneous. Define then $Y_t = \widetilde{\Lambda}([0, t])$, which is such that

$$Y_t = \Lambda\big(M_t \cap \pi^{-1}\big([0, t]\big)\big) = \Lambda(M_t) = X_t,$$

almost surely, i.e., Y_t and X_t are modifications. We want to verify that Y_t constitutes a homogeneous Lévy sheet. For $s, t \in \mathbb{R}_+^k$ with $s \leq t$ coordinate wise, the increments satisfy $\Delta_s^t Y = \widetilde{\Lambda}((s, t])$ with $(s, t] = \prod_{j=1}^k (s^{(j)}, t^{(j)}]$. Since $\widetilde{\Lambda}$ is independently scattered, increments over disjoint parameter rectangles are then independent. Furthermore, it holds that

$$\begin{aligned}C\big(u; \Delta_s^t Y\big) &= \alpha\lambda^k((s, t]) C\big(u; \Lambda'\big) \\ &= \alpha \prod_{j=1}^k \big(t^{(j)} - s^{(j)}\big) C\big(u; \Lambda'\big).\end{aligned}$$

As such, the distribution of $\Delta_s^t Y$ depends only on $\lambda^k((s, t])$ and it follows that $\Delta_{t+a}^{t+b} Y \overset{d}{=} \Delta_a^b Y$. Finally, for $t_n \to t$, it holds that

$$\lambda^k\big([0, t_n] \triangle [0, t]\big) \to 0,$$

which implies that $Y_{t_n} \to Y$ in probability. Proposition 4.1 in [29] then yields that there exists a modification $\widetilde{Y}$ of Y which has limits along monotone paths. Hence X_t is a homogeneous Lévy sheet, up to modification. □

Corollary 3.2. *Let the setting be as in Proposition 3.8. Suppose that $M = \mathbb{R}^k \times M'$ and is equipped with the Riemannian measure $\lambda_M = \lambda^k \otimes \lambda_{M'}$ where $\lambda_{M'}(M') < \infty$. Suppose also that M_t has the structure $M_t = ([0, t_1] \times \cdots \times [0, t_k]) \times M'$ for some Riemannian manifold M', then $t \mapsto \Lambda(M_t)$ is a homogeneous Lévy sheet.*

Example 3.4. In [23], a space-time ambit field model is proposed for day-ahead electricity prices of the form

$$Y_t(x) = \int_{M_t} f(t - s, x, \xi)\sigma_s(\xi)\Lambda(ds, d\xi). \tag{25}$$

In this setting, the manifold (M, ρ) is the cylinder surface parametrized as

$$M = \big\{\big(t, \cos(x), \sin(x)\big) \in \mathbb{R}^3 \mid t \in \mathbb{R},\ x \in (0, 2\pi]\big\}, \tag{26}$$

The subsets M_t are the truncated cylinder surfaces

$$M_t = \big\{\big(s, \cos(x), \sin(x)\big) \mid s \leq t,\ x \in (0, 2\pi]\big\}.$$

The Lévy basis Λ is then defined on $(M, \mathcal{M})$ with factorisable CQ $(\zeta, a, \nu, \lambda_M)$. If we instead start the time-truncation at $t = 0$, then Corollary 3.2 shows that $t \mapsto \Lambda(M_t)$ is a Lévy process.

Example 3.5. Setting $M = \mathbb{R}^d$ with $M_t = [0, t_1) \times [0, t_2) \times \cdots \times [0, t_d)$, it is simple to verify that the conditions of Proposition 3.8 hold. Indeed, set $M_\infty = \mathbb{R}^d_+$ and $\pi(x) = x$. Then

$$\pi^{-1}\big([0, t]\big) = \prod_{j=1}^{d} \big[0, t^{(j)}\big].$$

Since $M_t = \prod_{j=1}^d [0, t^{(j)})$, the two sets differ only on their boundaries which are λ^d-null sets so

$$\lambda^d\big(M_t \triangle \pi^{-1}\big([0, t]\big)\big) = 0.$$

Furthermore, $\pi_*(\lambda^d \restriction_{\mathbb{R}^d}) = \lambda^d$, so the pushforward condition holds with $\alpha = 1$. It follows that $X_t = \Lambda(M_t)$ has a modification that is a homogeneous Lévy sheet, which recovers the result of [29].

4 Lévy bases and stochastic integration on stochastically evolving manifolds

So far, we have only considered the case where the underlying geometric structures are deterministic. We will now explore the immediate consequences of evaluating a Lévy basis and related random fields along smooth curves and manifolds that are stochastic or stochastically evolving in time. We shall also restrict attention to the case where the randomness pertaining to the manifold is independent of the Lévy basis under consideration. This allows us to apply all of the theory from the deterministic setting, by adopting the convention that the stochastic curves and manifolds are defined on a separate probability field.

In the following, we therefore consider any Lévy bases as defined on some probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and all "stochastic manifolds" as being defined on a different

probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. We then think of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$. When the underlying space is non-deterministic or randomly evolving, it is unclear on which space to define the Lévy basis "directly". However, a rather tractable approach is obtained by fixing an underlying topological manifold M, and then letting the stochastic object be the metric ρ. This type of approach is standard when considering Ricci flows (see [35]), and such an approach is also considered in [18] to study stochastic partial differential equations on evolving manifolds. In our case, this perspective allows for a simple way of defining a Lévy basis Λ on the stochastic manifold, via the concept of *extended subordination*, where one substitutes the usual intensity measure for a random measure. Indeed, since the stochastic object is the metric, $\rho(\cdot;\omega_2)$, this is equivalent to a random Riemannian measure, $\mu(dz) = \lambda_M(dz;\omega_2)$. Hence we can simply define Λ on the space $(M, \mathcal{M})$, but with stochastic intensity measure μ, arising from the stochastic metric. This corresponds exactly to extended subordination as defined in [5] and we note that the stochastic intensity measure entails that Λ is in general not a Lévy basis. In the rest of the section we shall, with some abuse of terminology, refer to Λ as a Lévy basis with CQ (ζ, a, ν, μ), where it is tacitly understood that this CQ is not a proper quadruplet, and only satisfies Definition 2.3 for fixed $\omega_2 \in \Omega_2$, such that Λ is a Lévy basis for fixed ω_2.

We distinguish between two settings: First, where the intensity measure is simply a random Riemannian measure, dependent on ω_2, but not evolving through a temporal dimension. Secondly, we consider a particular case where the random intensity measure evolves over time, which allows for some additional structure.

4.1 Extended subordination by random Riemannian measures

Specifying the Riemannian measure as a random measure μ should, ω_2 by ω_2, preserve the properties that make it a valid Riemannian measure. This can, for example, be achieved by specifying

$$\mu(A) = \int_A f(z;\omega_2)\lambda_{M_0}(dz), \tag{27}$$

where λ_{M_0} is the Riemannian measure on M, corresponding to a deterministic "base metric" ρ_0, and $f(z;\omega_2) \in C^k(M)$ is a strictly positive random field. Note that, for a factorisable Lévy basis Λ on M with random intensity measure μ, extended subordination corresponds to subordination by a metatime T, due to Proposition 3.4.

Corollary 4.1. *Let M be a Riemannian manifold with Riemannian measure λ_M, and let μ be a random Riemannian measure of the form (27). Let Λ be a factorisable Lévy basis on $(M, \mathcal{M})$ with intensity measure $\mu(\cdot;\omega_2)$ for each $\omega_2 \in \Omega_2$. Then the following holds*

- *If Λ is Gaussian with CQ $(0, a, 0, \mu)$, it holds that*

$$\Lambda(A) \overset{f.d.d}{=} \int_A \sqrt{f(z)}W(dz), \quad A \in \mathcal{M}_b,$$

where $\overset{f.d.d}{=}$ denotes equality of finite-dimensional distributions, W is a Gaussian Lévy basis on $(M, \mathcal{M})$ with CQ $(0, a, 0, \lambda_M)$ and the stochastic integral is interpreted in the pathwise sense of [32], as discussed in Section 2.3.

- *If Λ is purely jumping with CQ $(0, 0, \nu, \mu)$ and $\lambda_M(M) = \infty$, it holds that*

$$\Lambda(A) \overset{f.d.d}{=} J|_T(A), \quad A \in \mathcal{M}_b,$$

where J is a Lévy basis on $(M, \mathcal{M})$ with CQ $(0, 0, \nu, \lambda_M)$ and T is a metatime such that

$$\lambda_M\big(T(A)\big) = \mu(A), \quad A \in \mathcal{M}_b. \tag{28}$$

Proof. Suppose Λ is Gaussian with CQ $(0, a, 0, \mu)$ and let $(A_j)_{j=1}^n$ be pairwise disjoint sets in $\mathcal{M}_b$. Since μ is the conditional intensity of Λ, we find that

$$\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^n u_j \Lambda(A_j)\right)\right] = \mathbb{E}\left[\exp\left(-\frac{a}{2}\sum_{j=1}^n u_j^2 \mu(A_j)\right)\right],$$

which is exactly the joint characteristic function of the integrals $\int_{A_j} \sqrt{f(z)} W(dz)$. When Λ is purely jumping, Proposition 3.4 yields that there exists a metatime such that (28) holds almost surely, and using that metatimes map disjoint sets to disjoint sets we find that

$$\begin{aligned}\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^n u_j \Lambda(A_j)\right)\right] &= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^n C\big(u_j; \Lambda'\big)\lambda_{M_0}\big(T(A_j)\big)\right)\right] \\ &= \mathbb{E}\left[\left(\mathrm{i}\sum_{j=1}^n u_j J|_T(A_j)\right)\right],\end{aligned}$$

from which it follows that $\Lambda(A) \overset{f.d.d}{=} J|_T(A)$. □

Example 4.1. Let the setting be as in Corollary 4.1 and suppose that Λ is a Gaussian Lévy basis with CQ $(0, a, 0, \mu)$. Then we can represent the law of Λ as a stochastic integral on the fixed base manifold M_0 as

$$\Lambda(A) \overset{d}{=} \int_A \sqrt{f(z)} W(dz),$$

where W is a Gaussian Lévy basis on M_0 with CQ $(0, a, 0, \lambda_{M_0})$. In particular, the conditional stochastic variance of $\Lambda(A)$ is

$$\mathbb{V}\big[\Lambda(A) \mid f\big] = a \int_A f(z) \lambda_{M_0}(dz).$$

We can extend this to consider laws of stochastic integrals on M since

$$\left(\int_A h(z)\Lambda(dz)\right) \overset{d}{=} \int_A h(z)\sqrt{f(z)} W(dz), \quad A \in \mathcal{M}_0,$$

for any Λ-integrable function $h : M_0 \to \mathbb{R}$. Interpreting $\sqrt{f(z)} W(dz)$ as a random measure, we can then model many quantities of the homogeneous noise on the stochastic manifold via non-homogeneous noise on the fixed manifold M_0.

4.2 Stochastically evolving manifolds and extended subordination

The goal is now to extend the analysis to stochastically *evolving* manifolds, such that the manifold is interpreted as being time-dependent, M_t. We take the view similar to [18], where the evolving manifold consists of a fixed topological k-manifold M, endowed with a stochastic and time-dependent metric $\rho_t(\cdot;\omega_2)$. For each t and ω_2, we impose that the evolving metric is of the form

$$\rho_t(z;\omega_2) = f(t;\omega_2)^{\frac{2}{k}}\rho_0(z), \tag{29}$$

where ρ_0 is a fixed base metric on M and $f(t;\omega_2) > 0$ is continuous in t and jointly measurable in (t,ω_2). Throughout, we assume that no topology changes happen due to the evolution of ρ_t, which can be ensured by working on a sufficiently small time-interval $[0,T]$. We can then define a temporo-spatial and factorisable Lévy basis Λ on the space $\mathbb{R}_+ \times M$ with CQ (ζ,a,ν,μ), where μ is the random Riemannian measure induced by the metric (29)

$$\mu(dt,dz;\omega_2) = f(t;\omega_2)\big(\lambda^1(dt)\otimes\lambda_{M_0}(dz)\big). \tag{30}$$

Once again, this simply corresponds to a factorisable Lévy basis on $\mathbb{R}_+ \times M$, time-changed in the sense of extended subordination. The cumulant function of Λ reads

$$C\big(u;\Lambda\big([0,t]\times A\big)\big) = \log\left(\mathbb{E}\left[\exp\left(C\big(u;\Lambda'\big)\lambda_{M_0}(A)\int_0^t f(s)ds\right)\right]\right), \quad A\in\mathcal{M}_b.$$

For fixed A, we therefore have that in the same Gaussian and pure-jump cases as Corollary 4.1, the process $t\mapsto\Lambda([0,t]\times A)$ is a Lévy process, time-changed by the absolutely continuous and increasing process $\int_0^t f(s)ds$. Due to the independence between Λ and μ, we may also construct stochastic integrals against Λ via the approach of [32], as summarized in Section 2.3, but we may in fact also define the stochastic integral in the sense of [37], for which we need the following proposition.

Proposition 4.1. *Let Λ be a factorisable Lévy basis on $(\mathbb{R}_+\times M,\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{M})$ with CQ (ζ,a,ν,μ) where μ is given by (30). Suppose that $\mathbb{E}[\int_0^t f(s)ds] < \infty$ for all $t<\infty$, and introduce the right continuous filtration $\mathcal{F}_t$ as*

$$\mathcal{F}_t = \sigma\big(\mathcal{F}_t^\Lambda \cup \mathcal{F}_t^\mu\big),$$

where $\mathcal{F}_t^\Lambda$ is defined in (9) and $\mathcal{F}_t^\mu$ is the filtration

$$\mathcal{F}_t^\mu = \sigma\big(f(s) : 0\le s\le t\big).$$

Then μ is predictable with respect to $\mathcal{F}_t$. Additionally, if it holds that $\mathbb{E}[\Lambda'] = 0$ and $\mathbb{E}[(\Lambda')^2]<\infty$, then Λ is an orthogonal martingale measure with respect to $\mathcal{F}_t$, which is to say that Λ satisfies the following properties

(i) $\Lambda(\{0\}\times A) = 0$ *almost surely for all* $A\in\mathcal{M}_b$.

(ii) For $t>0$, $\Lambda([0,t]\times\cdot)$ *is a* σ*-finite* L^2*-valued measure.*

(iii) For fixed $A\in\mathcal{M}_b$, *the process* $t\mapsto\Lambda([0,t]\times A)$ *is a martingale with respect to* $\mathcal{F}_t$.

(iv) For $A, B \in \mathcal{M}_b$ with $A \cap B = \emptyset$, it holds that $\mathbb{E}[\Lambda([0,t] \times A)\Lambda([0,t] \times B)] = 0$ for all t.

Proof. As f is continuous and adapted to its natural filtration, it follows that μ is also a $\mathcal{F}_t^\mu$-predictable random measure, and hence predictable with respect to the larger filtration $\mathcal{F}_t$. We now show that Λ is an orthogonal martingale measure in $\mathcal{F}_t$ under the additional moment assumptions on the Lévy seed. It is straightforward to see that $\Lambda(\{0\} \times A) = 0$ for all $A \in \mathcal{M}_b$, such that property (i) is satisfied. If the Lévy seed has mean zero, it follows that $\mathbb{E}[\Lambda([0,t] \times A) \mid \mathcal{F}_t^\mu] = 0$ and hence that $\mathbb{E}[\Lambda([0,t] \times A)] = 0$. Let then $(K_n)_{n=1}^\infty$ be a partition of M such that each $K_n \in \mathcal{M}_b$ (so $\lambda_{M_0}(K_n) < \infty$). For fixed $\omega_2 \in \Omega_2$, the set function $A \mapsto \Lambda([0,t] \times A; \omega_2)$ is a centered, square-integrable Lévy basis with deterministic intensity measure $\lambda_{M_0}(A) \int_0^t f(s; \omega_2) ds$ and due to the zero-mean property, it follows from Proposition 28 in [10] that $\Lambda([0,t] \times \cdot\,; \omega_2)$ can be taken σ-finite. Moreover, it holds unconditionally that

$$\sup_{A \in \mathcal{M} \restriction_{K_n}} \mathbb{E}\big[\big|\Lambda\big([0,t] \times A\big)\big|^2\big] \leq \mathbb{V}\big[\big(\Lambda'\big)^2\big] \mathbb{E}\left[\int_0^t f(s) ds\right] \lambda_{M_0}(K_n) < \infty.$$

Hence $\Lambda([0,t] \times A)$ defines a σ-finite $L^2(\Omega)$-valued set function. This can be upgraded to a measure if the set function is also countably additive, which we show. Let therefore $(A_j)_{j=1}^\infty$ be a sequence of disjoint sets with $\bigcup_j A_j \in \mathcal{M}_b$ and note that

$$\Lambda\big([0,t] \times A\big) - \sum_{j=1}^J \Lambda\big([0,t] \times A_j\big) = \Lambda\left([0,t] \times \left(A \setminus \bigcup_{j=1}^J A_j\right)\right),$$

such that

$$\begin{aligned}
&\left\| \Lambda\big([0,t] \times A\big) - \sum_{j=1}^J \Lambda\big([0,t] \times A_j\big) \right\|_{L^2(\Omega)}^2 \\
&\quad = \mathbb{V}\big[\Lambda'\big] \mathbb{E}\left[\int_0^t f(s) ds\right] \lambda_{M_0}\left([0,t] \times \left(A \setminus \bigcup_{j=1}^J A_j\right)\right) \\
&\quad \xrightarrow[J \to \infty]{} 0.
\end{aligned}$$

This shows property (ii). To show the martingale property, let $\mathcal{G} = \sigma(f(t) : t \geq 0)$ be the full σ-algebra generated by f. Then, conditionally on $\mathcal{G}$, Λ is a centered Lévy basis with deterministic intensity $\mu(\cdot; \omega_2)$ and it follows that

$$\mathbb{E}\big[\Lambda\big((s,t] \times A\big) \mid \mathcal{G} \vee \mathcal{F}_s^\Lambda\big] = 0.$$

Now, since $\mathcal{F}_s \subseteq \mathcal{G} \vee \mathcal{F}_s^\Lambda$, the tower property yields that $\mathbb{E}[\Lambda((s,t] \times A) \mid \mathcal{F}_s] = 0$ and therefore that $\mathbb{E}[\Lambda([0,t] \times A) \mid \mathcal{F}_s] = \Lambda([0,s] \times A)$, which is the property (iii). Finally, we note that Λ also satisfies property (iv) since for disjoint $A, B \in \mathcal{M}_b$, it holds that

$$\mathbb{E}\big[\Lambda\big([0,t] \times A\big)\Lambda\big([0,t] \times B\big)\big] = \mathbb{E}\big[\mathbb{E}\big[\Lambda\big([0,t] \times A\big)\Lambda\big([0,t] \times B\big) \mid \mathcal{F}_t^\mu\big]\big] = 0.$$

Thus, we conclude that Λ is an orthogonal martingale measure. □

The orthogonal martingale property of Λ as proven in Proposition 4.1 shows important dynamical properties of Λ on the randomly evolving manifold, and allows for defining stochastic integrals against Λ in the L^2-framework of [37, 17], thus entailing that the stochastic integral itself defines an orthogonal martingale measure. This framework heavily relies on the *covariance measure* of Λ, which is the random measure Q defined via

$$\begin{aligned} Q\big([0,t]\times A\big) &= \int_0^t \int_A \left(a(s,z) + \int_{\mathbb{R}} x^2 \nu(dx,s,z) \right) \mu(ds,dz) \\ &= \int_0^t \int_A \left(a(s,z) + \int_{\mathbb{R}} x^2 \nu(dx,s,z) \right) \lambda_{M_0}(dz) f(s) ds. \end{aligned}$$

The class of Λ-integrable random fields in this L^2 setting is then the collection of all $\mathcal{F}_t$-predictable fields h satisfying

$$\mathbb{E}\left[\int_0^t \int_M \big|h(s,z)\big|^2 Q(ds,dz) \right] < \infty,$$

where in particular h does not need to be independent of Λ. We can then define an ambit field $X_t(z)$ on M_t as

$$X_t(z) = \int_0^t \int_M h(t,s,z,y) \Lambda(ds,dy),$$

and if the kernel function h can be related to a time-varying differential operator $\mathcal{L}_t$ on M_t, the field $X_t(z)$ can be interpreted in the sense of so-called weak and mild solutions to stochastic partial differential equations on stochastically evolving manifolds. This point of view is thus parallel to the variational approach to stochastic partial differential equations on evolving manifolds in [18].

5 Trawl processes on manifolds

Trawl processes constitute some of the simplest possible ambit processes, but they exhibit a surprising amount of flexibility. They were originally studied in [6], and have seen considerable attention in recent years (see for example [11, 12, 28, 27, 34]). They are characterized by their interpretation as "the random measure of a moving geometric figure in a higher dimensional space" and enjoy several nice stationarity properties. They are essentially processes of the form $t \mapsto \Lambda(A_t)$ for a Lévy basis Λ and time-indexed family of sets A_t which are time translations of some fixed set A. As a result, trawl processes may be characterized via a property identical to (14), whenever $\lambda_M = \lambda^d$ (see Proposition 54 in [10]), and it is therefore natural to consider a more general setting where the sets A_t evolve on a manifold. This setting lends itself well to the machinery developed so far for Lévy bases on manifolds, and serves as a further step towards treating Lévy bases on evolving curves and manifolds. The goal of this section is to generalize trawl processes to so-called M-trawl processes (see Definition 5.2) and study some of their basic distributional properties. Compared to the usual case where the trawl set is always translated in the time dimension, M-trawl processes are defined in terms of a triplet (Λ, A, V), where V is a vector field that describes the movement of the set A over time and Λ is the Lévy basis.

Definition 5.1 (Trawl fields and processes)**.** Let $A \subseteq (-\infty, 0] \times \mathbb{R}^d$ be a Borel set with $\lambda^{d+1}(A) < \infty$. Define $A_t(x) = A + (t, x)$ and let Λ be a homogeneous Lévy basis on $(\mathbb{R}^{d+1}, \mathcal{B}(\mathbb{R}^{d+1}))$. Then the random field defined by

$$Y_t(x) = \int_{\mathbb{R}\times\mathbb{R}^d} \mathbf{1}_A(t-s, x-\xi)\Lambda(ds, d\xi) = \Lambda\big(A_t(x)\big),$$

is a trawl field. If $A_t(x) = A_t = A + (t, 0)$ does not depend on x, then the process

$$Y_t = \int_{\mathbb{R}\times\mathbb{R}^d} \mathbf{1}_A(t-s, \xi)\Lambda(ds, d\xi) = \Lambda(A_t),$$

is a trawl process.

Trawl fields and processes are $\mathbb{T}$-stationary by definition and the geometric interpretation of this is, that the volume of the set A is preserved through translations in time and space. This is not the case for the processes of the form $t \mapsto \Lambda(\varphi([0, t]))$ studied previously, where we evaluated the Lévy basis along an increasing family of subsets, and M-trawl processes are therefore fundamentally different. Stationarity properties are not guaranteed to hold for arbitrary sets moving along curves on a manifold, as the intrinsic curvature may deform the set, and we will therefore have to consider which operations are sensible for trawl fields on manifolds. We shall focus on the case of trawl processes (so only indexed by time) on manifolds, which we term M-trawl processes, and leave the general case of M-trawl fields for future work. We wish to maintain the fundamental stationarity property of the cumulant function of a trawl process, which reads

$$C\big(u, \Lambda(A_t)\big) = \lambda^{d+1}(A_t)C\big(u; \Lambda'\big) = \lambda^{d+1}(A)C\big(u; \Lambda'\big).$$

In a more general setting, it is most convenient to consider just the random measure of a moving set $\Lambda(A_t)$, and impose that the cumulant function has the form

$$C\big(u; \Lambda(A_t)\big) = c(A_t)C\big(u; \Lambda'\big) = c_A C\big(u; \Lambda'\big), \quad c_A > 0, \tag{31}$$

which means that the intensity measure should be invariant to the trawl operation. Arbitrary operations on sets $A \in \mathcal{M}$ are not guaranteed to preserve the volume of the set, but a natural generalization of moving the trawl set A by time-translations in Euclidean space is then given by volume preserving flows along a curve on the manifold. As we require a natural notion of time, we shall impose that M has the space-time structure $M = \mathbb{R} \times M'$ for some Riemannian manifold M'. Intuitively speaking, we want to move the set A smoothly along a curve on M over time, in a way that leaves the volume of A unchanged and this movement is prescribed via the velocity field, which generates a family of flows on M. The crucial point of volume preservation is then guaranteed by imposing some regularity on the velocity field. In our case, we shall impose that the velocity field V generates a one-parameter family of time-indexed flows $\gamma_t : M \to M$ which are isometries on M and such that

$$\frac{d}{dt}\gamma_t(p) = V\big(\gamma_t(p)\big), \quad \gamma_0(p) = p, \quad p \in M. \tag{32}$$

Such flows are globally well-defined at all times t if the Riemannian manifold M is complete as a metric space, which we therefore impose throughout the section. The

restriction to velocity fields that generate isometries may seem quite restrictive, but this corresponds to the class of so-called *Killing vector fields*, which have many connections to the topology and symmetries of M. We refer to [31, Chapter 8] for a treatment of Killing vector fields (see p. 316 for a discussion on the global well-definedness on complete manifolds).

Remark 5.1. It is possible to relax the assumptions on the velocity field V and still preserve the volume of the moving set A. In particular, it is sufficient to impose that

$$\operatorname{div}(V) = 0,$$

which is a weaker assumption that is always true for Killing vector fields. In this case, however, the metric is not necessarily preserved and thus the set A may be distorted in various ways as it is being moved, but in such a way that the volume is still constant.

Definition 5.2 (M-trawl process)**.** Let (M, ρ) be a Riemannian manifold of the form $M = \mathbb{R} \times M'$ where M' is a Riemannian manifold, and let V be a Killing vector field on M, and let γ_t be the flow generated by V, i.e.

$$\frac{d}{dt}\gamma_t(p) = V\big(\gamma_t(p)\big), \quad p \in M,\ t \in \mathbb{R}.$$

Let $A \subseteq (-\infty, 0] \times M'$ be a Borel set with $0 < \lambda_M(A) < \infty$ and define the ambit sets A_t as

$$A_t = \gamma_t(A) = \big\{\gamma_t(p) \mid p \in A\big\}, \quad t \in \mathbb{R}_+.$$

Finally, let Λ be a factorisable Lévy basis on $(M, \mathcal{M})$ with CQ $(\zeta, a, \nu, \lambda_M)$. Then we say that the triplet (Λ, A, V) is an M-trawl process, where A is the trawl set.

Example 5.1. The class of M-trawl processes generalizes the usual notion of trawl processes. To see this, assume that $Y_t = \Lambda(A_t)$ satisfies the conditions for Definition 5.1, i.e., $M = \mathbb{R} \times \mathbb{R}^d$, and let $A \subseteq (-\infty, 0] \times \mathbb{R}^d$ be translation invariant. Then (Λ, A, V) is an M-trawl process where V is the velocity field $V = \frac{\partial}{\partial t}$, where it is understood that t is the first coordinate of M. The associated flow is simply

$$\gamma_t(s, x_1, \ldots, x_d) = (s + t, x_1, \ldots, x_d).$$

Since Killing vector fields generate a one parameter group G of isometries om M, it follows that the Lévy basis Λ in Definition 5.2 is G-homogeneous, which means that the process Y_t is strongly stationary since

$$Y_t = \Lambda\big(\gamma_t(A)\big) \overset{f.d.d}{=} \Lambda(A) \overset{f.d.d}{=} \Lambda\big(\gamma_s(A)\big) = Y_s.$$

Furthermore, it is plain to see that the cumulant function of an M-trawl process satisfies (31) with $c_A = \lambda_M(A)$ and that consequently, their marginal law are determined uniquely by the Lévy seed Λ'. In particular we may obtain any infinitely divisible distribution as the stationary law of $Y_t = \Lambda(A_t)$ by selecting Λ'. Higher order cumulants are easily obtained as

$$\kappa_n = \mathrm{i}^n \frac{d^n C(u; Y_t)}{du^n}\bigg|_{u=0} = c_A \mathrm{i}^n \cdot \frac{d^n C(u; \Lambda')}{du^n}\bigg|_{u=0},$$

which provides a straightforward way of computing higher order moments of Y_t. The following Proposition generalizes the stationary increments property of trawl processes to M-trawl processes.

Proposition 5.1. *Let (Λ, A, V) be an M-trawl process. Then Y_t has stationary increments, i.e., for $0 \leq s \leq t$ we have*

$$Y_t - Y_s \overset{d}{=} Y_{t-s} - Y_0. \tag{33}$$

Proof. The cumulant function of $Y_t - Y_s$ reads

$$\begin{aligned} C(u; Y_t - Y_s) &= C\big(u; \Lambda(A_t \setminus A_s)\big) + C\big(-u; \Lambda(A_s \setminus A_t)\big) \\ &= \lambda_M(A_t \setminus A_s) C\big(u; \Lambda'\big) + \lambda_M(A_s \setminus A_t) C\big(-u; \Lambda'\big). \end{aligned} \tag{34}$$

The semigroup property of γ_t entails that

$$A_t \setminus A_s = \gamma_s\big(\gamma_{t-s}(A) \setminus A\big) = \gamma_s(A_{t-s} \setminus A_0),$$

and similarly $A_s \setminus A_t = \gamma_s(A_0 \setminus A_{t-s})$. Since each γ_t is an isometry, it preserves λ_M and so we find that

$$\lambda_M(A_t \setminus A_s) = \lambda_M(A_{t-s} \setminus A_0), \quad \lambda_M(A_s \setminus A_t) = \lambda_M(A_0 \setminus A_{t-s}).$$

We therefore find that

$$C(u; Y_{t-s} - Y_0) = \lambda_M(A_{t-s} \setminus A_0) C\big(u; \Lambda'\big) + \lambda_M(A_0 \setminus A_{t-s}) C\big(-u; \Lambda'\big) m$$

which coincides with the expression (34). We conclude that (33) holds. □

Trawl processes have a very tractable autocorrelation structure, which is determined by the trawl set. This is very useful in practice, as one may then impose natural conditions on the trawl set to shape the autocorrelation function to the problem at hand, in a manner that is decoupled from the stationary law of the process, which is governed only by the Lévy seed. Corollary 5.1 gives the autocovariance structure of a general M-trawl process in terms of the trawl set, and the proof is exactly the same as that of Proposition 56 in [10] and is therefore omitted.

Corollary 5.1. *Let (Λ, A, V) be an M-trawl process on a Riemannian manifold (M, ρ) satisfying $\mathbb{E}[(\Lambda')^2] < \infty$ and set $Y_t = \Lambda(A_t)$. Then it holds that*

$$\begin{aligned} \operatorname{Cov}(Y_t, Y_{t+s}) &= \lambda_M(A_t \cap A_{t+s}) \mathbb{V}\big[\Lambda'\big], \\ \operatorname{Corr}(Y_t, Y_{t+s}) &= \lambda_M(A_t \cap A_{t+s}) / \lambda_M(A_t). \end{aligned}$$

In particular, if γ_t is a measure-preserving flow, then it holds that

$$\begin{aligned} \operatorname{Cov}(Y_t, Y_{t+s}) &= \lambda_M(A \cap A_s) \mathbb{V}\big[\Lambda'\big], \\ \operatorname{Corr}(Y_t, Y_{t+s}) &= \lambda_M(A \cap A_s) / \lambda_M(A). \end{aligned}$$

Example 5.2. Consider the cylinder surface of Example 3.4 with factorisable Lévy basis Λ with CQ $(\zeta, a, \nu, \lambda_M)$. The cylinder surface can be represented as $M = \mathbb{R} \times \mathbb{S}^1$ with $\mathbb{S}^1$ denoting the unit circle and has Riemananian measure $\lambda_M(ds, d\theta) = ds d\theta$, where $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ is the angular parameter. Consider the vector field $V = \partial_s + \omega \partial_\theta$, which for $\theta \neq 0$ generates the flow

$$\gamma_t(s, \theta) = \big(t + s, \theta + \omega t \pmod{2\pi}\big).$$

Under the cylinder embedding (26) into $\mathbb{R}^3$, γ_t generates a helix that wraps around the cylinder (see also Example 3.2). Consider also the trawl set $A = [-T, 0] \times I_\alpha$ for some $T, \alpha > 0$, where

$$I_\alpha = \big\{\theta \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\theta, 0) \leq \alpha\big\}, \quad d_{\mathbb{S}^1}(\theta_1, \theta_2) = \min\big\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\big\}.$$

This trawl set corresponds to a rectangular patch on the cylinder surface containing all past points up to time $-T$ and all spatial points that are within an angular distance of α on the circle. The trawl set has volume $\lambda_M(A) = 2\alpha T < \infty$ and the induced process is $Y_t = \Lambda(A_t)$ where

$$A_t = \gamma_t(A) = [t - T, t] \times \big\{\theta \in \mathbb{S}^1 : d_{\mathbb{S}^1}(\theta, \omega t) \leq \alpha\big\},$$

corresponding to sliding the rectangular patch A along a helix on the cylinder, such that the set is both shifted and rotated. The covariance structure induced by the trawl geometry is obtained via Corollary 5.1 as

$$\mathrm{Cov}(Y_t, Y_{t+h}) = \mathbb{V}\big[\Lambda'\big]\big(T - |h|\big)^+\Big(2\alpha - \min_{k\in\mathbb{Z}}|\omega h - 2\pi k|\Big).$$

5.1 Trawl functions for M-trawl processes

In the Euclidean setting, it is often convenient to specify the trawl sets A_t via the graph of a *trawl function*, as this provides very natural interpretations and links between the trawl function and autocorrelation function. An interesting such class that leads to so-called monotonic trawls is to specify the trawl set to be of the form

$$A = \big\{(s, x) \in \mathbb{R}^2 \mid s \leq 0,\ 0 \leq x \leq \eta(s)\big\} \subseteq (-\infty, 0] \times [0, K]\}, \tag{35}$$

for some $K > 0$ and non-decreasing function $\eta : (-\infty, 0]$ with $\eta(0) < \infty$ absolutely continuous. Generalizing monotonic trawls to M-trawl processes can be done in several directions, but a natural extension is to leave the trawl function unchanged and with the same interpretation, and measure the distance between each point $x \in M$ and the point $(t, \mathbf{p})$, where $\mathbf{p} \in M'$ is some fixed reference point. We then have that

$$A = \big\{(s, x) \in (-\infty, 0] \times M' \mid s \leq 0,\ \rho_{M'}(x, \mathbf{p}) \leq \eta(s)\big\}. \tag{36}$$

For the trawl set (36), it holds that

$$\lambda_M(A) = \int_{-\infty}^{0} \lambda_{M'}\big(B\big(p, \eta(s)\big)\big)ds = \int_{-\infty}^{0} B_{M'}\big(\eta(s)\big)ds,$$

where $B_{M'}(r)$ represents the volume of the ball in M' with radius r, centered at p. If Λ has finite second moment, the autocorrelation structure is given by Corollary 5.1, and by imposing some regularity on the Killing field V, we can obtain explicit forms of $\frac{d}{dh}\mathrm{Corr}(Y_t, Y_{t+h})$ in terms of the trawl function. This is useful in applications to tailor the autocorrelation structure to the data at hand. If, for example, the flow γ_t is of the form $\gamma_t(s, x) = (s + t, \psi_s(x))$ where (ψ_t) is a one-parameter group of isometries on $(M', \rho_{M'})$, then

$$A_h = \big\{(s, x) : s \leq h,\ \rho_{M'}\big(\psi_{-h}(x), p\big) \leq \eta(s - h)\big\}.$$

If ψ_h preserves the distance to the origin p, i.e., $\rho_{M'}(\psi_h(x), p) = \rho_{M'}(x, p)$, then

$$\lambda_M(A \cap A_h) = \int_{-\infty}^{-h} B_{M'}\big(\eta(s)\big)ds,$$

and hence we find that

$$\frac{d}{dh}\lambda_M(A \cap A_h) = -B_{M'}\big(\eta(-h)\big),$$

and thus that

$$\frac{d}{dh}\mathrm{Corr}(Y_t, Y_{t+h}) = \frac{-B_{M'}(\eta(-h))}{\lambda_M(A)}. \tag{37}$$

This generalizes the expression derived in Chapter 8.3.1 of [10] for monotonic trawls on $\mathbb{R}^2$. Indeed, in the usual trawl setting on $\mathbb{R}^2$ and flows corresponding to time translations and with $\mathbf{p} = 0$, the expression (37) reduces to just $-\eta(h)/\lambda_M(A)$.

If the isometry ψ_h does not preserve p, then we find that

$$\lambda_M(A \cap A_h) = \int_{-\infty}^{-h} \lambda_{M'}\Big(B\big(p, \eta(s)\big) \cap B\big(\psi_h(p), \eta(s+h)\big)\Big)ds,$$

which means that the decay of the autocorrelation function is governed by the overlap of the moving balls $B(p, \eta(s))$ and $B(\psi_h(p), \eta(s+h))$.

Example 5.3. Consider the setting of Example 5.2. Defining the function $\eta(s) = \alpha \mathbf{1}_{[-T,0]}(s)$, it holds that $\eta(s) = \alpha$ exactly when $-T \leq s \leq 0$ and thus we have the trawl function representation

$$A = \big\{(s, \theta) \in (-\infty, 0] \times \mathbb{S}^1 : \eta(s) > 0,\ d_{\mathbb{S}^1}(\theta, 0) \leq \eta(s)\big\}.$$

Under the flow $\gamma_t(s, \theta) = (s + t, \theta + \omega t \pmod{2\pi})$, the trawled sets are

$$A_t = \big\{(s, \theta) : s \leq t,\ \eta(s - t) > 0,\ d_{\mathbb{S}^1}(\theta, \omega t) \leq \eta(s - t)\big\}.$$

Equivalently,

$$A_t = \big\{(s, \theta) : t - T \leq s \leq t,\ d_{\mathbb{S}^1}(\theta, \omega t) \leq \alpha\big\}.$$

6 Lévy noise on manifolds

In [8, 10] it is shown how Lévy bases may be considered as a generalization of white noise on $\mathbb{R}^d$ to that of *Lévy white noise*, but also how certain Lévy bases can be related to cylindrical random variables in a suitable Hilbert space. All of this is shown in a Euclidean setting, and we shall therefore consider some extensions to the case where the underlying space is a Riemannian manifold. Lévy type noise is typically studied in the context of stochastic partial differential equations (SPDE's), but only a relatively small part of the literature treats the manifold setting. Notable exceptions are those of [20, 21, 18, 19], who treat SPDE's in a setting very close to what we have in mind. In [18] the authors consider SPDE's of the form

$$dF(t) = A_M\big(t, F(t)\big)dt + B_M\big(t, F(t)\big)dW(t),$$

where A_M, B_M are suitably regular operators, and in particular A_M is a differential operator on M. A particular example is that of the stochastic heat equation where B_M is constant and $A_M(t, X(t)) = \Delta_M X(t)$ is the Laplace-Beltrami operator acting on X. In [18] the noise $W(t)$ is taken to be a cylindrical Wiener process which can then be generalized to Lévy processes by the machinery of [30, 8]. The case where W is a classical white noise, or more general Lévy white noise, is more intricate, since white noise is typically constructed on the space of tempered distributions on M and this is not easily generalized to arbitrary smooth manifolds. In the following, we show how to relate Lévy bases on manifolds to Hilbert space valued noise on general Riemannian manifolds, and sketch a construction of Lévy white noise in the narrower setting of affine Nash manifolds.

6.1 Lévy noise in Hilbert space

Following [8] we define the noise of an appropriate Lévy basis Λ as the map $x \mapsto \dot{\Lambda}(x)$ for $x \in M$. This relies mostly on Hilbert space theory, and as such the manifold setting is not of particular importance, but we go through the results to showcase the role of the Riemannian measure. Furthermore, we assume that the Hilbert space $L^2(\lambda_M)$ is separable and denote by $\{e_n\}_{n=1}^{\infty}$ a complete and orthonormal system in $L^2(\lambda_M)$. The key property of the Hilbert space setting is the *nuclear covariance condition*

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_M e_n(x)\Lambda(dx)\right)^2\right] < \infty. \tag{38}$$

In addition to (38), we also assume that the stochastic integral operator as initially defined on simple functions admits a Hilbert-Schmidt extension from $L^2(\lambda_M)$ into $L^2(\Omega)$, such that $\int_M \phi_n(x)\Lambda(dx) \to \int_M \phi(x)\Lambda(dx)$ in $L^2(\mathbb{P})$, whenever $\phi_n \to \phi$ in $L^2(\lambda_M)$. This assumption appears absent, but tacitly used, in [8].

Proposition 6.1. *Let Λ be a Lévy basis on $(M, \mathcal{M})$ with intensity measure λ_M and assume that Λ satisfies the nuclear covariance condition (38). Then there exists a $L^2(\lambda_M)$-valued square-integrable random variable $\dot{\Lambda}$ such that*

$$\int_M \phi(x)\Lambda(dx) = \int_M \phi(x)\dot{\Lambda}(x)\lambda_M(dx).$$

Thus $\dot{\Lambda}$ is the Radon-Nikodym derivative of Λ with respect to λ_M.

Proof. First we note that the mapping $\phi \to \langle \Lambda(\cdot), \phi\rangle_{L^2(\lambda_M)}$ defined by

$$\phi \mapsto \langle \Lambda, \phi\rangle_{L^2(\lambda_M)} = \int_M \phi(x)\Lambda(dx),$$

is linear and bounded on $L^2(M)$. Linearity is by definition of the stochastic integral, and boundedness follows by writing $\phi = \sum_{n=1}^{\infty} \phi_n e_n$ and then applying the Cauchy-Schwarz inequality to obtain

$$\mathbb{E}\left[\left|\int_M \phi(x)\Lambda(dx)\right|^2\right] \leq \sum_{n=1}^{\infty} \phi_n^2 \cdot \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_M e_n(x)\Lambda(dx)\right)^2\right] < \infty.$$

In fact, this shows that the operator is Hilbert-Schmidt from $L^2(\lambda_M)$ into $L^2(\mathbb{P})$ and it follows by standard Hilbert space theory that there exists a function $\dot{\Lambda} \in L^2(\lambda_M)$ such that

$$\big\langle \Lambda(A), \phi \big\rangle_{L^2(\lambda_M)} = \int_M \dot{\Lambda}(x)\phi(x)\lambda_M(dx).$$

□

Proposition 6.2. *Let Λ be a Lévy basis on $(M, \mathcal{M})$ satisfying the nuclear covariance condition (38) and such that $\Lambda(A) = \int_A \dot{\Lambda}(x)\lambda_M(dx)$. Set*

$$q(x,y) = \mathbb{E}\big[\big(\dot{\Lambda}(x) - \mathbb{E}\big[\dot{\Lambda}(x)\big]\big)\big(\dot{\Lambda}(y) - \mathbb{E}\big[\dot{\Lambda}(y)\big]\big)\big],$$

and define the linear operator Q via

$$Qf(x) = \int_M q(x,y)f(y)\lambda_M(dy), \quad f \in L^2(\lambda_M). \tag{39}$$

Then Q is a bounded, non-negative, self-adjoint, and trace class operator and maps $L^2(\lambda_M)$ into itself. Moreover, whenever $\mathbf{1}_A, \mathbf{1}_B \in L^2(\lambda_M)$, it holds that

$$\mathrm{Cov}\big(\Lambda(A), \Lambda(B)\big) = \langle Q\mathbf{1}_A, \mathbf{1}_B \rangle_{L^2(\lambda_M)}.$$

Proof. For brevity, define the $L^2(\lambda_M)$-valued and square integrable random variable $X = \dot{\Lambda} - \mathbb{E}[\dot{\Lambda}]$. We first show that $Qf \in L^2(\lambda_M)$ by noting that

$$\begin{aligned}
&\left| \int_M q(\cdot, y)f(y)\lambda_M(dy) \right|_{L^2(\lambda_M)} \\
&\quad \leq \int_M \big| q(\cdot, y)f(y) \big|_{L^2(\lambda_M)} \lambda_M(dy) \\
&\quad \leq \left(\int_M \int_M q(x,y)^2 \lambda_M(dx)\lambda_M(dy) \right)^{\frac{1}{2}} |f|_{L^2(\lambda_M)} \\
&\quad \leq \left(\int_M \int_M \mathbb{E}\big[\dot{\Lambda}(x)^2\big]\mathbb{E}\big[\dot{\Lambda}(y)^2\big] \lambda_M(dx)\lambda_M(dy) \right)^{\frac{1}{2}} |f|_{L^2(\lambda_M)} \\
&\quad = \mathbb{E}\big[|\dot{\Lambda}|^2_{L^2(\lambda_M)}\big] |f|_{L^2(\lambda_M)}.
\end{aligned}$$

By Parseval's identity and the nuclear covariance condition (38) it holds that

$$\mathbb{E}\big[|\dot{\Lambda}|^2_{L^2(\lambda_M)}\big] = \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_M e_n(x)\Lambda(dx) \right)^2 \right] < \infty.$$

Thus, it holds that $Qf \in L^2(\lambda_M)$. We then note that

$$\langle Qf, f \rangle_{L^2(\lambda_M)} = \mathbb{E}\big[\langle f, X \rangle^2_{L^2(\lambda_M)}\big] \geq 0,$$

which shows that the operator is non-negative. Self-adjointness follows by Fubini's theorem and observing that $q(x,y) = q(y,x)$, such that $\langle Qf, g \rangle_{L^2(\lambda_M)} = \langle f, Qg \rangle_{L^2(\lambda_M)}$ for $f, g \in L^2(\lambda_M)$. We then show that Q is trace class in the sense of Appendix A

in [30]. To this end, we note that Q is Hilbert-Schmidt, since $q \in L^2(\lambda_M \otimes \lambda_M)$ and therefore compact. As Q is also self-adjoint and non-negative, the spectral theorem ensures existence of an orthonormal basis $(e_n)_{n=1}^{\infty}$ and associated eigenvalues $\lambda_n \geq 0$ such that $Qe_n = \lambda_n e_n$ and $\lambda_n = \langle Qe_n, e_n \rangle_{L^2(\lambda_M)} = \mathbb{E}[|\langle X, e_n \rangle_{L^2(\lambda_M)}|^2]$. An application of Tonelli's theorem and Parseval's identity therefore yields

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n &= \mathbb{E}\left[\sum_{n=1}^{\infty} \left|\langle X, e_n \rangle_{L^2(\lambda_M)}\right|^2\right] \\ &= \mathbb{E}\left[\|X\|^2_{L^2(\lambda_M)}\right] \\ &< \infty.\end{aligned}$$

Setting $a_n(x) = \sqrt{\lambda_n} e_n(x)$ and $b_n(x) = \sqrt{\lambda_n} e_n(x)$, we have

$$Qf = \sum_{n=1}^{\infty} \langle f, a_n \rangle_{L^2(\lambda_M)} b_n$$

and

$$\sum_{n=1}^{\infty} \|a_n\|_{L^2(\lambda_M)} \|b_n\|_{L^2(\lambda_M)} = \sum_{n=1}^{\infty} \lambda_n = \mathbb{E}\left[\|X\|^2_{L^2(\lambda_M)}\right] < \infty,$$

which is exactly the trace class condition of [30]. Finally, whenever $\mathbf{1}_A, \mathbf{1}_B \in L^2(\lambda_M)$, the density representation of Λ gives

$$\begin{aligned}\operatorname{Cov}\big(\Lambda(A), \Lambda(B)\big) &= \mathbb{E}\left[\left(\int_A X(x) \lambda_M(dx)\right)\left(\int_B X(y) \lambda_M(dy)\right)\right] \\ &= \mathbb{E}\left[\langle X, \mathbf{1}_A \rangle_{L^2(\lambda_M)} \langle X, \mathbf{1}_B \rangle_{L^2(\lambda_M)}\right] \\ &= \langle Q\mathbf{1}_A, \mathbf{1}_B \rangle_{L^2(\lambda_M)}.\end{aligned}$$

□

6.2 *From white noise to Lévy noise*

Classically, white noise is considered as the time derivative of a Brownian motion, which only exist in a generalized sense, and this may also be extended to non-Gaussian Lévy white noise. A comprehensive treatment of white and Lévy white noise on $\mathbb{R}^d$ and their links to SPDE's may be found in [22].

Typically, white noise is defined on the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of smooth functions whose derivatives are all rapidly decreasing. The dual space of tempered distributions is denoted by $\mathcal{S}^*(\mathbb{R}^d)$. An immediate problem in defining white and Lévy white noise on Riemannian manifolds is, that it is not clear how to define these spaces in full generality. We will therefore restrict ourselves to a rather specific subset of Riemannian manifolds, namely *affine Nash manifolds*. These manifolds are distinguished by the property that they are globally defined in terms of polynomial equalities and inequalities, which makes it possible to define notions of rapid decay in a manner similar to the Euclidean setting. The following bare-bones definitions are all adapted from [3], which we refer to for a more comprehensive treatment of Nash manifolds. Note that any compact smooth manifold is diffeomorphic to a Nash manifold.

Definition 6.1 (Semi-algebraic sets)**.** A set $A \subseteq \mathbb{R}^d$ is called a semi-algebraic set if it can be represented as a finite union of sets defined by a finite number of polynomial equalities and inequalities of the form $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : p(x_1, \ldots, x_n) = 0\}$ and $\{(x_1, \ldots, x_d \in \mathbb{R}^d : p(x_1, \ldots, x_d) > 0\}$ where p represents some polynomial over $\mathbb{R}^d$.

Definition 6.2 (Affine Nash manifold)**.** An affine Nash manifold is a smooth manifold M embedded in $\mathbb{R}^d$, which is also a semi-algebraic set in $\mathbb{R}^d$.

The structure of affine Nash manifolds allows for defining Schwartz spaces and their duals analogously to the Euclidean setting. Indeed, [3] define the Schwartz space on M as

$$\mathfrak{S}(M) = \left\{\phi \in C^\infty(M) : D\phi \text{ is bounded for any Nash differential operator } D \text{ on } M\right\}.$$

Nash differential operators are defined in Section 3.5 of [3], and are essentially differential operators acting on functions whose graphs are semi-algebraic sets and such that the operator itself has such semi-algebraic coefficients. In [4] it is proven that $\mathfrak{S}(M)$ is a nuclear Fréchet space, and analogously to the classical theory of [22, 8] we can then invoke the Bochner-Minlos theorem to define a Lévy white noise probability space. This is done by defining $\Omega^* = \mathfrak{S}^*(M)$ with associated σ-algebra $\mathcal{F}^*$ and letting ν be a Lévy measure on $\mathbb{R} \setminus \{0\}$ such that

$$K := \int_{\mathbb{R}\setminus\{0\}} x^2 \nu(dx) < \infty.$$

By the Bochner-Minlos theorem on general nuclear spaces, there exists a probability measure $\mathbb{P}^*$ on $(\Omega^*, \mathcal{F}^*)$ such that

$$\begin{aligned}&\int_{\Omega^*} e^{\mathrm{i}\langle\omega,\phi\rangle}\mathbb{P}^*(d\omega)\\ &\quad= \exp\left(-\frac{1}{2}\sigma^2\|\phi\|^2_{L^2(\lambda_M)} + \int_M \int_{\mathbb{R}\setminus\{0\}} \left[e^{\mathrm{i}\phi(y)x} - 1 - \mathrm{i}\phi(y)x\right]\nu(dx)\lambda_M(dy)\right),\end{aligned}$$

where we use the suggestive notation $\langle\omega, \phi\rangle = \omega(\phi)$ for the action of ω on the Schwartz function ϕ. The probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ is called the Lévy white noise probability space. We can then define cylindrical random variables Ξ_ϕ on $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ via

$$\Xi_\phi(\omega) = \langle\omega, \phi\rangle, \quad \phi \in \mathfrak{S}(M),$$

which satisfy

$$\mathbb{E}^{\mathbb{P}^*}\left[\Xi_\phi(\omega)\right] = 0, \quad \mathbb{V}^{\mathbb{P}^*}\left[\Xi_\phi\right] = \left(\sigma^2 + K\right)\int_M \phi(y)^2\lambda_M(dy).$$

The extension to Ξ_ϕ for any $\phi \in L^2(\lambda_M)$ may be carried out by noting that $(\langle\omega, \phi_n\rangle)_{n=1}^\infty$ is a Cauchy sequence in $L^2(\mathbb{P}^*)$ and hence convergent in $L^2(\mathbb{P}^*)$, whenever $\phi_n \to \phi$ in $L^2(\lambda_M)$, independently of the approximating sequence $(\phi_n)_{n=1}^\infty$. In particular, for any $A \subseteq \mathcal{M}_b$, we can define the random variable $\Xi_A = \Xi_{\mathbf{1}_A}$, which leads to the following result as in [8, Proposition 2.16], the proof of which follows almost identically.

Proposition 6.3. *Define the random measure Λ on $\mathcal{M}_b$ as*

$$\Lambda(A) = \Xi_A, \quad A \in \mathcal{M}_b.$$

Then Λ is a factorisable Lévy basis on $(M, \mathcal{M})$ with intensity measure c given by

$$c(A) = \sigma^2 \lambda_M(A) + \int_{\mathbb{R}} \min\big(1, x^2\big) \nu(dx) \lambda_M(A).$$

In particular, if M has a space-time structure of the form $M = \mathbb{R} \times M'$, then Λ is a finite variance orthogonal martingale measure with respect to its natural filtration (9).

Proof. We show that Λ is a Lévy basis, by showing that it satisfies an equivalent definition to that of Definition 2.1. First we show that Λ has the additivity property that $\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)$ for disjoint A and B. Let therefore $A, B \in \mathcal{M}_b$ be disjoint and let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be two sequences of functions in $L^2(\lambda_M)$ such that $a_n \to \mathbf{1}_A$ and $b_n \to \mathbf{1}_B$. Then $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$, and since $a_n + b_n \to \mathbf{1}_A + \mathbf{1}_B$ in $L^2(\lambda_M)$ it holds that $a_n + b_n \to \mathbf{1}_{A\cup B}$ in $L^2(\lambda_M)$. Then $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$, and since $a_n + b_n \to \mathbf{1}_A + \mathbf{1}_B$ in $L^2(\lambda_M)$ it holds that $a_n + b_n \to \mathbf{1}_{A\cup B}$ in $L^2(\lambda_M)$. It then holds that

$$\Xi_{a_n+b_n} = \langle \omega, a_n + b_n \rangle = \langle \omega, a_n \rangle + \langle \omega, b_n \rangle = \Xi_{a_n} + \Xi_{b_n}.$$

But it also holds, by independence of the approximating sequence, that $\Xi_{a_n+b_n} \to \Lambda(A \cup B)$ in $L^2(\mathbb{P}^*)$, which implies that $\Lambda(A \cup B) = \Lambda(A) + \Lambda(B)$, thus proving the additive property of Λ. We then show that Λ is independently scattered. To this end we take two approximating sequences $\{a_n\}_{n=1}^\infty, \{b_n\}_{n=1}^\infty \subseteq \mathfrak{S}(M)$ converging to $\mathbf{1}_A$ and $\mathbf{1}_B$ in $L^2(\lambda_M)$. We then have

$$\begin{aligned}
&\log\big(\mathbb{E}\big[e^{\mathrm{i}u_A \Xi_{a_n}} e^{\mathrm{i}u_B \Xi_{b_n}}\big]\big) \\
&\quad = \log\big(\mathbb{E}\big[e^{\mathrm{i}\langle \cdot, u_A a_n + u_B b_n \rangle}\big]\big) \\
&\quad = -\frac{1}{2}\sigma^2 \|u_A a_n + u_B b_n\|^2_{L^2(\lambda_M)} \\
&\qquad + \int_M \int_{\mathbb{R}\setminus\{0\}} \big[e^{\mathrm{i}x(u_A a_n(z) + u_B b_n(z))} - 1 - \mathrm{i}x\big(u_A a_n(z) + u_B b_n(z)\big)\big] \nu(dx) \lambda_M(dz).
\end{aligned} \tag{40}$$

As $\mathbf{1}_A$ and $\mathbf{1}_B$ are orthogonal in $L^2(\lambda_M)$, it holds that $\|u_A a_n + u_B b_n\|^2_{L^2(\lambda_M)} \to \|u_A \mathbf{1}_A + u_B \mathbf{1}_B\|^2_{L^2(\lambda_M)}$, which shows the claim for the Gaussian part. For the jump part, set $r(u) = e^{\mathrm{i}u} - 1 - \mathrm{i}u$ and $\psi(u) = \int_{\mathbb{R}\setminus\{0\}} r(xu)\nu(dx)$. Note that $|r'(u)| \le |u|$, which yields the estimate

$$\big|r(ux) - r(vx)\big| = \left| \int_v^u xr'(sx)ds \right| \le |u - v|\big(|u| + |v|\big)x^2.$$

From this, we find that for $u, v \in \mathbb{C}$,

$$\big|\psi(u) - \psi(v)\big| \le K|u - v|\big(|u| + |v|\big).$$

For any $f, g \in L^2(\lambda_M)$, we therefore find by an application of the Cauchy-Schwarz inequality that

$$\int_M \left|\psi\big(f(z)\big) - \psi\big(g(z)\big)\right| \lambda_M(dz) \leq K \|f - g\|_{L^2(\lambda_M)} \big(\|f\|_{L^2(\lambda_M)} + \|g\|_{L^2(\lambda_M)}\big),$$

which yields continuity in the sense that $\int \psi(f_n(z))\lambda_M(dz) \to \int \psi(f(z))\lambda_M(dz)$ whenever $f_n \to f$ in $L^2(\lambda_M)$. It follows that the jump-part on right hand side of (40) converges to

$$\lambda_M(A) \int_{\mathbb{R}\setminus\{0\}} \big[e^{\mathrm{i}xu_A} - 1 - \mathrm{i}xu_A\big] \nu(dx) + \lambda_M(B) \int_{\mathbb{R}\setminus\{0\}} \big[e^{\mathrm{i}xu_B} - 1 - \mathrm{i}xu_B\big] \nu(dx),$$

and hence that

$$\mathbb{E}\big[e^{\mathrm{i}u_A\Lambda(A)} e^{\mathrm{i}u_B\Lambda(B)}\big] = \mathbb{E}\big[e^{\mathrm{i}u_A\Lambda(A)}\big]\mathbb{E}\big[e^{\mathrm{i}u_B\Lambda(B)}\big],$$

which proves that $\Lambda(A)$ and $\Lambda(B)$ are independent, but also that Λ is infinitely divisible since

$$\log\big(\mathbb{E}\big[e^{\mathrm{i}u\Lambda(A)}\big]\big) = \lambda_M(A)\left(-\frac{1}{2}u\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} \big[e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\big]\nu(dx)\right).$$

By Proposition 27 in [10], Λ is then a factorisable Lévy basis with CQ $(0, \sigma^2, \nu, c)$. It is shown in Chapter 5.2.2 of [10] that square-integrable Lévy bases on $\mathbb{R} \times E$ are orthogonal martingale measure when $M' = \mathbb{R}^d$, but it is straightforward to extend this statement to the manifold case. □

Remark 6.1. By setting $\sigma = 1$ and $\nu \equiv 0$ in Proposition 6.3, we obtain a white noise, which then corresponds to a Lévy basis with CQ $(0, 1, 0, \lambda_M)$. This is in line with the interpretation of Lévy bases as an extension of white noise as defined in [37], which is also discussed in Chapter 5.1.2 of [10].

From now on, we assume that Λ is a Lévy basis as constructed in Proposition 6.3. Since Λ has finite second moment and admits a Lévy-Itô representation per Proposition 3.3, we can represent Λ of the form

$$\Lambda(A) = \zeta\lambda_M(A) + \sigma W(A) + \int_{\mathbb{R}\setminus\{0\}} y\widetilde{N}(dy, A),$$

where $\widetilde{N}$ is a compensated Poisson random measure with compensator $\lambda_M(dz)\nu(dy)$ and ζ is chosen accordingly. We assume that $\sigma = 1$, so the Gaussian random measure W defines an isonormal Gaussian process X via the stochastic integral $X(h) = \int_M h(z)W(dz)$, for which it holds that $\mathbb{E}[X(h)X(g)] = \langle h, g\rangle_{L^2(\lambda_M)}$. Letting $\mathcal{F}_X = \sigma(X(h) : h \in L^2(\lambda_M))$, it is then well-known that the space $L^2(\mathbb{P}^*; \mathcal{F}_X^*)$ decomposes as

$$L^2\big(\mathbb{P}^*; \mathcal{F}_X^*\big) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the n'th homogeneous Wiener chaos and that every $F \in L^2(\mathbb{P}^*; \mathcal{F}_X^*)$ can be represented as

$$F = \mathbb{E}[F] + \sum_{n \geq 1} I_n(f_n),$$

where f_n are symmetric square integrable functions over the space $M^n = \bigtimes_{j=1}^n M$ equipped with the measure $\lambda_M^{\otimes n}$ and $I_n(f) = \int_{M^n} f(z_1, \ldots, z_n) W^{\otimes n}(dz_1, \ldots, dz_n)$ is the n'th iterated Wiener-Itô integral with respect to W, which satisfies an isometry property of the form

$$\mathbb{E}\big[I_n(f) I_m(g)\big] = \begin{cases} 0 & \text{if} \quad n \neq m, \\ n! \langle f, g \rangle_{L^2(\lambda_M^{\otimes n})} & \text{if} \quad n = m. \end{cases}$$

A similar result holds for the compensated Poisson random measure $\widetilde{N}$. Setting $Y = M \times \mathbb{R} \setminus \{0\}$, $\mu = \lambda_M \otimes \nu$, and $\mathcal{F}_N^* = \sigma(N(A) : A \in \mathcal{B}(Y))$, the space $L^2(\mathbb{P}; \mathcal{F}_N^*)$ admits the decomposition (see, e.g., [24])

$$L^2(\mathbb{P}; \mathcal{F}_N) = \bigoplus_{n=0}^{\infty} \mathcal{N}_n,$$

where

$$\mathcal{N}_n = \overline{\big\{\widetilde{I}_n(f) : f \in L^2_{\mathrm{sym}}\big(\mu^{\otimes n}\big)\big\}},$$

with $\widetilde{I}_n$ denoting the n'th multiple Wiener-Itô integral of the form

$$\widetilde{I}_n(f) = \int_{Y^n} f(y_1, \ldots, y_n) \widetilde{N}(dy_1) \cdots \widetilde{N}(dy_n).$$

Here, we adopt the convention that any function f in the iterated Wiener-Itô integral is taken as an L^2-representative which is zero on the diagonal of Y^n. The iterated Wiener-Itô integrals $\widetilde{I}_n$ also have the isometry property

$$\mathbb{E}\big[\widetilde{I}_n(f) \widetilde{I}_m(g)\big] = \begin{cases} 0 & \text{if} \quad n \neq m, \\ n! \langle f, g \rangle_{L^2(\mu^{\otimes n})} & \text{if} \quad n = m, \end{cases}$$

and it holds that any $F \in L^2(\mathbb{P}^*; \mathcal{F}_N^*)$ can be represented as

$$F = \mathbb{E}[F] + \sum_{n \geq 1} \widetilde{I}_n(f_n).$$

Combining these two representations, we set $\mathcal{F}_\Lambda^* = \sigma(\mathcal{F}_X \cup \mathcal{F}_N)$ and find that the space $L^2(\mathbb{P}^*; \mathcal{F}_\Lambda^*)$ has the mixed chaos decomposition

$$L^2\big(\mathbb{P}^*; \mathcal{F}_\Lambda^*\big) = \bigoplus_{p=0}^{\infty} \bigoplus_{q=0}^{\infty} \mathcal{H}_p \otimes \mathcal{N}_q,$$

and every $F \in L^2(\mathbb{P}^*; \mathcal{F}_\Lambda^*)$ admits a unique representation of the form

$$F = \sum_{p,q \geq 0} \mathcal{I}_{p,q}(h_{p,q}), \quad h_{p,q} \in L^2_{\mathrm{sym}}\big(\lambda_M^{\otimes p} \otimes \mu^{\otimes q}\big), \tag{41}$$

where for $h_{p,q} = f_p \otimes g_q$, the multiple Wiener-Itô integral is $\mathcal{I}_{p,q} = I_p(f_p)\widetilde{I}_q(g_q)$. Hence the Lévy basis Λ has a chaotic representation property. In view of this, we may then follow [22] and define the space $\mathfrak{S}_H(M)$ of stochastic test functions as

$$\mathfrak{S}_H(M) := \left\{ X = \sum_{p,q\geq 0} I_p(f_p)\widetilde{I}_q(g_q) : f_p \in L^2_{\mathrm{sym}}\big(\lambda_M^{\otimes p} \otimes \mu^{\otimes q}\big),\ \|X\|_{\mathfrak{S}} < \infty \right\},$$

where $\|\cdot\|_{\mathfrak{S}}$ is an appropriate seminorm, which should ensure that $\mathfrak{S}_H(M) \subseteq L^2(\mathbb{P}^*)$ is a nuclear space. Supposing that $\mathfrak{S}_H(M)$ is in fact a nuclear space (see Remark 6.2 below), we can then define the noise $\dot{\Lambda}$ by the linear map

$$\langle \dot{\Lambda}, f\rangle := \Xi_f, \quad f \in L^2(\lambda_M).$$

The notation $\int_M f(x)\dot{\Lambda}(x)\lambda_M(dx)$ is then understood in a distributional sense and satisfies

$$\langle \dot{\Lambda}, f\rangle = \Xi_f = \int_M f(x)\Lambda(dx),$$

thus admitting an interpretation of $\dot{\Lambda}$ as the noise of the Lévy basis Λ.

Remark 6.2. The nuclearity of $\mathfrak{S}(M)$ alone does not imply the nuclearity of the stochastic test-function space proposed above. Indeed, its first Poisson chaos contains the factor $L^2(\nu)$. If this space is infinite-dimensional, retaining its Hilbert-space norm prevents the relevant embeddings from being Hilbert-Schmidt. A nuclear stochastic test-function space should instead be constructed by choosing a suitable nuclear Gel'fand triple

$$E_\nu \subset L^2(\nu) \subset E_\nu'$$

and combining it with the nuclear space $\mathfrak{S}(M)$ in the corresponding Fock-space construction. We do not pursue this construction here.

Remark 6.3. The Lévy basis Λ treated above violates the nuclear covariance condition (38), which means it is not regular enough to admit a Hilbert space valued Radon-Nikodym derivative $\dot{\Xi}_x$. This is easily seen from the computation

$$\sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_M e_n(x)\Lambda(dx)\right)^2\right] = \sum_{n=1}^{\infty} \mathbb{E}\big[\Xi_{e_n}^2\big] = \sum_{n=1}^{\infty} |e_n|_2^2 = \sum_{n=1}^{\infty} 1 = \infty.$$

Example 6.1. Consider the ambit field (25) of Example 3.4. The cylinder surface parametrized by (26) clearly satisfies Definition 6.2 and is thus an affine Nash manifold. Suppose that $\sigma_t(x) \equiv 1$. Then $Y_t(x)$ is a linear functional of the driving Lévy basis Λ and it is then simple to verify that Y has chaos representation (41) given by

$$Y_t(x) = \mathbb{E}\big[Y_t(x)\big] + \mathcal{I}_{1,0}\big(h_{1,0}^{(t,x)}\big) + \mathcal{I}_{0,1}\big(h_{0,1}^{(t,x)}\big),$$

where we define $\mathbb{S}^1 = \{((x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and set

$$h_{1,0}^{(t,x)}(s,y) = \sqrt{a}\mathbf{1}_{(0,t]}(s)f(t-s,x,y),$$
$$h_{0,1}^{(t,x)}(s,y) = z\mathbf{1}_{(0,t]}(s)f(t-s,x,y).$$

A Basic geometric concepts

In this appendix, we recall some basic definitions and properties of Riemannian manifolds, which are used throughout the paper. All of the results can be found in [25, 26].

A.1 Riemannian manifolds and embeddings

A smooth manifold is essentially a space that locally behaves like Euclidean space, but whereas Euclidean space is globally flat, the smooth manifold is not. The dimension of a manifold is equal to the dimension of the "local Euclidean space", such that a k-manifold locally looks like $\mathbb{R}^k$. This property is described by countable collections of sets U_α that cover the full manifold, together with injective maps ϕ_α that give a local coordinate description of U_α in terms of coordinates in $\mathbb{R}^k$.

Definition A.1 (Smooth manifold)**.** We say that M is a smooth manifold of dimension k if M is a second countable Hausdorff space and there exists a collection of subsets $\{U_\alpha\}_\alpha$ of M together with an injective map $\phi_\alpha : U_\alpha \to \mathbb{R}^d$ for each α such that the following hold.

1. For each α, the set $\phi(U_\alpha)$ is an open subset of $\mathbb{R}^d$.
2. For each α, β the sets $\phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^d$.
3. Whenever $U_\alpha \cap U_\beta \neq \emptyset$ the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism.
4. Countably many of the sets U_α cover M, i.e. there exists a countable collection $\{U_\alpha\}_{\alpha \in I}$ such that $\bigcup_{\alpha \in I} U_\alpha = M$.
5. For $p \neq q$ where $p, q \in M$ there either exists U_α with $p, q \in U_\alpha$ or disjoint U_α, U_β such that $p \in U_\alpha$ and $q \in U_\beta$.

We say that each (U_α, ϕ_α) is a smooth chart, i.e. $U_\alpha \subseteq M$ is open and $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha)$ is a homeomorphism.

A Riemannian manifold is essentially a smooth manifold M with an associated metric. This metric is defined on the tangent space T_pM. A smooth manifold M is in general not a vector space, but T_pM constitutes a k-dimensional vector space with basis $(\frac{\partial}{\partial x^i})_{i=1}^k$, where x^i are local coordinates of the manifold M.

Definition A.2 (Tangent space)**.** Let M be a smooth manifold and let $p \in M$. A linear map $F : C^\infty(M) \to \mathbb{R}$ is called a derivation map at p if $F(gh) = g(p)F(h) + h(p)F(g)$ for every $g, h \in C^\infty(M)$. The set of all such derivations at p is a k-dimensional vector space and is called the tangent space at p and is denoted by T_pM.

Definition A.3 (Differential of a map)**.** Let M, N be smooth manifolds, and let $F : M \to N$ be a smooth map. For each $p \in M$, define the map

$$dF_p : T_pM \to T_{F(p)}N,$$

as the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ as

$$dF_p(v)(f) = v(f \circ F).$$

This is a linear operator and a derivation map at $F(p)$, and we call dF_p the differential of F at p.

Definition A.4 (Metric)**.** Let $\rho_p : T_pM \times T_pM \to \mathbb{R}$ be symmetric and positive definite for each $p \in M$, such that $\rho(u,v) = \rho(v,u)$ for every $u, v \in T_pM$ and $\rho(u,u) > 0$ for each non-zero $u \in T_pM$. Then we say that ρ is a metric on T_pM, and this induces a positive definite symmetric matrix $(\varrho_{ij}) \in \mathbb{R}^{k\times k}$ given by

$$\varrho_{ij}(p) = \rho_p\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right), \quad 1 \le i, j \le k. \tag{42}$$

Definition A.5 (Riemannian manifold)**.** We say that (M, ρ) is a Riemannian manifold if M is a smooth manifold and ρ is a metric.

Riemannian manifolds are abstractly defined in terms of their associated metric, and in many cases it is simpler to study them from this intrinsic point of view. In applications such as temporo-spatial modelling, it is sometimes possible to envision the manifold under consideration, and potentially as a subspace in a larger Euclidean space. In this case, the Riemannian metric is implicitly fixed via the choice of parametrization, which we now elaborate on. An example is that of [23], where the manifold is parametrized as a cylinder surface sitting in the ambient Euclidean space $\mathbb{R}^3$. The ambient dimension is 3, but the cylinder surface is described by only two coordinates and hence is a 2-manifold. In this extrinsic view, we consider M as an *embedded submanifold* in $\mathbb{R}^d$, and in this case the Riemannian metric is inherited from the way that M sits in $\mathbb{R}^d$, i.e., the particular parametrization of M. The formal definition of an embedded submanifold and a description of the induced metric is given below.

Definition A.6 (Immersion and embedding)**.** Let M, N be smooth manifolds, and $F : M \to N$ a smooth map. We say that F is an immersion if its differential dF_p is injective at each point p. We say that F is an embedding, if F is an immersion and a homeomorphism onto its image.

Definition A.7 (Embedded submanifold)**.** Let M be a smooth manifold. An embedded submanifold of M is a set $A \subseteq M$ which is a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $A \hookrightarrow M$ is an embedding.

If M is a smooth manifold embedded in a Riemannian manifold (N, ρ) with embedding ι, then the pullback $\rho^* = \iota^*\rho$ is a Riemannian metric on M, and we say that ρ^* is the metric on M induced by ι. When working with a Riemannian manifold (M, ρ) embedded in some higher dimensional space, we shall always assume that ρ is the metric induced by the embedding.

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