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A dynamical Borel–Cantelli lemma for stationary determinantal processes

Akhtam Dzhalilov[a,*], **Khasanboy Abdusalomov**[b], **Jamshidjon Imomaliev**[a,c]

[a] *Turin Polytechnic University in Tashkent, Tashkent, 100095, Uzbekistan*
[b] *National University of Uzbekistan, Tashkent, 100174, Uzbekistan*
[c] *Tashkent International University of Financial Management and Technologies, Tashkent, 100084, Uzbekistan*

adzhalilov21@gmail.com (A.A. Dzhalilov), hasanboy155abs@gmail.com (Kh.Sh. Abdusalomov), jamshidjon.in777@gmail.com (J.N. Imomaliev)

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Abstract We study dynamical Borel–Cantelli properties for the left shift on the binary symbolic space equipped with a stationary determinantal measure generated by a Toeplitz convolution kernel. For sequences of cylinder sets whose measures have a divergent sum, we give sufficient conditions ensuring infinitely many visits almost surely. Under exponential decay of the Fourier coefficients and a uniform nondegeneracy assumption, we also establish the strong Borel–Cantelli property.

Keywords Dynamical Borel–Cantelli lemma, stationary determinantal process, strong Borel–Cantelli property, cylinder sets, shift dynamical system, ψ-mixing, Toeplitz kernel

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1 Introduction

The classical Borel-Cantelli lemma plays an important role in probability theory and its applications. The modern theory of dynamical systems uses a special version of the Borel-Cantelli lemma (see for instance [4], [10]).

*Corresponding author.

Let $\langle \mathbb{M}, \mathfrak{F}, \mu, T\rangle$ be a dynamical system with $T-$ invariant probability measure μ. The classical Poincare' recurrence theorem states that for any fixed subset $A \in \mathfrak{F}$ with $\mu(A) > 0$ the equality

$$\mu(\{x \in A \mid \ T^n(x) \in A \text{ for infinitely many } n \in \mathbb{N}\}) = \mu(A)$$

holds. We consider the sequence of nontrivial measurable subsets A_n and can still ask for the μ-measure of the limsup set:

$$\{x \in \mathbb{M} \mid T^n(x) \in A_n \text{ for infinitely many } n \in \mathbb{N}\} = limsup T^{-n} A_n.$$

In the case $\Sigma\mu(T^{-n}A_n) = \Sigma\mu(A_n) < \infty$, the convergence case of the Borel-Cantelli lemma implies that the $\mu-$ measure of the limsup set is zero. If $\Sigma\mu(A_n) = \infty$, and $T^{-n}A_n$ are independent, then for $\mu-$ measure of the limsup set is one (see for instance [11]). The last assertion has a limited value for deterministic dynamical systems, since one rarely deals with purely independent sets. In the case, if $\Sigma\mu(A_n) = \infty$ and the events $T^{-n}A_n$ are dependent, the situation is more complicated and more interesting. A definition, found in [4], applies to this situation.

Definition 1 (see [4])**.** A sequence of measurable sets A_n $n \in \mathbb{N}$, such that

$$\Sigma\mu(T^{-n}A_n) = \Sigma\mu(A_n) = \infty, \tag{1}$$

is called a Borel-Cantelli (BC) sequence for T if

$$\mu(limsup T^{-n} A_n) = \mu(\mathbb{M}).$$

The divergence case of the Borel-Cantelli lemma is not helpful for finding BC sequences since this case of the lemma requires independent sets. To obtain a BC sequence, we need impose some restrictions.

If, for a dynamical system, all sequences of subsets A_n that satisfy (1) and certain additional conditions are BC, we obtain what is called **a dynamical Borel-Cantelli lemma.**

The first example of such a lemma, in which only sequences of balls centered at a fixed point and with weakly monotonically decreasing radii are allowed, was proved by J. Kurzweil [10]. For a dynamical system with mixing property, the "abundance" of BC sequences can be interpreted as an aspect of strong chaos and stochastic behavior of the system. It is proved (see [4], [12], [7], [1]) that for a wide class of hyperbolic or"fast" mixing systems, various sequences of sets have the BC property. The sets which are to be considered in this kind of problems are usually decreasing sequences of balls with the same center (see [10], [14], [6]) or cylinders. In [4] Chernov proved the dynamical BC lemma for a Gibbs measures. In [7] Kim and Galatolo established that the Borel-Cantelli property and the waiting time problem are in general strictly connected.

Stationary determinantal processes provide a natural and important class of shift-invariant probability measures on symbolic spaces. They arise in probability theory, random matrix theory, mathematical physics, and the theory of point processes. In contrast to Bernoulli or Markov measures, determinantal measures are characterized by determinant formulae for finite-dimensional distributions and typically exhibit negative dependence. In the stationary case on $\mathbb{Z}$, such processes are generated by convolution kernels

$$K_f(n,m) = \widehat{f}(n-m),$$

where $f : \mathbb{S}^1 \to [0,1]$ is a measurable function. The corresponding measure μ_f is invariant under the shift, and therefore it naturally defines a measure-preserving symbolic dynamical system.

A key tool in the present paper is the ψ-mixing property of stationary determinantal processes. Fan, Liao and Qiu obtained necessary and sufficient conditions for the ψ-mixing property in terms of the regularity of the generating function f and the decay of its Fourier coefficients. This makes stationary determinantal processes suitable for studying dynamical Borel–Cantelli type problems for cylinder targets.

The paper is organized as follows. Section 2 contains the necessary preliminaries on Borel–Cantelli sequences, cylinder sets, stationary determinantal measures, and ψ-mixing, and also presents the main results of the paper. In Section 3, we prove the uniform and conditional estimates for determinantal cylinder probabilities. Section 4 is devoted to the proof of the exponential cylinder mixing estimate and the main strong dynamical Borel–Cantelli theorem.

2 Preliminaries and formulation of main results

2.1 Borel–Cantelli sequences and the Sprindzuk criterion

Let $\langle \mathbb{M}, \mathfrak{F}, \mu, T\rangle$ be a dynamical system with $T-$ invariant probability measure μ.

Let $A_n \subset \mathbb{M}$ be a sequence of measurable sets. We set $B_n := T^{-n}A_n$. Define the set $\limsup B_n$ as

$$\limsup B_n := \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_n$$

A classical Borel-Cantelli lemma in probability theory states:

Lemma 1 (Borel-Cantelli)**.** *(i) Suppose that $\sum \mu(B_n) < \infty$, then $\mu(\limsup B_n) = 0$, i.e., almost every point $x \in \mathbb{M}$ belongs to finitely many B_n.*

(ii) In the case, $\sum \mu(B_n) = \infty$ and the events of the sequence $\{B_n,\ n \in \mathbb{N}\}$ are pairwise independent, then $\mu(\limsup B_n) = 1$, i.e., almost every point $x \in \mathbb{M}$ belongs to infinitely many B_n.

The last lemma can be restated as follows:

Lemma 2. *(i) Suppose that the sum of all $\mu(A_n)$ converges, then for almost every point $x \in \mathbb{M}$, there are only finitely many values of $n \in \mathbb{N}$ such that $T^n x \in A_n$.*

(ii) In the case, $\sum \mu(B_n) = \infty$ and the events of the sequence $\{B_n,\ n \in \mathbb{N}\}$ are pairwise independent, then for almost every point $x \in \mathbb{M}$, there are infinitely many values of $n \in \mathbb{N}$ such that $T^n x \in A_n$.

Next we introduce several necessary definitions.

A sequence of subsets $A_n \subset \mathbb{M}$ is called a **Borel-Cantelli (BC) sequence** if for $\mu-$ almost every $x \in \mathbb{M}$ there are infinitely many values of $n \in \mathbb{N}$, such that, $T^n x \in A_n$.

Denote by $\chi_n(x)$ the indicator function of the set $B_n := T^{-n}A_n$. For every $N > 0$ we define the following two sums:

$$S_N(x) := \sum_{n=1}^{N} \chi_n(x), \quad E_N := \sum_{n=1}^{N} \mu(A_n).$$

Definition 2. A sequence $\{A_n \subset \mathbb{M},\ n \in \mathbb{N}\}$ is said to be a **strongly Borel-Cantelli (sBC) sequence** if for μ-almost every $x \in \mathbb{M}$ we have

$$\lim_{N\to\infty} \frac{S_N(x)}{E_N} = 1.$$

We define the quantities R_{mn} which characterizes the dependence of two events B_m and B_n:

$$R_{mn} := \mu(B_m \cap B_n) - \mu(B_m)\mu(B_n) = \mu(T^{-m}A_m \cap T^{-n}A_n) - \mu(A_m)\mu(A_n).$$

A sufficient condition for a sequence $\{A_n\}$ to be an sBC sequence, in terms of R_{mn}, was first found by W. Schmidt, and the proof was later provided by Sprindzuk [13] in the context of Diophantine approximations. This condition was recently adapted to dynamical systems by D. Kleinbock and G. Margulis [9].

We suppose that the numbers R_{mn} satifies the following condition:

$$\sum_{n=M}^{N} \sum_{m=M}^{N} |R_{mn}| \le C \cdot \sum_{n=M}^{N} \mu(A_n),$$

for some constant $C > 0$ and for all $N > M > 1$. The last condition is called $(SP)-$ **condition.**

Theorem 1 ([13], Chapter I, Lemma 10)**.** *If the sequence $\{A_n\}$ satisfies (SP), then it is an sBC sequence; moreover, for μ-almost every $x \in \mathbb{X}$ one has*

$$S_N = E_N + O(E_N^{1/2} \log^{3/2+\varepsilon} E_N). \tag{2}$$

2.2 Cylinder sets and geometry of supports

We now introduce cylinder sets in symbolic spaces. An alphabet $\mathcal{A}$ is a finite set of symbols; for example, $\mathcal{A} = \{a_1, a_2, \ldots, a_k\}$.

A cylinder $C \subset \{0,1\}^{\mathbb{Z}}$ is obtained by fixing symbols on a finite interval $L = [m,k] \subset \mathbb{Z}$, i.e., for some $\omega = \{\omega_m, \ldots, \omega_k\} \in \{0,1\}^L$, we set

$$C_L(\omega) = \{x \in \{0,1\}^{\mathbb{Z}} : x_i = \omega_i, i \in L\}. \tag{3}$$

Each cylinder is open and closed in $\{0,1\}^{\mathbb{Z}}$. We call m and k the left and right endpoints of an interval L, respectively, and $(m+k)/2$ the center of L. Following [4], we introduce the relevant geometry explicitly. If $L = [m,k] \subset \mathbb{Z}$ is a finite lattice interval and $D \in \mathbb{N}_0$, its *D-neighborhood* is

$$N_D(L) := [m-D, k+D] \cap \mathbb{Z}.$$

We say that an interval L_2 lies in the D-neighborhood of L_1 if $L_2 \subset N_D(L_1)$.

Definition 3 (see [4]). Two lattice intervals L_1 and L_2 are called ***D*-nested** if either $L_1 \subset N_D(L_2)$ or $L_2 \subset N_D(L_1)$.

Definition 4 (see [4]). For two lattice intervals $L_1 = [m_1, k_1]$ and $L_2 = [m_2, k_2]$, not necessarily disjoint, the **asymmetric distance from L_1 to L_2** is

$$\delta(L_1, L_2) := \min\{D \in \mathbb{N}_0 : L_2 \subset N_D(L_1)\}.$$

Equivalently,

$$\delta(L_1, L_2) = \max\{k_2 - k_1,\ m_1 - m_2,\ 0\}.$$

2.3 Stationary determinantal measures

Let $\mathbb{S}^1 = [0, 1)$ be the unit circle and let $f : \mathbb{S}^1 \to [0, 1]$ be a Borel function with Fourier coefficients

$$\widehat{f}(k) = \int_0^1 f(t) e^{-2\pi i k t}\, dt. \tag{4}$$

Define the convolution kernel

$$K_f(n, m) = \widehat{f}(n - m), \qquad n, m \in \mathbb{Z}.$$

Then K_f induces a stationary determinantal measure μ_f on $\{0, 1\}^{\mathbb{Z}}$ via

$$\mu_f\big(\{x \in \{0, 1\}^{\mathbb{Z}} \mid x_{n_1} = \cdots = x_{n_k} = 1\}\big) = \det\big(\widehat{f}(n_i - n_j)\big)_{1 \le i, j \le k}. \tag{5}$$

The process is shift invariant under $\sigma(x)_n = x_{n+1}$.

Suppose that a function f is given in the form (4). The $n \times n$ Toeplitz matrix generated by f is defined by $T_n(f) = \big(\widehat{f}(i - j)\big)_{1 \le i, j \le n}$. Equivalently, $T_n(f)$ has the explicit form

$$T_n(f) = \begin{pmatrix} \widehat{f}(0) & \widehat{f}(-1) & \widehat{f}(-2) & \cdots & \widehat{f}(-(n-1)) \\ \widehat{f}(1) & \widehat{f}(0) & \widehat{f}(-1) & \cdots & \widehat{f}(-(n-2)) \\ \widehat{f}(2) & \widehat{f}(1) & \widehat{f}(0) & \cdots & \widehat{f}(-(n-3)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widehat{f}(n-1) & \widehat{f}(n-2) & \widehat{f}(n-3) & \cdots & \widehat{f}(0) \end{pmatrix}.$$

We denote by $\mathbf{1} \in \mathbb{R}^n$ the vector whose all components are equal to 1.

For a $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$, we denote by $D_n(v)$ the diagonal matrix associated with v, defined by

$$D_n(v) := \operatorname{diag}(v) = \begin{pmatrix} v_1 & 0 & 0 & \cdots & 0 \\ 0 & v_2 & 0 & \cdots & 0 \\ 0 & 0 & v_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_n \end{pmatrix}.$$

Corollary 1 (see [5]). *Let $f : \mathbb{S}^1 \to [0,1]$ be an integrable function. For every $n \geq 1$, define the Toeplitz matrix $T_n(f) := \big(\widehat{f}(i-j)\big)_{1\leq i,j\leq n}$. Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n$, and $C[\varepsilon] := \{x \in \{0,1\}^{\mathbb{Z}} : x_1 = \varepsilon_1, \ldots, x_n = \varepsilon_n\}$ be a cylinder.*

Then the determinantal measure μ_f of $C[\varepsilon]$ can be written as

$$\mu_f(C[\varepsilon]) = \det\Big(D_n(2\varepsilon - \mathbf{1}) \cdot T_n(f) + D_n(\mathbf{1} - \varepsilon)\Big). \tag{6}$$

Let $\varepsilon = (\varepsilon_i)_{i=1}^n \in \{0,1\}^N$ and let $T_n(f) = (\widehat{f}(i-j))_{1\leq i,j\leq n}$ be the Toeplitz matrix generated by f. Then

$$\mu_f(C[\varepsilon]) = \det\Big(D_n(2\varepsilon - 1)\, T_n(f) + D_n(1-\varepsilon)\Big) = \det A_n(C[\varepsilon]),$$

where $A_n(C[\varepsilon])$ is the $n \times n$ matrix

$$A_n(C[\varepsilon]) = \begin{pmatrix} (2\varepsilon_1 - 1)\widehat{f}(0) + (1-\varepsilon_1) & \cdots & (2\varepsilon_1 - 1)\widehat{f}(1-n) \\ (2\varepsilon_2 - 1)\widehat{f}(1) & \cdots & (2\varepsilon_2 - 1)\widehat{f}(2-n) \\ \vdots & \ddots & \vdots \\ (2\varepsilon_n - 1)\widehat{f}(n-1) & \cdots & (2\varepsilon_n - 1)\widehat{f}(0) + (1-\varepsilon_n) \end{pmatrix}.$$

2.4 *The ψ-mixing coefficient*

Let $(\xi_n)_{n\in\mathbb{Z}}$ be the coordinate process on $\Omega = \{0,1\}^{\mathbb{Z}}$, that is, $\xi_n(x) = x_n$. For integers $n \leq m$, define

$$\mathcal{F}_n^m := \vee(\xi_n, \xi_{n+1}, \ldots, \xi_m),$$

the sigma-algebra generated by the coordinate maps $\xi_n, \ldots, \xi_m$. Similarly, set

$$\mathcal{F}_{-\infty}^n := \vee(\ldots, \xi_{n-1}, \xi_n), \qquad \mathcal{F}_n^{+\infty} := \vee(\xi_n, \xi_{n+1}, \ldots).$$

Then, for each $\ell \geq 1$, the ψ–*mixing coefficient* of μ_f is defined by

$$\psi_{\mu_f}(\ell) = \sup_{\substack{A\in\mathcal{F}_{-\infty}^0,\, B\in\mathcal{F}_\ell^{+\infty} \\ \mu_f(A)\mu_f(B)>0}} \left| \frac{\mu_f(A\cap B)}{\mu_f(A)\mu_f(B)} - 1 \right|.$$

We say that μ_f is ψ–*mixing* if

$$\lim_{\ell\to\infty} \psi_{\mu_f}(\ell) = 0.$$

Theorem 2 (Fan–Liao–Qiu, 2022 see [5]). *Let $f : S^1 \to [0,1]$ be an integrable function which is not identically* 0 *or* 1. *Then for every integer $\ell \geq 1$, the ψ–mixing coefficient of the stationary determinantal process μ_f satisfies*

$$\psi_{\mu_f}(\ell) \geq 1 - \exp\left(- \sum_{n=\ell+1}^{\infty} |n|\, |\widehat{f}(n)|^2 \right). \tag{7}$$

In particular, if μ_f is ψ–mixing, then $f \in H^{1/2}(\mathbb{T})$.

Conversely, if there exists a constant $\tau > 0$ such that

$$f \in H^{1/2}(\mathbb{T}) \qquad \textit{and} \qquad \tau \le f(t) \le 1-\tau \quad \textit{for all } t \in \mathbb{T}, \tag{8}$$

then μ_f is ψ–mixing and its ψ–function satisfies the upper bound

$$\psi_{\mu_f}(\ell) \le \frac{2}{\tau^2}\left(\sum_{n=\ell+1}^{\infty} |n|\,|\widehat{f}(n)|^2\right)\exp\left(1+\frac{2}{\tau^2}\sum_{n=\ell+1}^{\infty} |n|\,|\widehat{f}(n)|^2\right). \tag{9}$$

2.5 Main results

Lemma 3. *Let $A \in M_n$ and let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of A counted with their (algebraic) multiplicities (see [8, Definition 1.2.5]). Then*

$$\det(I+A) = \prod_{i=1}^{n}\big(1+\lambda_i\big).$$

Theorem 3. *Let A be a Hermitian matrix of order n, and let B be a principal submatrix of A of order $n-1$. If*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \qquad \textit{and} \qquad \mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1}$$

are the eigenvalues of A and B, respectively, then

$$\lambda_i \ge \mu_i \ge \lambda_{i+1}, \qquad i = 1, \ldots, n-1.$$

Lemma 4 (see [3, page 75])**.** *Let A be an $n \times n$ Hermitian matrix with eigenvalues*

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n,$$

and let B be a $k \times k$ principal submatrix of A with eigenvalues

$$\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k.$$

Then, for each $s = 1, \ldots, k$, one has

$$\lambda_{n-k+s} \;\le\; \mu_s \;\le\; \lambda_s.$$

We formulate the main results of our work. The first result gives a uniform exponential estimate for cylinder probabilities. It shows that, under the non-degeneracy condition $\tau \le f \le 1-\tau$, the measure of every n-cylinder decays exponentially in n. This estimate will be used repeatedly to control the size of cylinder targets.

Theorem 4. *Let $f : \mathbb{S}^1 \to [0,1]$ be an integrable Borel function, and let μ_f be the associated stationary determinantal probability measure on $\{0,1\}^{\mathbb{Z}}$. Assume that there exists a constant $\tau \in (0, 1/2)$ such that*

$$\tau \le f \le 1-\tau \qquad \textit{almost everywhere on } \mathbb{S}^1 \textit{ with respect to the Lebesgue measure.}$$

For every cylinder $C[\varepsilon] := C[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$, $\quad \varepsilon \in \{0,1\}^n$, the following bounds hold:

$$\tau^n \;\le\; \mu_f\big(C[\varepsilon]\big) \;\le\; (1-\tau)^n.$$

The next result is a conditional version of the previous cylinder estimate. It says that fixing additional coordinates decreases the measure at an exponential rate depending only on the number of newly fixed coordinates. This is the determinantal analogue of the finite-energy property for Gibbs measures.

Theorem 5. *Let $f : \mathbb{S}^1 \to [0,1]$ be an integrable Borel function. Assume that there exists a constant $\tau \in (0,1/2)$ such that*

$$\tau \le f \le 1-\tau \qquad \textit{almost everywhere on } \mathbb{S}^1.$$

Let $\mu = \mu_f$ be the stationary determinantal probability measure associated with f. Then for any pair of cylinders $C_1(\omega) \subset C(\omega)$ supported on finite intervals $L_1 \supset L$, one has

$$\tau^{\,|L_1|-|L|} \;\le\; \frac{\mu_f(C_1(\omega))}{\mu_f(C(\omega))} \;\le\; (1-\tau)^{\,|L_1|-|L|}.$$

We now state the main dynamical Borel–Cantelli result. The geometric assumption on the supports is the D-nested condition, while the probabilistic input is the quantitative mixing of the stationary determinantal measure. Together they imply the Sprindzuk condition and hence the strong Borel–Cantelli property.

Theorem 6. *Let $f : S^1 \to [0,1]$ be a Borel function, and let $\langle \{0,1\}^{\mathbb{Z}}, \mathcal{B}, \mu_f, \sigma\rangle$ be the dynamical system generated by the stationary determinantal point process associated with f, where μ_f is the σ-invariant determinantal probability measure induced by the convolution kernel*

$$K_f(n,m) = \widehat{f}(n-m), \qquad n,m \in \mathbb{Z},$$

and σ is the shift map.

Assume that there exist constants $C, a > 0$ and $\tau \in (0,1/2)$ such that

$$|\widehat{f}(n)| \le Ce^{-a|n|} \quad (n \in \mathbb{Z}), \qquad \tau \le f(t) \le 1-\tau \quad \textit{for a.e. } t \in \mathbb{T}.$$

Suppose that there exists a fixed constant $D \in \mathbb{N}_0$ such that the cylinders $C_{L_n}(\omega)$, $n \ge 1$, are supported on finite intervals

$$L_n = [m^{(n)}, k^{(n)}] \subset \mathbb{Z}$$

and every pair L_m, L_n is D-nested in the sense of Definition 3. Then,

1. *the sequence of cylinders $\{C_{L_n}(\omega)\}_{n\ge 1}$ satisfies* (SP)*;*
2. *if $\sum_{n=1}^{\infty} \mu_f\big(C_{L_n}(\omega)\big) = \infty$, $\{C_{L_n}(\omega)\}_{n\ge 1}$ is an sBC sequence.*

3 Proof of Theorems 4 and 5

In this section we prove the cylinder estimates stated in the previous section. The first estimate gives a uniform exponential upper and lower bound for the measure of every finite cylinder. The second estimate is a finite-energy type bound, controlling the change of measure when a cylinder is refined by fixing additional coordinates. These estimates will be used in the proof of the Sprindzuk condition for D-nested cylinder targets.

3.1 Uniform bounds for cylinder probabilities

We first prove the uniform cylinder estimate. The key point is that the Shirai–Takahashi formula allows us to express each cylinder probability as a finite Toeplitz determinant. The non-degeneracy condition $\tau \le f \le 1-\tau$ then gives a uniform bound on the corresponding finite-dimensional operator.

Proof. Fix $n \ge 1$ and a binary word

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{0,1\}^n.$$

Consider the corresponding cylinder set

$$C[\varepsilon] := \{x \in \{0,1\}^{\mathbb{Z}} : x_1 = \varepsilon_1, \dots, x_n = \varepsilon_n\}.$$

By shift-invariance of the measure μ_f, it suffices to consider cylinders supported on the index set $\{1, \dots, n\}$.

Let

$$T_n(f) := \big(\widehat{f}(i-j)\big)_{1 \le i,j \le n}$$

be the Toeplitz matrix generated by f. For a vector $v \in \mathbb{C}^n$, denote by $D_n(v)$ the diagonal matrix with diagonal entries $v_1, \dots, v_n$.

Using the determinantal representation of cylinder measures, we have

$$\mu_f(C[\varepsilon]) = \det\Big(D_n(2\varepsilon - \mathbf{1})\, T_n(f) + D_n(\mathbf{1} - \varepsilon)\Big), \tag{10}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$.

We first rewrite the matrix in (10) into a more convenient form. Set

$$\eta := 2\varepsilon - \mathbf{1} \in \{-1,1\}^n, \qquad g := 2f - 1.$$

Then the following identity holds:

$$D_n(2\varepsilon - \mathbf{1})\, T_n(f) + D_n(\mathbf{1} - \varepsilon) = \frac{1}{2}\Big(I_n + D_n(\eta)\, T_n(g)\Big), \tag{11}$$

where I_n denotes the $n \times n$ identity matrix.

Indeed, since

$$T_n(f) = \frac{1}{2}\big(T_n(g) + I_n\big),$$

and

$$1 - \varepsilon_i = \frac{1}{2}(1 - \eta_i), \qquad i = 1, \dots, n,$$

we obtain

$$D_n(\mathbf{1} - \varepsilon) = \frac{1}{2}\big(I_n - D_n(\eta)\big), \qquad D_n(2\varepsilon - \mathbf{1}) = D_n(\eta).$$

Substituting these expressions into the left-hand side of (11), we compute

$$\begin{aligned} D_n(2\varepsilon - \mathbf{1})\, T_n(f) + D_n(\mathbf{1} - \varepsilon) &= D_n(\eta) \cdot \frac{1}{2}\big(T_n(g) + I_n\big) + \frac{1}{2}\big(I_n - D_n(\eta)\big) \\ &= \frac{1}{2}\Big(D_n(\eta)\, T_n(g) + I_n\Big), \end{aligned}$$

which proves (11).

Combining (10) and (11), we obtain

$$\mu_f(C[\varepsilon]) = \det\Big(2^{-1}(I_n + D_n(\eta)\, T_n(g))\Big) = 2^{-n} \det(I_n + A), \tag{12}$$

where

$$A := D_n(\eta)\, T_n(g).$$

We now estimate $\det(I_n + A)$. Since $D_n(\eta)$ is diagonal with entries ± 1, we have

$$\|D_n(\eta)\| = 1.$$

Therefore,

$$\|A\| = \|D_n(\eta)\, T_n(g)\| = \|T_n(g)\|. \tag{13}$$

The matrix $T_n(g)$ is a finite compression of the bi-infinite Toeplitz operator $T(g)$ acting on $\ell^2(\mathbb{Z})$, hence

$$\|T_n(g)\| \leq \|T(g)\|.$$

Moreover, $T(g)$ is unitarily equivalent to the multiplication operator by g on $L^2(\mathbb{T})$ (see, for example, [2], p. 179, Corollary 7.8), which yields

$$\|T(g)\| = \|g\|_\infty.$$

Using the assumption $\tau \leq f \leq 1 - \tau$ almost everywhere, we obtain

$$\|g\|_\infty = \|2f - 1\|_\infty \leq 1 - 2\tau < 1.$$

Consequently,

$$\|A\| \leq 1 - 2\tau < 1. \tag{14}$$

Let $\lambda_1(A), \ldots, \lambda_n(A)$ denote the eigenvalues of A. Since the spectral radius satisfies $\rho(A) \leq \|A\|$, it follows from (14) that

$$|\lambda_i(A)| \leq 1 - 2\tau, \qquad i = 1, \ldots, n.$$

The eigenvalues of $I_n + A$ are $1 + \lambda_i(A)$, and hence

$$\det(I_n + A) = \prod_{i=1}^{n} \big(1 + \lambda_i(A)\big).$$

By the triangle inequality,

$$1 - |\lambda_i(A)| \leq |1 + \lambda_i(A)| \leq 1 + |\lambda_i(A)|, \qquad i = 1, \ldots, n.$$

Combining these inequalities, we obtain

$$2^n\, \tau^n \leq |\det(I_n + A)| \leq 2^n\, (1 - \tau)^n.$$

By (12), we have

$$\tau^n \leq \mu_f(C[\varepsilon]) \leq (1 - \tau)^n.$$

This completes the proof of Theorem 4. □

3.2 Conditional cylinder estimates

We next prove a conditional version of the previous estimate. This result shows that the measure of a cylinder decreases exponentially when additional coordinates are fixed. Such a bound is needed later when two shifted cylinders overlap, since their intersection is again a cylinder supported on the union of the two intervals.

Proof. Let

$$\Lambda = \varepsilon_1 \cdots \varepsilon_m \in \{0,1\}^m.$$

We present the proof in the two pure cases, namely when the defining word consists only of ones or only of zeros. In these cases the matrices M_Λ and M_{Λ_1} are Hermitian positive definite, and the proof follows directly from the Cauchy interlacing theorem.

Define

$$M_\Lambda := D_m(2\varepsilon - 1)\, T_m(f) + D_m(1-\varepsilon),$$

where $T_m(f)$ is the $m \times m$ Toeplitz matrix generated by f. By the Shirai–Takahashi formula for stationary determinantal processes,

$$\mu_f([\Lambda]) = \det(M_\Lambda).$$

Let Λ_1 be an extension of Λ:

$$\Lambda_1 = (\varepsilon_1^{(1)}, \ldots, \varepsilon_{m+k}^{(1)}) = \underbrace{(\varepsilon_1^l, \ldots, \varepsilon_{k_1}^l)}_{\Lambda_l}\ \underbrace{(\varepsilon_1, \ldots, \varepsilon_m)}_{\Lambda}\ \underbrace{(\varepsilon_1^r, \ldots, \varepsilon_{k_2}^r)}_{\Lambda_r},$$

where $k_1 + k_2 = k$. Set

$$A := D_{k_1}(2\varepsilon^l - 1), \quad B := D_m(2\varepsilon - 1), \quad C := D_{k_2}(2\varepsilon^r - 1),$$

and

$$E_l := D_{k_1}(1-\varepsilon^l), \quad E := D_m(1-\varepsilon), \quad E_r := D_{k_2}(1-\varepsilon^r).$$

Then the matrix M_{Λ_1} has the block form

$$M_{\Lambda_1} = \begin{pmatrix} AT_{ll} + E_l & AT_l & AT_{lr} \\ BT_{ml} & BT + E & BT_{mr} \\ CT_{rl} & CT_r & CT_{rr} + E_r \end{pmatrix},$$

with

$$BT + E = M_\Lambda.$$

Consequently,

$$\mu_f([\Lambda_1]) = \det(M_{\Lambda_1}),$$

and therefore

$$\frac{\mu_f(C_1)}{\mu_f(C)} = \frac{\det(M_{\Lambda_1})}{\det(M_\Lambda)}.$$

By Theorem 4,

$$\tau^{m+k} \le \det(M_{\Lambda_1}) \le (1-\tau)^{m+k}.$$

Moreover,

$$\tau \le \lambda_1^{(\Lambda_1)} \le \cdots \le \lambda_{m+k}^{(\Lambda_1)} \le 1 - \tau. \tag{15}$$

Since M_Λ is a principal submatrix of M_{Λ_1} and both matrices are Hermitian, the Cauchy interlacing theorem (Lemma 4) yields

$$\lambda_i^{(\Lambda_1)} \;\le\; \lambda_i^{(\Lambda)} \;\le\; \lambda_{i+k}^{(\Lambda_1)}, \qquad i = 1, \dots, m.$$

Using the product representation of determinants,

$$\frac{\mu_f(C_1)}{\mu_f(C)} = \frac{\prod_{i=1}^{m+k} \lambda_i^{(\Lambda_1)}}{\prod_{i=1}^{m} \lambda_i^{(\Lambda)}} \ge \prod_{i=1}^{k} \lambda_i^{(\Lambda_1)} \ge \tau^k.$$

Similarly,

$$\frac{\mu_f(C_1)}{\mu_f(C)} \le \prod_{i=m+1}^{m+k} \lambda_i^{(\Lambda_1)} \le (1-\tau)^k.$$

Since $k = |\Lambda_1| - |\Lambda|$, we obtain

$$\tau^{|\Lambda_1|-|\Lambda|} \;\le\; \frac{\mu_f(C_1)}{\mu_f(C)} \;\le\; (1-\tau)^{|\Lambda_1|-|\Lambda|}.$$

It remains to consider the mixed case, when both symbols 0 and 1 occur in the word ε. In this case the matrix

$$M_\Lambda = D_\Lambda(2\varepsilon - \mathbf{1})T_\Lambda(f) + D_\Lambda(\mathbf{1} - \varepsilon)$$

is not Hermitian in general. Put

$$S_\Lambda := D_\Lambda(2\varepsilon - \mathbf{1}).$$

Then

$$H_\Lambda := S_\Lambda M_\Lambda = T_\Lambda(f) - D_\Lambda(\mathbf{1} - \varepsilon)$$

is Hermitian. Similarly,

$$H_{\Lambda_1} := S_{\Lambda_1} M_{\Lambda_1} = T_{\Lambda_1}(f) - D_{\Lambda_1}(\mathbf{1} - \varepsilon_1)$$

is Hermitian. Since $C_{\Lambda_1}(\varepsilon_1) \subset C_\Lambda(\varepsilon)$, the matrix H_Λ is a principal submatrix of H_{Λ_1}. Hence the Cauchy interlacing theorem applies. Moreover,

$$\det M_\Lambda = \det S_\Lambda \ \det H_\Lambda, \qquad \det M_{\Lambda_1} = \det S_{\Lambda_1} \ \det H_{\Lambda_1}.$$

Thus the signs coming from the zero symbols are completely accounted for by $\det S_\Lambda$ and $\det S_{\Lambda_1}$. Therefore, applying the same eigenvalue-ratio argument as above together with the Cauchy interlacing theorem, we obtain

$$\tau^{|\Lambda_1|-|\Lambda|} \le \frac{\mu_f(C_{\Lambda_1}(\varepsilon_1))}{\mu_f(C_\Lambda(\varepsilon))} \le (1-\tau)^{|\Lambda_1|-|\Lambda|}.$$

This completes the proof of Theorem 5. □

4 Mixing estimates and proof of Main Theorem 6

In this section we prove the main dynamical Borel–Cantelli result. The proof has two steps. First, using the quantitative ψ-mixing estimate for stationary determinantal processes, we derive an exponential correlation bound for separated cylinder events. Second, this bound is combined with the geometric D-nested condition on the supports to verify the Schmidt–Sprindzuk condition.

Theorem 7 (Exponential cylinder mixing bound)**.** *Let $f : \mathbb{S}^1 \to [0,1]$ satisfy the following assumptions:*

(A1) *there exist constants $C, a > 0$ such that*

$$|\widehat{f}(n)| \le Ce^{-a|n|} \qquad (n \in \mathbb{Z});$$

(A2) *there exists $\tau \in (0, 1/2)$ such that*

$$\tau \le f(t) \le 1 - \tau \qquad (t \in S^1).$$

Then there exist constants $c_3 > 0$ and $\theta \in (0,1)$ (depending only on a, C, τ) such that for any cylinder sets C_1 and C_2 whose supports are separated by a gap $d := n_2^- - n_1^+ \in \mathbb{N}$, one has

$$\Big|\mu_f(C_1 \cap C_2) - \mu_f(C_1)\mu_f(C_2)\Big| \;\le\; c_3\,\theta^d\,\mu_f(C_1)\mu_f(C_2). \tag{16}$$

Proof. Let $\psi(\ell)$ denote the ψ–mixing coefficient of μ_f at separation ℓ (as in Fan–Liao–Qiu, Def. of ψ–mixing). By the definition of $\psi(\ell)$, for any $A \in \mathcal{F}_{-\infty}^0$ and $B \in \mathcal{F}_\ell^{+\infty}$ with $\mu_f(A)\mu_f(B) > 0$,

$$\Big|\mu_f(A \cap B) - \mu_f(A)\mu_f(B)\Big| \;\le\; \psi(\ell)\,\mu_f(A)\mu_f(B). \tag{17}$$

If C_1 and C_2 are cylinder sets supported on disjoint coordinate blocks separated by $d := n_2^- - n_1^+ \ge 1$, then $C_1 \in \mathcal{F}_{-\infty}^0$ and $C_2 \in \mathcal{F}_d^{+\infty}$. Thus (17) with $\ell = d$ gives

$$\Big|\mu_f(C_1 \cap C_2) - \mu_f(C_1)\mu_f(C_2)\Big| \;\le\; \psi(d)\,\mu_f(C_1)\mu_f(C_2). \tag{18}$$

Under (A2), Theorem 2 provides, for every $\ell \ge 1$,

$$\psi(\ell) \;\le\; \frac{2}{\tau^2}\,S(\ell)\,\exp\!\left(1 + \frac{2}{\tau^2}S(\ell)\right), \qquad S(\ell) := \sum_{n=\ell+1}^{\infty} n\,|\widehat{\{f\}}(n)|^2. \tag{19}$$

Assumption (A1) gives $|\widehat{\{f\}}(n)| \le Ce^{-an}$ for $n \ge 0$. Let $r := e^{-2a} \in (0,1)$. Then

$$S(\ell) \;\le\; C^2 \sum_{n=\ell+1}^{\infty} nr^n \;=\; C^2\,\frac{r^{\ell+1}\big((\ell+1) - \ell r\big)}{(1-r)^2} \;\le\; K_1\,(\ell+1)\,e^{-2a(\ell+1)},$$

with $K_1 := C^2(1 - e^{-2a})^{-2}$. Substituting this in (19) we obtain

$$\psi(\ell) \;\le\; K_2\,(\ell+1)\,e^{-2a(\ell+1)}, \tag{20}$$

for some $K_2 = K_2(a, C, \tau) > 0$ (since $S(\ell) \to 0$, the exponential factor in (19) stays bounded).

Fix any $c \in (0, 2a)$ and set $\theta := e^{-c} \in (0, 1)$. Using the standard bound $\sup_{x>0} xe^{-\gamma x} = 1/(e\gamma)$ for $\gamma > 0$, we have

$$
\begin{aligned}
(\ell+1)e^{-2a(\ell+1)} &= e^{-c(\ell+1)} \cdot (\ell+1)e^{-(2a-c)(\ell+1)} \leq \\
&\leq \frac{1}{e(2a-c)}\, e^{-c(\ell+1)} = \frac{1}{e(2a-c)}\, \theta^{\ell+1}.
\end{aligned}
$$

Hence, from (20),

$$
\psi(\ell) \;\leq\; \frac{K_2}{e(2a-c)}\, \theta^{\ell+1} \;\leq\; K_3\, \theta^{\ell},
$$

for a constant $K_3 = K_3(a, C, \tau, c) > 0$ (absorbing the extra factor θ into K_3). Putting $\ell = d$ in (18) yields

$$
\left|\mu_f(C_1 \cap C_2) - \mu_f(C_1)\mu_f(C_2)\right| \;\leq\; K_3\, \theta^{d}\, \mu_f(C_1)\mu_f(C_2).
$$

Finally, enlarging the constant if necessary to cover finitely many small d, we obtain (16) with $c_3 := K_3$. □

Lemma 5. *Assume that f satisfies assumptions* (A1)–(A2) *of Theorem 7. Let C_1 and C_2 be cylinders supported on intervals L_1 and L_2, respectively. Then there exist constants $c_4 > 0$ and $\theta_4 \in (0, 1)$ such that*

$$
\left|\mu_f(C_1 \cap C_2) - \mu_f(C_1)\mu_f(C_2)\right| \leq c_4 \theta_4^{\delta(L_1, L_2)} \mu_f(C_1).
$$

Proof. There are two cases depending on the position of L_1 and L_2: disjoint and intersecting.

Put

$$
\theta_4 := \max\{\theta, 1-\tau\} \in (0, 1), \qquad c_4 := \max\{c_3, 2\}.
$$

1) Let L_1 and L_2 be disjoint. First assume that $k_1 < m_2$. Applying Theorem 7, we have

$$
|\mu\,(C_1 \cap C_2) - \mu\,(C_1)\,\mu\,(C_2)| \leq c_3 \theta^{m_2 - k_1} \mu\,(C_1)\,\mu\,(C_2)\,.
$$

By Theorem 4,

$$
\mu(C_2) \leq (1-\tau)^{|L_2|} = (1-\tau)^{k_2 - m_2 + 1}.
$$

Therefore,

$$
\begin{aligned}
|\mu\,(C_1 \cap C_2) - \mu\,(C_1)\,\mu\,(C_2)| &\leq c_3 \theta^{m_2 - k_1} (1-\tau)^{k_2 - m_2 + 1} \mu\,(C_1) \\
&\leq c_4 \theta_4^{k_2 - k_1} \mu\,(C_1) \\
&= c_4 \theta_4^{\delta(L_1, L_2)} \mu\,(C_1)\,.
\end{aligned}
$$

Here we used the fact that, for $k_1 < m_2$,

$$
\delta(L_1, L_2) = k_2 - k_1.
$$

The case $k_2 < m_1$ is treated similarly. In this case,

$$
\delta(L_1, L_2) = m_1 - m_2,
$$

and the same argument gives

$$|\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| \le c_4 \theta_4^{\delta(L_1,L_2)} \mu(C_1)\,.$$

2) Let L_1 and L_2 be intersecting. If $C_1 \cap C_2 = \emptyset$, then

$$|\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| = \mu(C_1)\mu(C_2).$$

Using Theorem 4, we obtain

$$\mu(C_2) \le (1-\tau)^{|L_2|}.$$

Since $L_1 \cap L_2 \ne \emptyset$, we have

$$\delta(L_1, L_2) \le |L_2|.$$

Hence

$$|\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| \le c_4 \theta_4^{\delta(L_1,L_2)} \mu(C_1).$$

Now suppose that $C_1 \cap C_2 \ne \emptyset$. Then $C_1 \cap C_2$ is a cylinder supported on $L_1 \cup L_2$, and

$$C_1 \cap C_2 \subset C_1.$$

Applying Theorem 5, we have

$$\mu(C_1 \cap C_2) \le (1-\tau)^{|L_1 \cup L_2| - |L_1|} \mu(C_1).$$

Since

$$|L_1 \cup L_2| - |L_1| \ge \delta(L_1, L_2),$$

we get

$$\mu(C_1 \cap C_2) \le (1-\tau)^{\delta(L_1,L_2)} \mu(C_1).$$

On the other hand, by Theorem 4,

$$\mu(C_1)\mu(C_2) \le (1-\tau)^{|L_2|} \mu(C_1) \le \theta_4^{\delta(L_1,L_2)} \mu(C_1).$$

Therefore,

$$\begin{aligned} |\mu(C_1 \cap C_2) - \mu(C_1)\mu(C_2)| &\le \mu(C_1 \cap C_2) + \mu(C_1)\mu(C_2) \\ &\le c_4 \theta_4^{\delta(L_1,L_2)} \mu(C_1). \end{aligned}$$

The proof is complete. □

Proof of Theorem 6. Suppose that the intervals $L_n = [m_n, k_n]$, $n \in \mathbb{N}$, satisfy the D-nested condition. It is important to note that no assumptions are made regarding the relationship between n and m, nor between the measures of $C_n(\omega)$ and $C_m(\omega)$.

We denote by $L_n - (m-n)$ the segment formed by shifting L_n to $(m-n)$ to the left, i.e.

$$L_n - (m-n) = [m_n - (m-n),\ k_n - (m-n)].$$

Without loss of generality, assume that L_m lies in the D-neighborhood of L_n, that is, $L_m \subset N_D(L_n)$. The opposite case is obtained by interchanging m and n. We estimate the δ-asymmetric distance between L_m and $L_n - (m-n)$. By the D-nested property, we have

$$\delta\left(L_m, L_n - (m-n)\right) \geq |m-n| - D.$$

Consider the quantity

$$\begin{aligned} R_{mn} &= \mu(\sigma^{-m}C_m(\omega) \cap \sigma^{-n}C_n(\omega)) - \mu(\sigma^{-n}C_n(\omega))\mu(\sigma^{-m}C_m(\omega)) \\ &= \mu(C_m(\omega) \cap \sigma^{m-n}C_n(\omega)) - \mu(C_n(\omega))\mu(C_m(\omega)). \end{aligned}$$

By applying Lemma 5 to the cylinders $C_m(\omega)$ and $\sigma^{m-n}C_n(\omega)$, we obtain

$$|R_{mn}| \leq c_4\theta_4^{\delta(L_m, L_n-(m-n))}\mu(C_m(\omega)).$$

Hence

$$|R_{mn}| \leq c_4\theta_4^{|m-n|-D}\mu(C_m(\omega)).$$

Since D is fixed, replacing c_4 by a larger constant if necessary, we get

$$|R_{mn}| \leq c_4\theta_4^{|m-n|}\mu(C_m(\omega)).$$

We first sum over the pairs satisfying $L_m \subset N_D(L_n)$. Since $R_{mn} = R_{nm}$, the remaining pairs are treated by interchanging m and n. Absorbing the resulting factor 2 into c_4, for $N > M$ we have

$$\begin{aligned} \sum_{m=M}^{N}\sum_{n=M}^{N}|R_{mn}| &\leq c_4\sum_{m=M}^{N}\mu(C_m(\omega))\sum_{n=M}^{N}\theta_4^{|m-n|} \\ &\leq c_4\sum_{m=M}^{N}\mu(C_m(\omega))\sum_{j=-\infty}^{\infty}\theta_4^{|j|}. \end{aligned}$$

Since

$$\sum_{j=-\infty}^{\infty}\theta_4^{|j|} = 1 + 2\sum_{j=1}^{\infty}\theta_4^{j} = \frac{1+\theta_4}{1-\theta_4} < \infty,$$

we obtain, again increasing c_4 if necessary,

$$\sum_{m=M}^{N}\sum_{n=M}^{N}|R_{mn}| \leq c_4\sum_{m=M}^{N}\mu(C_m(\omega)).$$

Thus the condition (SP) is satisfied. Therefore, if

$$\sum_{n=1}^{\infty}\mu(C_n(\omega)) = \infty,$$

then the sequence $\{C_n(\omega)\}_{n\geq 1}$ is strongly Borel–Cantelli. Theorem 6 is proved. □

Example 1. Let $r \in (0,1)$ and choose $\varepsilon \in \mathbb{R}$ such that

$$0 < |\varepsilon| < \frac{1-r}{2r}.$$

Define

$$f(t) := \frac{1}{2} + \varepsilon \sum_{k=1}^{\infty} r^k \cos(2\pi k t), \qquad t \in \mathbb{T}.$$

Then f satisfies the assumptions of the Theorem 7.

Indeed, since

$$\left| \sum_{k=1}^{\infty} r^k \cos(2\pi k t) \right| \le \sum_{k=1}^{\infty} r^k = \frac{r}{1-r},$$

we have

$$\frac{1}{2} - \frac{|\varepsilon| r}{1-r} \le f(t) \le \frac{1}{2} + \frac{|\varepsilon| r}{1-r}.$$

Hence, for any

$$\tau \in \left(0, \frac{1}{2} - \frac{|\varepsilon| r}{1-r}\right],$$

one gets

$$\tau \le f(t) \le 1 - \tau \qquad (t \in \mathbb{T}).$$

Also,

$$\widehat{f}(0) = \frac{1}{2}, \qquad \widehat{f}(\pm k) = \frac{\varepsilon}{2} r^k \quad (k \ge 1).$$

If we set

$$a := -\log r > 0, \qquad C := \max\left\{\frac{1}{2}, \frac{|\varepsilon|}{2}\right\},$$

then

$$|\widehat{f}(n)| \le C e^{-a|n|} \qquad (n \in \mathbb{Z}).$$

Thus assumptions (A1) and (A2) are satisfied.

Therefore, by the theorem, there exist constants $c_3 > 0$ and $\theta \in (0, 1)$ such that for any cylinder sets C_1, C_2 whose supports are separated by a gap $d \in \mathbb{N}$,

$$\left|\mu_f(C_1 \cap C_2) - \mu_f(C_1)\mu_f(C_2)\right| \le c_3\, \theta^d\, \mu_f(C_1)\mu_f(C_2).$$

In particular, the corresponding ψ–mixing upper bound decays exponentially.

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