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Multiscale asymptotic analysis of kernel-smoothed solutions to fractional Riesz-Bessel equations with random initial conditions

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The paper is dedicated to the 75th anniversary of Professor Nikolai Leonenko.

Abstract This paper investigates fractional Riesz–Bessel equations with random initial conditions that exhibit either classical or cyclic long-range dependence. It studies zoom-in asymptotics for the corresponding solutions and establishes multiscaling limit theorems. It is known that for similar problems, non-degenerate multiscaling limits may not exist in general. The paper develops a kernel-smoothing approach for these equations and obtains non-degenerate limit fields under suitable normalisation and rescaling. It proves that the kernel-smoothed solutions converge weakly to Gaussian random fields, which are non-stationary in both time and space. Their stochastic integral representations and covariance functions are derived. The paper also analyses the regularity and dependence structure of the limit fields. In particular, under appropriate general assumptions on the smoothing kernel, the limits exhibit long-range dependence in time and short-range dependence in space. Numerical examples for the case of Matérn-type kernels are provided to illustrate the theoretical results.

Keywords Fractional Riesz-Bessel equations, random partial differential equations, cyclic long memory, spectral singularities, multiscaling limit theorems, Matérn kernel
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1 Introduction

Fractional Riesz-Bessel equations extend classical kinetic models by using a composite fractional operator that combines the non-local behaviour of the fractional Laplacian with the regularising effect of the Bessel potential operator. Such equations are widely used in the study of transport in disordered media, multifractal dynamics in financial systems, and high-frequency turbulence, see [24, 37]. The work [19] initiated the development of this area, while [38] and [42] established detailed mathematical frameworks for fractional operators and integrals that are widely used today. Earlier publications considered these equations with random initial conditions given by Gaussian random fields [8, 9]. For such models, the asymptotic behaviour of the solution is determined not only by the differential operators but also by the spectral structure of the initial condition.

The first probabilistic approaches to the heat equation with random initial conditions were introduced by De Fériet and Rosenblatt [21, 44]. In recent years, there has been considerable interest in equations of this type with general random initial conditions. Several publications have demonstrated how the structure of these initial conditions influences the resulting solutions and their properties. The dependence structure of the initial random field is typically characterised by its spectral density: singularities at zero frequency correspond to classical long-range dependence, whereas singularities at nonzero frequencies are associated with cyclic long-memory behaviour [5]. Later work extended this analysis to broad classes of random environments and non-homogeneous settings [15, 47]. For classical diffusion equations, scaling limits of random solutions were derived in [2, 30], while related methods were applied to Burgers-type equations in [27, 29]. These studies established general principles for investigating asymptotics of rescaled random solution fields.

More recent research has focused on fractional operators and long-range dependence. Gay and Heyde [23] showed that fractional Laplace operators generate random fields with long-range dependence. Further studies of stochastic heat equations with fractional Laplace operators were carried out in [7, 13]. The results by Anh and Leonenko [8] proved that when the initial spectrum has a singularity at the origin, suitably normalised solutions converge to non-Gaussian limits. This established a direct link between fractional operators, singular spectral densities, and limit theorems. The analysis was further developed by renormalisation and homogenization methods for fractional-in-time/space diffusion equations with random input [9, 11]. These results belong to the non-central limit theory for strongly dependent random fields, see [22, 46]. Related asymptotic problems have been studied for Airy and Korteweg–de Vries equations [16, 25]. Stochastic models on spheres and hyperbolic diffusion equations were analysed in [14, 18]. Further results for diffusion in expanding space–time were recently obtained in [17].

Contrary to the classical increasing domain asymptotic settings, multiscaling refers to asymptotic zoom-in regimes in which the normalisation and the space–time rescaling are used to study local behaviour. For stochastic fractional diffusion equations, these settings lead to multiscaling limit theorems for their solutions, as well as the classification of the limit behaviour depending on the initial conditions [4]. The analysis of fractional Riesz–Bessel equations with cyclic long-memory initial conditions in [32] demonstrated that such rescaled solutions converge to Gaussian random fields. Related scaling and homogenization questions have also been studied for time-fractional relativistic diffusion equations and for coupled parabolic and reaction–diffusion–wave systems [33–35]. These works demonstrated how the limit behaviour depends on the fractional operator and the initial spectrum.

Kernel smoothing is often introduced to obtain improved behaviour and non-degenerate limits by controlling high-frequency contributions [4]. In numerous applications, this approach functions as a filter applied to realisations of random processes or fields [6, 12]. For example, the work of [50] considered kernel smoothing methods for stochastic partial differential equations, where solutions to the Fokker–Planck equation were approximated using adaptive Gaussian kernels. The approach was combined with stochastic filtering to estimate probability distributions from noisy observations. This established a direct link between kernel smoothing, types of considered equations and properties of the limits.

The existing literature (see [3–5] and the references therein) provides multiscaling theorems for several classes of equations with long-range and cyclically dependent initial conditions, including fractional and higher-order equations. However, the results in [3, 4] demonstrate that multiscaling limits do not exist in all cases and suggest using kernel smoothing to obtain non-degenerate asymptotic behaviour. To the best of our knowledge, no existing work has considered fractional Riesz–Bessel equations in conjunction with kernel smoothing and multiscale asymptotic analysis under general spectral settings. This study considers both classical and cyclic long-memory initial conditions, extending earlier results on the scaling limits of the corresponding solution fields. It also investigates the smoothness properties and dependence structures of these limit fields.

The paper is organised as follows. Section 2 provides the main definitions, notations, and some basic assumptions used throughout the paper. Section 3 derives limit theorems for filtered random fields with random initial conditions, for both classical and cyclic long-range dependence cases. It also presents numerical examples with realisations of the limit fields and their covariance structures. Hölder continuity and long-range dependence properties of the obtained multiscaling limit fields are studied in Section 4. Section 5 concludes the paper with a brief discussion and directions for future research.

All numerical computations and plotting in this paper were performed using the software R (version 4.6.0). The corresponding R code is freely available in the folder “Research materials” from the website https://sites.google.com/site/olenkoandriy/.

2 Definitions and notations

This section provides the main notations and background material required in the following sections.

Standard notations such as $\overline{c}$ for the complex conjugate of c, $\mathcal{B}(\mathbb{R})$ for the σ-field of Borel sets on $\mathbb{R}$, $\Gamma(\cdot)$ for the Gamma function, and $\mathbf{1}_A(\cdot)$ for the indicator function of a set A are used throughout. The symbol C denotes constants, which exact values are not important for this exposition and may vary, even within the same proof.

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R})$ consists of all functions $f \in L^2(\mathbb{R})$ such that $\|f\|^2_{H^s} := \int_{\mathbb{R}} (1 + |\lambda|^2)^s |\widehat{f}(\lambda)|^2 \, d\lambda < +\infty$.

Let $u(t,x)$ be a real-valued function of the two arguments $x \in \mathbb{R}^d$ and $t > 0$. The time derivative of order $\beta \in (0,1]$ is defined by

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} := \begin{cases} \dfrac{\partial u}{\partial t}(t,x), & \text{if } \beta = 1, \\ \left(\mathcal{D}_t^\beta u\right)(t,x), & \text{if } \beta \in (0,1), \end{cases}$$

where

$$\left(\mathcal{D}_t^\beta u\right)(t,x) := \frac{1}{\Gamma(1-\beta)} \left[\frac{\partial}{\partial t} \int_0^t (t-\tau)^{-\beta} u(\tau,x)\, d\tau - \frac{u(0,x)}{t^\beta}\right], \qquad t > 0,$$

is the fractional derivative in the Caputo–Djrbashian sense [42, (2.138)].

The publications [3, 8] and [9] studied fractional Riesz–Bessel equations (FRBE)

$$\frac{\partial^\beta u(t,x)}{\partial t^\beta} = -\mu(I-\Delta)^{\gamma/2}(-\Delta)^{\alpha/2} u(t,x), \quad t > 0, \tag{1}$$

subject to a random initial condition

$$u(0,x) = \xi(x), \tag{2}$$

where $\alpha \geq 0$, $\gamma > 0$, and $\mu > 0$, while $\xi(x)$, $x \in \mathbb{R}^d$, is a zero-mean Gaussian random field. For a function φ in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, with Fourier transform $\widehat{\varphi}(\lambda) := \int_{\mathbb{R}^d} e^{-i\langle x,\lambda\rangle}\varphi(x)dx$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by $\widehat{(-\Delta)^{\alpha/2}\varphi}(\lambda) = |\lambda|^\alpha \widehat{\varphi}(\lambda)$, and the Bessel fractional differential operator $(I-\Delta)^{\gamma/2}$ is defined by $\widehat{(I-\Delta)^{\gamma/2}\varphi}(\lambda) = (1+|\lambda|^2)^{\gamma/2}\widehat{\varphi}(\lambda)$. Thus, $(-\Delta)^{\alpha/2}$ is the inverse of the Riesz potential operator $(-\Delta)^{-\alpha/2}$, whose Fourier multiplier is $|\lambda|^{-\alpha}$, while $(I-\Delta)^{\gamma/2}$ is the inverse of the Bessel potential operator $(I-\Delta)^{-\gamma/2}$, whose Fourier multiplier is $(1+|\lambda|^2)^{-\gamma/2}$. When $\alpha = 0$, the operator $(-\Delta)^{\alpha/2}$ is understood as the identity operator. See, for example, [8, 9].

The Cauchy problem (1) with the non-random initial condition given by the Dirac delta function, $u(0,x) = \delta(x)$, has a unique solution determined by $\widehat{G}(t,\lambda)$, $t > 0$, $\lambda \in \mathbb{R}^d$, the Fourier transform with respect to the second argument of the fundamental solution (i.e., the Green function) of the non-random Cauchy problem.

The solution to the initial value problem (1) and (2) in convolution form is defined as

$$u(t,x) = \int_{\mathbb{R}^d} G(t,x-y)\, u(0,y), dy, \tag{3}$$

where the Green function $G(t,x)$ is specified by its Fourier transform.

In the literature this solution is called a Green-type solution or a mean-square solution, as it can be interpreted in the mean-square sense. For more details and justifications of this approach, consult [10, 12].

For simplicity, this paper focuses on the one-dimensional case of $d = 1$. To make the notations consistent with multidimensional cases considered in the mentioned publications, the operator $\Delta = \partial^2/\partial x^2$ denotes the one-dimensional Laplacian.

Let $\xi(x)$, $x \in \mathbb{R}$, be a measurable stationary real-valued zero-mean Gaussian random process defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $Z(\cdot)$ and $W(\cdot)$ denote respectively, an orthogonal random measure of $\xi(\cdot)$ and a Gaussian white-noise random measure on $\mathbb{R}$, with $\mathbb{E}|Z(d\lambda)|^2 = F(d\lambda)$, where $F(\cdot)$ is the spectral measure of $\xi(\cdot)$. In this paper, the random measures $Z(\cdot)$ and $W(\cdot)$ are assumed to be symmetric in the sense of complex conjugate symmetry, that is, $Z(A) = \overline{Z(-A)}$ and $W(A) = \overline{W(-A)}$ for all $A \in \mathcal{B}(\mathbb{R})$, which ensures that the process $\xi(x)$ is real-valued.

The covariance function of $\xi(x)$ can be written as

$$r(x) := \mathrm{Cov}(\xi(0), \xi(x)) = \int_{\mathbb{R}} e^{ix\lambda} F(d\lambda), \tag{4}$$

where $F(\cdot)$ is the spectral measure.

If the spectral measure is absolutely continuous, it can be represented as

$$F(\Delta) = \int_{\Delta} f_\xi(\lambda) d\lambda, \quad \Delta \in \mathcal{B}(\mathbb{R}),$$

where the function $f_\xi(\lambda), \lambda \in \mathbb{R}$ is called the spectral density function of the stationary process $\xi(\cdot)$. Since the stationary process $\xi(\cdot)$ is real-valued, its spectral density is necessarily even.

Then, the following spectral representation of the random process $\xi(x)$ holds true

$$\xi(x) = \int_{\mathbb{R}} e^{i\lambda x} Z(d\lambda) = \int_{\mathbb{R}} e^{i\lambda x} \sqrt{f_\xi(\lambda)} W(d\lambda).$$

The Fourier transform of the Green function $G(t, x)$ is equal to, see [10, 12],

$$\widehat{G}(t, \lambda) = E_\beta\left(-\mu t^\beta |\lambda|^\alpha \left(1 + |\lambda|^2\right)^{\gamma/2}\right),$$

Here $E_\beta(\cdot)$ denotes the Mittag–Leffler function, defined by

$$E_\beta(s) := \sum_{k=0}^{\infty} \frac{s^k}{\Gamma(1+\beta k)}, \qquad s \in \mathbb{R},\ 0 < \beta \le 1.$$

For $\beta = 1$, $E_1(s) = e^s$. More details on the Mittag–Leffler function and its properties can be found in [36].

For $0 < \beta < 1$ and negative values of its argument, the Mittag–Leffler function satisfies the following two-sided estimate (see [45, Theorem 4, inequality (6.8)]):

$$\frac{1}{1 + \Gamma(1-\beta)s} \le E_\beta(-s) \le \frac{1}{1 + \frac{s}{\Gamma(1+\beta)}}, \quad s \ge 0. \tag{5}$$

The solution (3) of the initial value problem (1) and (2) admits the following stochastic integral representation, see [12],

$$u(t, x) = \int_{\mathbb{R}} e^{ix\lambda} E_\beta(-\mu|\lambda|^\alpha (1 + |\lambda|^2)^{\gamma/2} t^\beta) Z(d\lambda). \tag{6}$$

The covariance function of the solution field $u(t, x)$ is

$$\mathrm{Cov}(u(t, x), u(t', x')) = \int_{\mathbb{R}} e^{i(x-x')\lambda} E_\beta(-\mu|\lambda|^\alpha (1 + |\lambda|^2)^{\gamma/2} t^\beta)$$

$$\times E_\beta(-\mu|\lambda|^\alpha(1+|\lambda|^2)^{\gamma/2}(t')^\beta)F(d\lambda).$$

To further specify the class of Gaussian random processes $\xi(x)$, $x \in \mathbb{R}$, used as the initial condition (2), the following assumption is used, see [5] for more details.

Assumption 1. The covariance function (4) has the form

$$r(x) = \sum_{j=0}^{n} \frac{\cos(w_j x)}{(1+x^2)^{\kappa_j/2}} A_j, \quad x \in \mathbb{R}, \tag{7}$$

where $\sum_{j=0}^{n} A_j = 1$, $A_j \geq 0$, $\kappa_j \in (0,1)$, $w_j > 0$ (except $w_0 = 0$), $j = 0, \ldots, n$.

The frequency parameters w_j determine the locations of the singularities of the spectral density. In particular, for each $j = 1, \ldots, n$ with $A_j > 0$, the corresponding singularities are located at $\lambda = \pm w_j$, while, when $A_0 > 0$, $w_0 = 0$ corresponds to a singularity at the origin.

Let us define the next constants

$$C(\bar{\kappa}, \bar{w}, \bar{A}) := \sum_{j=1}^{n} c_{1,j}(\kappa_j) A_j K_{\frac{\kappa_j - 1}{2}}(|w_j|)|w_j|^{\frac{\kappa_j - 1}{2}},$$

$$C(\kappa_0, A_0) := \frac{A_0}{2\Gamma(\kappa_0)\cos(\kappa_0 \pi/2)},$$

where $c_{1,j}(\kappa_j) := 2^{\mathbf{1}_{\{0\}}(j)} 2^{\frac{1-\kappa_j}{2}} / \left(\sqrt{\pi}\Gamma\left(\kappa_j/2\right)\right)$, $j = 0, 1, \ldots, n$, $\bar{\kappa} = (\kappa_1, \ldots, \kappa_n)$, $\bar{w} = (w_1, \ldots, w_n)$, and $\bar{A} = (A_1, \ldots, A_n)$. The symbol $K_\nu(\cdot)$ denotes the modified Bessel function of the second kind, which can be defined, see [49, p. 78], by

$$K_\nu(z) := \frac{1}{2} \int_0^{+\infty} s^{\nu - 1} \exp\left(-\frac{1}{2}\left(s + \frac{1}{s}\right) z\right) ds, \quad z > 0, \quad \nu \in \mathbb{R}.$$

The covariance function in (7) is non-integrable and exhibits an oscillating behaviour, which corresponds to the cyclic long-range dependence scenario. It follows from (7) that the corresponding spectral density has the representation

$$\begin{aligned} f_\xi(\lambda) := \sum_{j=1}^{n} \frac{c_{1,j}(\kappa_j)}{2} A_j \Big(K_{\frac{\kappa_j - 1}{2}} \left(|\lambda + w_j|\right) |\lambda + w_j|^{\frac{\kappa_j - 1}{2}} \\ + K_{\frac{\kappa_j - 1}{2}} \left(|\lambda - w_j|\right) |\lambda - w_j|^{\frac{\kappa_j - 1}{2}} \Big) + \frac{c_{1,0}(\kappa_0)}{2} A_0 K_{\frac{\kappa_0 - 1}{2}}(|\lambda|)|\lambda|^{\frac{\kappa_0 - 1}{2}}, \end{aligned} \tag{8}$$

3 Multiscaling limit theorems for filtered random fields

The publications [4, 5] showed that, for the Cauchy problem with random initial conditions, there are cases where multiscaling limits do not exist. They also proposed kernel smoothing as a way to obtain non-degenerate limits in the case of higher-order heat equations. The following results apply this approach to the fractional Riesz–Bessel equations studied in [3]. For the same class of random initial conditions, the multiscaling limit results in [3] were obtained under the restriction $\alpha > 1/2$ when

$A_0 = 0$ and under the restriction $\alpha > \kappa_0/2$ when $A_0 > 0$. The filtering approach developed in the present paper provides additional integrability through the kernel multiplier and allows these restrictions on α to be removed.

Consider a spatial kernel $g(x_1,x)$, $x_1,x \in \mathbb{R}$, and define the centered spatially averaged random field

$$U_\varepsilon^g(t,x) := \varepsilon^{\rho_1} \int_{\mathbb{R}} g(x_1,x)\, u\Big(\frac{t}{\varepsilon^{\rho_2}}, \frac{x_1}{\varepsilon^{\rho_3}}\Big)\; dx_1, \tag{9}$$

where $\rho_i \in \mathbb{R}$, $i = 1,2,3$, are some scaling parameters. As the initial condition is given by a zero-mean Gaussian random field, the fields $u(t,x)$ and $U_\varepsilon^g(t,x)$ are also zero-mean.

We will use real-valued spatial kernel functions $g(x_1,x)$, $x_1,x \in \mathbb{R}$, such that for all x it holds $g(\cdot,x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. Their Fourier transforms with respect to the first argument are given by

$$\widehat{g}(\lambda,x) := \int_{\mathbb{R}} e^{-i\lambda x_1} g(x_1,x)\, dx_1.$$

To obtain non-degenerate limits, we assume that $\widehat{g}(\cdot,x) \neq 0$ on a set of positive Lebesgue measure. Since the Fourier transform of these kernel functions is taken with respect to the first variable, we will use the notation $\|g(z,x)\|_{H_z^s}$ for the corresponding Sobolev norm in the z-variable.

Theorem 1. *Consider the random field $u(t,x)$, $t > 0$, $x \in \mathbb{R}$, defined by (1) and the random initial condition given by (2), satisfying Assumption 1 with $A_0 = 0$.*

For $\rho_1 = -\rho_3/2$, $\rho_2 = \alpha\rho_3/\beta$, and $\rho_3 > 0$, when $\varepsilon \to 0$, the finite-dimensional distributions of $U_\varepsilon^g(t,x)$ converge weakly to those of a zero-mean Gaussian random field $U_0^g(t,x)$ given by

$$U_0^g(t,x) := \sqrt{C(\bar{\kappa},\bar{w},\bar{A})} \int_{\mathbb{R}} \widehat{g}(\lambda,x)\, E_\beta\big(-\mu t^\beta |\lambda|^\alpha\big)\; W(d\lambda). \tag{10}$$

The limit random field $U_0^g(t,x)$ has the covariance function

$$\begin{aligned}\operatorname{Cov}\big(U_0^g(t,x), U_0^g(t',x')\big) = C(\bar{\kappa},\bar{w},\bar{A}) \int_{\mathbb{R}} \widehat{g}(\lambda,x)\overline{\widehat{g}(\lambda,x')}& \\ \times E_\beta(-\mu t^\beta |\lambda|^\alpha) E_\beta(-\mu (t')^\beta |\lambda|^\alpha)\, d\lambda.&\end{aligned} \tag{11}$$

Proof. Using the stochastic integral representation (6) of the solution and substituting t with t/ε^{ρ_2} and x with x_1/ε^{ρ_3}, one gets

$$u\Big(\frac{t}{\varepsilon^{\rho_2}}, \frac{x_1}{\varepsilon^{\rho_3}}\Big) = \int_{\mathbb{R}} e^{i\frac{x_1}{\varepsilon^{\rho_3}}\lambda} E_\beta\left(-\mu|\lambda|^\alpha (1+|\lambda|^2)^{\gamma/2} \Big(\frac{t}{\varepsilon^{\rho_2}}\Big)^\beta\right) \sqrt{f_\xi(\lambda)}\, W(d\lambda). \tag{12}$$

The application of kernel smoothing (9) to (12) results in

$$\begin{aligned}U_\varepsilon^g(t,x) = \varepsilon^{\rho_1} \int_{\mathbb{R}}\int_{\mathbb{R}} g(x_1,x) e^{i\frac{x_1}{\varepsilon^{\rho_3}}\lambda} E_\beta\Big(-\mu|\lambda|^\alpha (1+|\lambda|^2)^{\gamma/2} t^\beta \varepsilon^{-\rho_2\beta}\Big)& \\ \times \sqrt{f_\xi(\lambda)}\, W(d\lambda)\, dx_1.&\end{aligned}$$

As the Mittag-Leffler function is bounded and $g(x_1,x)\sqrt{f_\xi(\lambda)} \in L_2(\mathbb{R}^2)$ for each fixed x, the stochastic Fubini theorem can be applied, which allow the order of integration to be interchanged; see, for example, [20, Chapter 2, Section 2.4] and [48].

As the Mittag-Leffler function is bounded and $g(x_1,x)\sqrt{f_\xi(\lambda)} \in L_2(\mathbb{R}^2)$ for each fixed x, the stochastic Fubini theorem can be applied, allowing to interchange the order of integration. Then, applying the change of variable $\tilde{\lambda} := \lambda/\varepsilon^{\rho_3}$ and using the scaling property of Gaussian white noise $W(\varepsilon^{\rho_3} d\tilde{\lambda}) \stackrel{d}{=} \varepsilon^{\rho_3/2} W(d\tilde{\lambda})$, we obtain

$$U_\varepsilon^g(t,x) = \varepsilon^{\rho_1+\rho_3/2} \int_{\mathbb{R}} \widehat{g}(\tilde{\lambda},x) E_\beta\left(-\mu|\tilde{\lambda}|^\alpha \left(1+|\tilde{\lambda}|^2\varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta \varepsilon^{\alpha\rho_3-\rho_2\beta}\right) \times \sqrt{f_\xi(\tilde{\lambda}\varepsilon^{\rho_3})}\, W(d\tilde{\lambda}).$$

The values of the scaling parameters for which the normalisation factor is constant and the Mittag-Leffler function remains nondegenerate in the limit are $\rho_1 = -\rho_3/2$, $\rho_3 > 0$, and $\alpha\rho_3 - \rho_2\beta = 0$. In this case

$$U_\varepsilon^g(t,x) = \int_{\mathbb{R}} \widehat{g}(\tilde{\lambda},x) E_\beta\left(-\mu|\tilde{\lambda}|^\alpha \left(1+|\tilde{\lambda}|^2\varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta\right) \sqrt{f_\xi(\tilde{\lambda}\varepsilon^{\rho_3})}\, W(d\tilde{\lambda}).$$

Since $\rho_3 > 0$, if $\varepsilon \to 0$, one has

$$E_\beta\left(-\mu|\tilde{\lambda}|^\alpha \left(1+|\tilde{\lambda}|^2\varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta\right) \longrightarrow E_\beta\left(-\mu t^\beta |\tilde{\lambda}|^\alpha\right).$$

It suggests that the limit random field is given by

$$U_0^g(t,x) = \sqrt{C(\bar{\kappa},\bar{w},\bar{A})} \int_{\mathbb{R}} \widehat{g}(\lambda,x) E_\beta\left(-\mu t^\beta|\lambda|^\alpha\right) W(d\lambda).$$

To justify the convergence, consider $R_\varepsilon(t,x) := \mathbb{E}\left(U_\varepsilon^g(t,x) - U_0^g(t,x)\right)^2$. Let us denote

$$Q_\varepsilon(\tilde{\lambda}) := \sum_{j=1}^n \frac{c_{1,j}(\kappa_j)}{2} A_j \Big(K_{\frac{\kappa_j-1}{2}}\left(|\tilde{\lambda}\varepsilon^{\rho_3}+w_j|\right)|\tilde{\lambda}\varepsilon^{\rho_3}+w_j|^{\frac{\kappa_j-1}{2}} + K_{\frac{\kappa_j-1}{2}}\left(|\tilde{\lambda}\varepsilon^{\rho_3}-w_j|\right)|\tilde{\lambda}\varepsilon^{\rho_3}-w_j|^{\frac{\kappa_j-1}{2}}\Big).$$

By the Itô isometry,

$$R_\varepsilon(t,x) = \int_{\mathbb{R}} |\widehat{g}(\tilde{\lambda},x)|^2 E_\beta^2\left(-\mu t^\beta|\tilde{\lambda}|^\alpha\right) \left(\frac{E_\beta\left(-\mu|\tilde{\lambda}|^\alpha \left(1+|\tilde{\lambda}|^2\varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta\right)}{E_\beta\left(-\mu t^\beta|\tilde{\lambda}|^\alpha\right)} \times \sqrt{Q_\varepsilon(\tilde{\lambda})} - \sqrt{C(\bar{\kappa},\bar{w},\bar{A})}\right)^2 d\tilde{\lambda}.$$

Since

$$\frac{E_\beta\left(-\mu|\tilde{\lambda}|^\alpha \left(1+|\tilde{\lambda}|^2\varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta\right)}{E_\beta\left(-\mu t^\beta|\tilde{\lambda}|^\alpha\right)} \to 1 \quad \text{and} \quad Q_\varepsilon(\tilde{\lambda}) \to C(\bar{\kappa},\bar{w},\bar{A}),$$

when $\varepsilon \longrightarrow 0$, the integrand converges pointwise to zero.

By the properties of the Fourier transforms, $\widehat{g}(\cdot, x) \in L_2(\mathbb{R})$ and the integrand is dominated by an integrable majorant for each ε. As the functions Q_ε have unbounded peaks at the moving locations, the classical dominated convergence theorem is not directly applicable. Instead, one must apply the generalized Lebesgue dominated convergence theorem, as in [3]. Then, the remaining part of the proof uses the same arguments as in the proof of [3, Theorem 3.1] and analogously results in $R_\varepsilon(t, x) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, for arbitrary $a_1, \ldots, a_m \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} \mathbb{E} \left(\sum_{j=1}^{m} a_j \left(U_\varepsilon^g(t_j, x_j) - U_0^g(t_j, x_j) \right) \right)^2 = 0.$$

Therefore, the Cramér-Wold theorem gives the convergence of finite-dimensional distributions.

Because $\widehat{g}(\cdot, x) \in L_2(\mathbb{R})$ and, by (5), the function $E_\beta^2(\cdot)$ is bounded, the following integral is finite for all $\alpha \geq 0$

$$\int_{\mathbb{R}} |\widehat{g}(\lambda, x)|^2 E_\beta^2(-\mu t^\beta |\lambda|^\alpha) d\lambda < \infty.$$

Therefore, compared to [3, Theorem 3.1], the condition $\alpha > 1/2$ is not required.

The covariance representation (11) follows directly from (10) by the orthogonality property of the random measure $W(\cdot)$. □

Remark 1. The Matérn kernel is a widely used class of continuous kernels in many recent applications, see, for example, [31, 43] and the references therein. For the parameters $\nu > 0$ and $a > 0$ it is defined by

$$h(x) := \frac{(a|x|)^\nu \, K_\nu(a|x|)}{2^{\nu-1}\Gamma(\nu)}. \tag{13}$$

Its Fourier transform is given by

$$\widehat{h}(\lambda) = \frac{2\sqrt{\pi} a^{2\nu} \, \Gamma\left(\nu + \frac{1}{2}\right)}{\Gamma(\nu)} \left(a^2 + \lambda^2\right)^{-\left(\nu + \frac{1}{2}\right)}. \tag{14}$$

Some important special cases of the simplified expressions of (13) and (14), that are frequently used in applications, are:

- $\nu = \frac{1}{2}$: $h(x) = e^{-a|x|}$, $\widehat{h}(\lambda) = \frac{2a}{a^2+\lambda^2}$;
- $\nu = \frac{3}{2}$: $h(x) = (1 + a|x|)\, e^{-a|x|}$, $\widehat{h}(\lambda) = \frac{4a^3}{(a^2+\lambda^2)^2}$;
- $\nu = \frac{5}{2}$: $h(x) = \left(1 + a|x| + \frac{(ax)^2}{3}\right) e^{-a|x|}$, $\widehat{h}(\lambda) = \frac{16a^5}{3(a^2+\lambda^2)^3}$.

When the Matérn kernels are used as covariance functions, the parameter ν controls the smoothness of the corresponding stochastic processes. The case $\nu = 1/2$ produces rougher trajectories, whereas the cases $\nu = 3/2$ and $\nu = 5/2$ yield increasingly smoother sample paths.

Consider the kernel function used in (9) of the form

$$g(x_1, x) := h(x_1 - x) = \frac{(a|x_1 - x|)^{\nu} K_{\nu}(a|x_1 - x|)}{2^{\nu-1}\Gamma(\nu)}.$$

Then, for any $x \in \mathbb{R}$, by the properties of the Fourier transform,

$$\widehat{g}(\lambda, x) = e^{-i\lambda x}\,\widehat{h}(\lambda) = e^{-i\lambda x}\frac{2\sqrt{\pi}a^{2\nu}\,\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu)}\,\left(a^2+\lambda^2\right)^{-\left(\nu+\frac{1}{2}\right)}. \tag{15}$$

Since the fields are real-valued and the integrand in (10) is symmetric with respect to λ, for a real-valued measure $w(\cdot)$, the complex exponential term $e^{i\lambda x}$ in the expression above for $\widehat{g}(\lambda, x)$ can be replaced by $\cos(\lambda x)$ without altering the limit field and its covariance structure.

Example 1. This numerical example illustrates Theorem 1. The parameter vectors were chosen as $\bar{\kappa} = (0.2, 0.6, 0.8)$ and $\bar{w} = (0.8, 1.2, 2.0)$, with weights $\bar{A} = (0.4, 0.35, 0.25)$ satisfying the normalization condition $\sum_{j=1}^{n} A_j = 1$, since $A_0 = 0$. The parameters of the FRBE were selected as $\alpha = 1$, $\beta = 0.5$, and $\mu = 1$.

The random field $U_0^g(t, x)$ was defined by using its stochastic integral representation (10), where the stochastic integral was approximated by a Riemann-type sum over a uniform symmetric grid in the frequency domain:

$$U_0^g(t, x) \approx \sqrt{C(\bar{\kappa}, \bar{w}, \bar{A})}\sum_{j=-N}^{N}\widehat{g}(\lambda_j, x)\,E_{\beta}\left(-\mu t^{\beta}|\lambda_j|^{\alpha}\right)\,W(\Delta\lambda_j),$$

with $\lambda_j = j\Delta$, $j = -N, \ldots, N$.

The selected discretisation step $\Delta = 0.01$ and the truncation level $N = 1000$ correspond to the approximation of the integrals over the interval $[-10, 10]$. This range was chosen because, outside this interval, the integrands are sufficiently small and make a negligible contribution to the integrals. The Gaussian increments $W(\Delta\lambda_j)$ are taken to be independent normally distributed random variables, $W(\Delta\lambda_j) \sim N(0, \Delta)$. They are set symmetrically with respect to the origin due to the real-valued initial-condition random field. The random field $U_0^g(t, x)$ is then simulated in the spatio-temporal domain $x \in [0, 40]$, $t \in [0, 2]$.

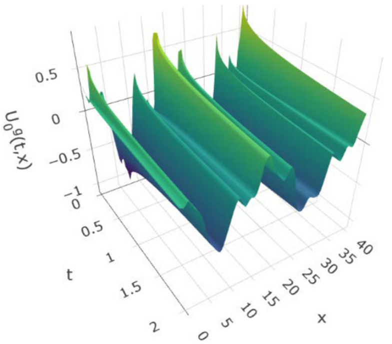
For $a = 1$, and for the Matérn smoothness parameters $\nu = 1/2$ and $\nu = 3/2$, the kernels in Remark 1 reduce, respectively, to

$$g(x_1, x) = e^{-|x-x_1|}, \qquad \widehat{g}(\lambda, x) = \frac{2e^{-i\lambda x}}{1+\lambda^2}, \tag{16}$$

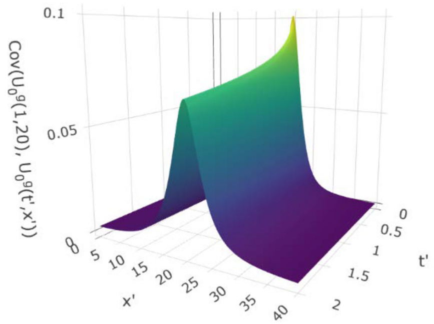
and

$$g(x_1, x) = (1+|x-x_1|)\,e^{-|x-x_1|}, \qquad \widehat{g}(\lambda, x) = \frac{4e^{-i\lambda x}}{(1+\lambda^2)^2}. \tag{17}$$

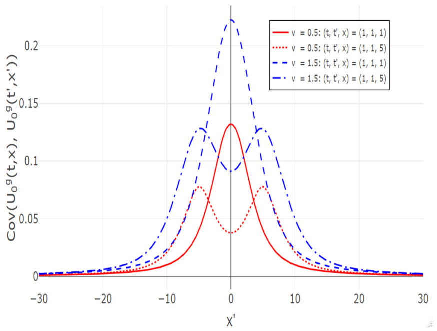
Figure 1a presents a realisation of $U_0^g(t, x)$ for $\nu = 1/2$. Similar plots were obtained for $\nu = 3/2$ and $\nu = 5/2$, which exhibited smoothed realisations, but they are omitted for brevity. The field varies mainly along the spatial direction x, exhibiting repeated ridge-like structures. For small values of t, the field exhibits oscillations of



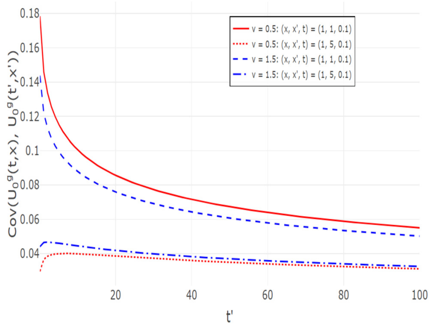
(a) Realisation of $U_0^g(t,x)$



(b) Covariance function of $U_0^g(t,x)$



(c) Examples of spatial covariance functions, for fixed t, t' and $\nu = 0.5$ and 1.5



(d) Examples of temporal covariance functions, for fixed x, x' and $\nu = 0.5$ and 1.5

Fig. 1. Example of the limit field $U_0^g(t,x)$ from Theorem 1

larger magnitudes. As t increases, the Mittag–Leffler factor progressively damps these oscillations, leading to a gradual smoothing of the same spatial pattern.

As the limit random field is non-stationary, several plots are provided to illustrate the behaviour of its covariance function at different spatial and temporal locations. The covariance function $\mathrm{Cov}\big(U_0^g(t,x), U_0^g(t',x')\big)$ shown in Figure 1b is obtained by approximating the integral in (11) at the fixed point $(t,x) = (1,20)$. As can be seen, the covariance attains its maximum at this point and decreases as either the temporal or spatial separation increases, although the decay is considerably slower in time.

Figure 1c presents examples of spatial covariance functions plotted for $x' \in [-30,30]$, with the temporal variables fixed at $t = t' = 1$. Two spatial reference locations, $x = 1$ and $x = 5$, and two Matérn smoothness parameters, $\nu = 1/2$ and $\nu = 3/2$, were used. In both cases, the covariance attains its largest values near the selected point x and decreases with x' as the spatial separation between the two points increases. The temporal covariance shown in Figure 1d was obtained by fixing $t = 0.1$ and plotting the covariance as a function of $t' \in [0,100]$. The covariance takes high values when the temporal separation between t and t' is small and gradually decreases as the separation increases. The slow temporal decay of the covariance suggests possible long-memory behaviour between the values at fixed time $t = 0.1$ and varying times t'.

As expected, the covariance corresponding to the closer spatial locations $(x, x') = (1, 1)$ is larger than that for the more separated locations $(x, x') = (1, 5)$.

Theorem 2. *Consider the random field $u(t, x)$, $t > 0$, $x \in \mathbb{R}$, defined by (1) and the random initial condition given by (2), satisfying Assumption 1 with $A_0 \neq 0$.*

For the scaling parameters $\rho_1 = -\kappa_0\rho_3/2$, $\rho_2 = \alpha\rho_3/\beta$, and $\rho_3 > 0$, when $\varepsilon \to 0$, the finite-dimensional distributions of $U_\varepsilon^g(t, x)$ converge weakly to those of a zero-mean Gaussian random field $\tilde{U}_0^g(t, x)$ given by

$$\tilde{U}_0^g(t,x) := \sqrt{C(\kappa_0, A_0)} \int_{\mathbb{R}} \widehat{g}(\lambda, x)\, E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right) |\lambda|^{\frac{\kappa_0 - 1}{2}}\, W(d\lambda). \tag{18}$$

The limit random field $\tilde{U}_0^g(t, x)$ has the covariance function

$$\operatorname{Cov}\left(\tilde{U}_0^g(t,x), \tilde{U}_0^g(t',x')\right) = C(\kappa_0, A_0) \int_{\mathbb{R}} \widehat{g}(\lambda, x)\overline{\widehat{g}(\lambda, x')} \\ \times E_\beta(-\mu t^\beta |\lambda|^\alpha) E_\beta(-\mu (t')^\beta |\lambda|^\alpha) |\lambda|^{\kappa_0 - 1}\, d\lambda. \tag{19}$$

Remark 2. In contrast to Theorem 1, the limit in Theorem 2 is completely determined by the behaviour of the spectral density near the origin. This shows that the power-law singularity of the spectrum at zero dominates the contributions from spectral singularities at all other points.

Proof. Using the same arguments as in the proof of Theorem 1, we get

$$U_\varepsilon^g(t,x) = \varepsilon^{\rho_1 + \rho_3/2} \int_{\mathbb{R}} \widehat{g}(\tilde{\lambda}, x) E_\beta\left(-\mu |\tilde{\lambda}|^\alpha \left(1 + |\tilde{\lambda}|^2 \varepsilon^{2\rho_3}\right)^{\gamma/2} t^\beta \varepsilon^{\alpha\rho_3 - \rho_2\beta}\right) \\ \times \sqrt{f_\xi(\tilde{\lambda}\varepsilon^{\rho_3})}\, W(d\tilde{\lambda}).$$

Since $A_0 > 0$, the spectral density satisfies

$$f_\xi(\lambda) \sim C(\kappa_0, A_0)\, |\lambda|^{\kappa_0 - 1}, \qquad \lambda \to 0.$$

Therefore, to obtain a non-trivial limit, one can choose

$$\alpha\rho_3 - \rho_2\beta = 0, \quad \rho_1 = -\frac{\kappa_0\rho_3}{2}, \quad \rho_3 > 0.$$

Then, the smoothed field can be written as

$$U_\varepsilon^g(t,x) = \int_{\mathbb{R}} \widehat{g}(\tilde{\lambda}, x) E_\beta\left(-\mu |\tilde{\lambda}|^\alpha (1 + |\tilde{\lambda}|^2 \varepsilon^{2\rho_3})^{\gamma/2} t^\beta\right) \sqrt{\frac{f_\xi(\varepsilon^{\rho_3}\tilde{\lambda})}{\varepsilon^{\rho_3(\kappa_0 - 1)}}}\, W(d\tilde{\lambda}).$$

Since $\rho_3 > 0$, when $\varepsilon \to 0$, the integrand converges pointwise to

$$\sqrt{C(\kappa_0, A_0)}\, \widehat{g}(\tilde{\lambda}, x) E_\beta\left(-\mu t^\beta |\tilde{\lambda}|^\alpha\right) |\tilde{\lambda}|^{\frac{\kappa_0 - 1}{2}},$$

which suggests that the limit field is

$$\tilde{U}_0^g(t,x) = \sqrt{C(\kappa_0, A_0)} \int_{\mathbb{R}} \widehat{g}(\lambda, x) E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right) |\lambda|^{\frac{\kappa_0-1}{2}} W(d\lambda).$$

Note that $g(\cdot, x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ implies that $\widehat{g}(\cdot, x) \in L_2(\mathbb{R})$ and is bounded. Since $\kappa_0 \in (0,1)$, it follows that $\widehat{g}(\cdot, x)| \cdot |^{(\kappa_0-1)/2} \in L_2(\mathbb{R})$ for every $x \in \mathbb{R}$. Thus, using (5), $E_\beta(-z) \in (0,1]$ for $z \geq 0$, and the limit field is well defined and has finite variance.

Let us justify the convergence

$$R_\varepsilon(t,x) = \mathbb{E}\left(U_\varepsilon^g(t,x) - \tilde{U}_0^g(t,x)\right)^2 \to 0, \quad \text{when } \varepsilon \to 0.$$

By the Itô isometry,

$$R_\varepsilon(t,x) = \int_{\mathbb{R}} |\widehat{g}(\lambda,x)|^2 \left(E_\beta\left(-\mu|\lambda|^\alpha (1+|\lambda|^2 \varepsilon^{2\rho_3})^{\gamma/2} t^\beta\right) \sqrt{\frac{f_\xi(\varepsilon^{\rho_3}\lambda)}{\varepsilon^{\rho_3(\kappa_0-1)}}} \right.$$
$$\left. - \sqrt{C(\kappa_0, A_0)}\, E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right) |\lambda|^{\frac{\kappa_0-1}{2}} \right)^2 d\lambda.$$

The integrand converges pointwise to zero. Noting that for sufficiently small ε, $f_\xi(\varepsilon^{\rho_3}\lambda)/\varepsilon^{\rho_3(\kappa_0-1)} \leq C|\lambda|^{\kappa_0-1}$, one can see that in a neighbourhood of the origin, the integrand is dominated by $C_1|\widehat{g}(\lambda,x)|^2|\lambda|^{\kappa_0-1}$, and therefore has an integrable majorant on $\mathbb{R}$. However, as the functions $f_\xi(\varepsilon^{\rho_3}\lambda)$ have unbounded peaks at the moving locations, the classical dominated convergence theorem is not directly applicable. Instead, one must apply the generalized Lebesgue dominated convergence theorem, as in [3]. Similar to Theorem 1, in contrast to [3, Theorem 3.2], the condition $\alpha > \kappa_0/2$ is not required due to the presence of the kernel multiplier.

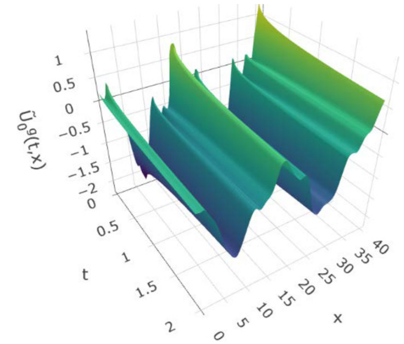
The covariance representation follows directly from the Itô isometry, which completes the proof. □

Remark 3. Note that, in general, the product $\widehat{g}(\lambda,x)\overline{\widehat{g}(\lambda,x')}$ is not a function of $x - x'$. Consequently, in contrast to the results in [5], the limit fields $U_0^g(t,x)$ and $\tilde{U}_0^g(t,x)$ are, in general, non-stationary in both space and time. For translation kernels $g(x_1,x) = h(x_1 - x)$, however, the product equals $e^{-i\lambda(x-x')}|\widehat{h}(\lambda)|^2$, so the fields are stationary in space.

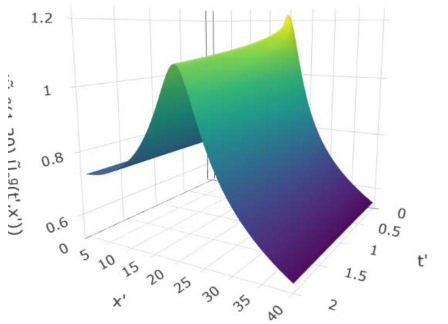
Example 2. In this example, we considered the limit random field $\tilde{U}_0^g(t,x)$, $t > 0$, $x \in \mathbb{R}$, obtained in Theorem 2 under Assumption 1 with $A_0 \neq 0$. The numerical parameters were kept the same as in Example 1. In addition, for this case, the values $A_0 = 0.4$ and $\kappa_0 = 0.2$ were used. By Theorem 2, the values of other nonzero-frequency weights A_j, $j \neq 0$, do not affect the limit field and can be renormalised to sum to 0.6. The same kernels as in (16) and (17) were employed. The field was evaluated on the space–time grid described in Example 1.

The limit field is simulated using the stochastic integral representation in (18) and its Riemann-type approximation

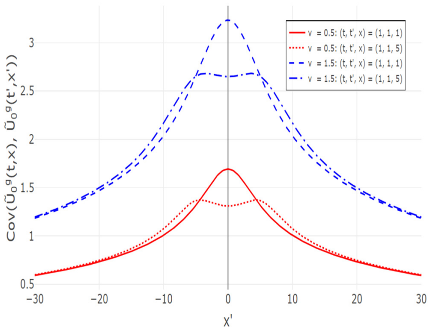
$$\tilde{U}_0^g(t,x) \approx \sqrt{C(\kappa_0, A_0)} \sum_{\substack{j=-N \\ j\neq 0}}^{N} \widehat{g}(\lambda_j, x)\, E_\beta\left(-\mu t^\beta |\lambda_j|^\alpha\right) |\lambda_j|^{(\kappa_0-1)/2} W(\Delta\lambda_j).$$



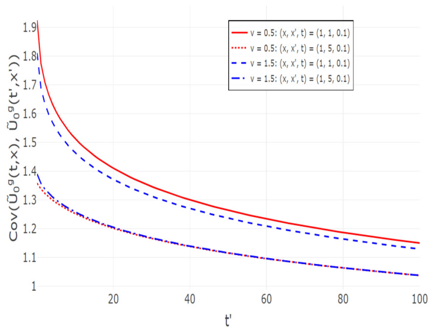
(a) Realisation of $\tilde{U}_0^g(t,x)$



(b) Covariance function of $\tilde{U}_0^g(t,x)$



(c) Examples of spatial covariance functions, for fixed t, t' and $\nu = 0.5$ and 1.5



(d) Examples of temporal covariance functions, for fixed x, x' and $\nu = 0.5$ and 1.5

Fig. 2. Example of the limit field $\tilde{U}_0^g(t,x)$ from Theorem 2

The additional factor $|\lambda_j|^{(\kappa_0-1)/2}$ is the main numerical difference from the example for Theorem 1. It gives stronger weight to low frequencies, and therefore, the behaviour of the field is more influenced by the neighbourhood of the origin in the spectral domain. Because $(\kappa_0 - 1)/2 < 0$, the sum in the discretisation of $\tilde{U}_0^g(t,x)$ excludes zero frequency.

Figure 2a shows the realization of $\tilde{U}_0^g(t,x)$ for $\nu = 1/2$. Similar to Example 1, the main variation of the field remains in the spatial direction and it gradually decreases over the temporal domain. The covariance function in Figure 2b is largest near the selected spatial reference location and decreases as the spatial or temporal separation increases. The covariance surface is more elevated compared to Figure 1b. Figures 2c and 2d show the spatial and temporal covariance functions for fixed temporal and spatial variables, respectively. Their shapes are similar to those in Example 1, but with higher values. This difference in magnitude can be attributed to the stronger contribution of low-frequency components in the case $A_0 > 0$.

4 Some properties of multiscaling limit fields

This section studies properties of the multiscaling limit fields introduced in the previous section, demonstrating how they depend on the smoothing kernels and parameters of the FRBE.

Let us define

$$\eta_* = \sup\left\{\eta \in (0,1] : \int_{\mathbb{R}} |\lambda|^{2\alpha\eta} |\widehat{g}(\lambda,x)|^2 d\lambda < \infty \text{ for all } x \in \mathbb{R}\right\}$$

and

$$\tilde{\eta}_* = \sup\left\{\eta \in (0,1] : \int_{\mathbb{R}} |\lambda|^{2\alpha\eta+\kappa_0-1} |\widehat{g}(\lambda,x)|^2 d\lambda < \infty \text{ for all } x \in \mathbb{R}\right\}.$$

Example 3. For the case of the Matérn kernels, using $\widehat{g}(\lambda,x)$ given by (15), one obtains

$$\int_{\mathbb{R}} |\lambda|^{2\alpha\eta} |\widehat{g}(\lambda,x)|^2\, d\lambda = C \int_{\mathbb{R}} \frac{|\lambda|^{2\alpha\eta}}{(a^2+\lambda^2)^{2\nu+1}}\, d\lambda < +\infty.$$

As $\nu > 0$ and $\alpha > 0$, this integral is finite when $4\nu + 2 - 2\alpha\eta > 1$. Therefore, $\eta_* = \min\left(1, \frac{4\nu+1}{2\alpha}\right)$. Similarly, one obtains $\tilde{\eta}_* = \min\left(1, \frac{4\nu+2-\kappa_0}{2\alpha}\right)$.

First, we investigate the Hölder continuity of the limit fields in time.

Theorem 3. *Assume that $\eta_* > 0$ for the kernel $g(\cdot,\cdot)$. Then, for each fixed x, the limit field $U_0^g(t,x)$ in Theorem 1 is mean-square Hölder continuous in t of any order $\gamma_t \in (0, \eta_*\beta)$. It has a continuous modification with sample paths that are almost surely Hölder continuous in t of order γ_t.*

If $\tilde{\eta}_* > 0$ for the kernel $g(\cdot,\cdot)$, then, for each fixed x, the limit field $\tilde{U}_0^g(t,x)$ in Theorem 2 is mean-square Hölder continuous in t and has a continuous modification with sample paths that are almost surely Hölder continuous in t of any order $\gamma_t \in (0, \tilde{\eta}_*\beta)$.

Proof. Let $x \in \mathbb{R}$ be fixed. Using the stochastic integral representation of $U_0^g(t,x)$ in Theorem 1 and the Itô isometry for stochastic integrals with respect to the Wiener measure, one obtains that for $t, s > 0$ it holds that

$$\mathbb{E}\left|U_0^g(t,x) - U_0^g(s,x)\right|^2 = C(\bar{\kappa}, \bar{w}, \bar{A}) \times \int_{\mathbb{R}} |\widehat{g}(\lambda,x)|^2 \left|E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right) - E_\beta\left(-\mu s^\beta |\lambda|^\alpha\right)\right|^2 d\lambda. \tag{20}$$

For $0 < \beta < 1$, complete monotonicity of the Mittag-Leffler functions $E_\beta(-v)$ on $[0,\infty)$ implies that its derivative is bounded in absolute value by its value at zero. For $\beta = 1$ the same immediately follows from $E_1(-v) = e^{-v}$. Hence, for $\eta \in (0,1]$,

$$|E_\beta(-v_1) - E_\beta(-v_2)| \le C|v_1 - v_2|^\eta.$$

Taking $v_1 = \mu t^\beta |\lambda|^\alpha$ and $v_2 = \mu s^\beta |\lambda|^\alpha$, since $\beta \in (0,1]$, we get

$$\left|E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right) - E_\beta\left(-\mu s^\beta |\lambda|^\alpha\right)\right| \le C|\lambda|^{\eta\alpha} |t^\beta - s^\beta|^\eta \le C|\lambda|^{\eta\alpha} |t-s|^{\eta\beta}.$$

Using the Itô isometry, we obtain

$$\mathbb{E}\left|U_0^g(t,x)-U_0^g(s,x)\right|^2 \le C|t-s|^{2\eta\beta}\int_{\mathbb{R}}|\lambda|^{2\eta\alpha}|\widehat{g}(\lambda,x)|^2 d\lambda.$$

Thus, as the last integral is finite for $\eta<\eta_*$, $U_0^g(\cdot,x)$ is mean-square Hölder continuous in t of order $\eta\beta$.

Moreover, since $U_0^g(t,x)$ is Gaussian, for every $p\ge 2$,

$$\mathbb{E}\left|U_0^g(t,x)-U_0^g(s,x)\right|^p = C_p\left(\mathbb{E}\left|U_0^g(t,x)-U_0^g(s,x)\right|^2\right)^{p/2}\le C|t-s|^{\eta p\beta}.$$

By Kolmogorov's continuity theorem, the field has a continuous modification with sample paths that are almost surely Hölder continuous in t of any order $\gamma_t\in(0,\eta_*\beta-1/p)$. As p can be selected arbitrary large, we obtain that the Hölder exponent in t can be chosen as $\gamma_t\in(0,\eta_*\beta)$.

For the limit field $\tilde{U}_0^g(t,x)$ in Theorem 2 and $\gamma_t\in(0,\tilde{\eta}_*\beta)$ the proof is identical, noting that

$$\mathbb{E}\left|\tilde{U}_0^g(t,x)-\tilde{U}_0^g(s,x)\right|^2 = C(\kappa_0,A_0)\int_{\mathbb{R}}|\widehat{g}(\lambda,x)|^2\left|E_\beta\left(-\mu t^\beta|\lambda|^\alpha\right)\right.$$
$$\left.-E_\beta\left(-\mu s^\beta|\lambda|^\alpha\right)\right|^2|\lambda|^{\kappa_0-1}\,d\lambda$$

and

$$\mathbb{E}\left|\tilde{U}_0^g(t,x)-\tilde{U}_0^g(s,x)\right|^2 \le C|t-s|^{2\eta\beta}\int_{\mathbb{R}}|\lambda|^{2\eta\alpha+\kappa_0-1}|\widehat{g}(\lambda,x)|^2 d\lambda. \qquad \square$$

The next result examines the Hölder continuity of the limit field with respect to the spatial variable.

Theorem 4. *Assume that the kernel function $g(\cdot,\cdot)$ has a finite Sobolev norm of order $-\alpha$ with respect to the first variable, and assume that this norm is Hölder continuous of order $\theta\in(0,1]$ with respect to the second variable, that is,*

$$\|g(z,x)-g(z,y)\|_{H_z^{-\alpha}}\le C|x-y|^\theta.$$

Then, for each fixed $t>0$, the limit random field $U_0^g(t,x)$ in Theorem 1 is mean-square Hölder continuous in x of any order $\gamma_x\in(0,\theta)$. It has a continuous modification with sample paths that are almost surely Hölder continuous in x of order γ_x.

If the following assumption holds true

$$\left\|(-\Delta)_z^{(\kappa_0-1)/4}(g(z,x)-g(z,y))\right\|_{H_z^{-\alpha}}\le C|x-y|^{\theta^*},$$

then, for each fixed $t>0$, the limit random field $\tilde{U}_0^g(t,x)$ in Theorem 2 is mean-square Hölder continuous in x of any order $\gamma_x\in(0,\theta^*)$, and it admits a modification whose sample paths are almost surely Hölder continuous in x of order γ_x.

Proof. Let $t>0$ be fixed. By the Itô isometry, we have

$$\mathbb{E}\left|U_0^g(t,x)-U_0^g(t,y)\right|^2 = C(\bar{\kappa},\bar{w},\bar{A})\int_{\mathbb{R}}|\widehat{g}(\lambda,x)-\widehat{g}(\lambda,y)|^2\left|E_\beta\left(-\mu t^\beta|\lambda|^\alpha\right)\right|^2 d\lambda.$$

For $0 < \beta < 1$, using the estimate (5), one obtains that there exists a constant $C > 0$ such that

$$\left|E_\beta\left(-\mu t^\beta |\lambda|^\alpha\right)\right|^2 \le \left(1 + \frac{\mu t^\beta |\lambda|^\alpha}{\Gamma(1+\beta)}\right)^{-2} \le \frac{C}{(1+|\lambda|^\alpha)^2} \le \frac{C}{\left(1+|\lambda|^2\right)^\alpha}.$$

For $\beta = 1$ and the fixed $t > 0$, the same bound follows from the identity $E_1(-s) = e^{-s}$.

Hence,

$$\mathbb{E}\left|U_0^g(t,x) - U_0^g(t,y)\right|^2 \le C \int_{\mathbb{R}} \frac{|\widehat{g}(\lambda,x) - \widehat{g}(\lambda,y)|^2}{\left(1+|\lambda|^2\right)^\alpha}\, d\lambda.$$

The last integral is the squared negative Sobolev norm with respect to the first variable of the function $g(\cdot,x) - g(\cdot,y)$. Therefore, by the conditions of the theorem,

$$\mathbb{E}\left|U_0^g(t,x) - U_0^g(t,y)\right|^2 \le C\, \|g(z,x) - g(z,y)\|^2_{H_z^{-\alpha}} \le C|x-y|^{2\theta}.$$

Thus, $U_0^g(t,x)$ is mean-square Hölder continuous in x of any order $\gamma_x \in (0,\theta)$. By Kolmogorov's continuity theorem, for each fixed $t > 0$ it admits a modification whose sample paths are Hölder continuous of order γ_x.

In the case of the limit field $\tilde{U}_0^g(t,x)$ in Theorem 2 we analogously obtain that

$$\mathbb{E}\left|\tilde{U}_0^g(t,x) - \tilde{U}_0^g(t,y)\right|^2 \le C \int_{\mathbb{R}} \frac{|\widehat{g}(\lambda,x) - \widehat{g}(\lambda,y)|^2}{\left(1+|\lambda|^2\right)^\alpha} |\lambda|^{\kappa_0 - 1}\, d\lambda. \tag{21}$$

By the properties of the Fourier transforms of the Riesz potential

$$\widehat{(-\Delta)_z^{(\kappa_0-1)/2} g}(z,x) = |z|^{(\kappa_0-1)/4}\, \widehat{g}(z,x),$$

one obtains that the last integral in (21) is equal to

$$\left\|(-\Delta)_z^{(\kappa_0-1)/4}(g(z,x) - g(z,y))\right\|^2_{H_z^{-\alpha}}.$$

Thus, the assumption of the theorem implies the required statement. □

Example 4. For the Matérn kernel case, it follows by the formula (15) that

$$I(x,y) := \int_{\mathbb{R}} \frac{|\widehat{g}(\lambda,x) - \widehat{g}(\lambda,y)|^2}{\left(1+|\lambda|^2\right)^\alpha}\, d\lambda = C \int_{\mathbb{R}} \frac{\left|e^{i\lambda x} - e^{i\lambda y}\right|^2 d\lambda}{\left(1+|\lambda|^2\right)^\alpha \left(a^2+\lambda^2\right)^{2\nu+1}}$$

$$\le C \int_{\mathbb{R}} \frac{\left|e^{i\lambda x} - e^{i\lambda y}\right|^2 d\lambda}{\left(1+|\lambda|^2\right)^{\alpha+2\nu+1}}.$$

Using the elementary estimate $\left|e^{i\lambda x} - e^{i\lambda y}\right| \le C \min\{|\lambda||x-y|, 1\}$ one obtains

$$I(x,y) \le C \int_{\mathbb{R}} \frac{\min(\lambda^2|x-y|^2, 1)}{(1+\lambda^2)^{\alpha+2\nu+1}}\, d\lambda. \tag{22}$$

Consider $0 < |x-y| \leq 1$ and split the integral in (22) at $|\lambda| = |x-y|^{-1}$. Then

$$I(x,y) \leq C|x-y|^2 \int_{|\lambda|\leq|x-y|^{-1}} \frac{\lambda^2\, d\lambda}{(1+\lambda^2)^{\alpha+2\nu+1}} + C\int_{|\lambda|>|x-y|^{-1}} \frac{d\lambda}{(1+\lambda^2)^{\alpha+2\nu+1}}.$$

Noting that α and $\nu > 0$, then $\alpha + 2\nu + 1 > 1/2$, the second integral is finite and

$$\int_{|\lambda|>|x-y|^{-1}} \frac{d\lambda}{(1+\lambda^2)^{\alpha+2\nu+1}} \leq C\int_{|\lambda|>|x-y|^{-1}} |\lambda|^{-2(\alpha+2\nu+1)}\, d\lambda \leq C|x-y|^{2\alpha+4\nu+1}.$$

Note that for all λ it holds

$$\frac{\lambda^2}{(1+\lambda^2)^{\alpha+2\nu+1}} \leq |\lambda|^{-2\alpha-4\nu} \quad \text{and} \quad \frac{\lambda^2}{(1+\lambda^2)^{\alpha+2\nu+1}} \leq |\lambda|^2.$$

By applying the first bound in the first two cases below and, in the third case, splitting the integral at $|\lambda| = 1$ and applying the corresponding bound on each subinterval, one obtains

$$|x-y|^2 \int_{|\lambda|\leq|x-y|^{-1}} \frac{\lambda^2\, d\lambda}{(1+\lambda^2)^{\alpha+2\nu+1}} \leq C \begin{cases} |x-y|^{2\alpha+4\nu+1}, & 2\alpha+4\nu < 1, \\ -|x-y|^2 \log(|x-y|), & 2\alpha+4\nu = 1, \\ |x-y|^2, & 2\alpha+4\nu > 1. \end{cases}$$

Thus, by combining the estimates for the two integrals,

$$I(x,y) \leq C \begin{cases} |x-y|^{2\alpha+4\nu+1}, & 2\alpha+4\nu < 1, \\ -|x-y|^2 \log(|x-y|), & 2\alpha+4\nu = 1, \\ |x-y|^2, & 2\alpha+4\nu > 1. \end{cases}$$

On the other hand, if $|x-y| \geq 1$, then, for any $\theta > 0$, it holds

$$I(x,y) \leq C\int_{\mathbb{R}} \frac{d\lambda}{\left(1+|\lambda|^2\right)^{\alpha+2\nu+1}} \leq C|x-y|^{2\theta}.$$

Consequently, in the Matérn kernel case, Theorem 4 holds for the limit random field $U_0^g(t,x)$ provided that $\theta = \min(1, \alpha+2\nu+1/2)$, except at the critical boundary $2\alpha+4\nu = 1$, where the logarithmic factor prevents the estimate with $\theta = 1$. In that case every $\theta < 1$ is admissible. Analogously, by (21), for the limit random field $\tilde{U}_0^g(t,x)$, it is required that $\theta^* = \min(1, \alpha+2\nu+1-\kappa_0/2)$, with the same exception of $\theta < 1$ at the corresponding critical case of $2\alpha+4\nu = \kappa_0$.

Now we investigate the dependence structure of the limit fields. Recall that, in general, these random fields are nonstationary in both space and time. Therefore, we characterise their short- and long-range dependence separately for each variable and at each space–time location.

We say that a random field $\zeta(t,x)$ is short-range dependent in time at a point (t_0,x_0), $x_0 \in \mathbb{R}$, $t_0 > 0$, if its covariance function is absolutely integrable with respect to time lag

$$\int_0^{+\infty} |\operatorname{Cov}\big(\zeta(t_0,x_0), \zeta(t_0+h,x_0)\big)|\, dh < +\infty. \tag{23}$$

It is short-range dependent in space at (t_0, x_0) if

$$\int_{\mathbb{R}} |\operatorname{Cov}\big(\zeta(t_0, x_0), \zeta(t_0, x_0 + h)\big)| dh < +\infty. \tag{24}$$

If the above integrals are divergent, the fields are called long-range dependent.

Theorem 5. *Let $\beta \in (0,1)$ and the kernel function $g(x_1, x)$ have a nondegenerate Fourier transform $\hat{g}(\lambda, x_0)$, i.e., $\hat{g}(\lambda, x_0) \neq 0$ on a set of positive measure. Then, the limit fields $U_0^g(t,x)$ and $\tilde{U}_0^g(t,x)$ are long-range dependent in time at (t_0, x_0).*

If $\alpha > 1/4$ and for almost all $\lambda \in \mathbb{R}$ it holds that uniformly $\hat{g}(\lambda, \cdot) \in L_1(\mathbb{R})$, then the limit fields $U_0^g(t,x)$ and $\tilde{U}_0^g(t,x)$ are short-range dependent in space at (t_0, x_0).

Proof. Note that, by Theorem 1 and the positivity of the Mittag-Leffler functions for negative arguments, it holds

$$\begin{aligned}
&\int_0^{+\infty} \left|\operatorname{Cov}\big(U_0^g(t_0, x_0), U_0^g(t_0 + h, x_0)\big)\right| dh \\
&\quad = C(\bar{\kappa}, \bar{w}, \bar{A}) \int_0^{+\infty} \left| \int_{\mathbb{R}} |\widehat{g}(\lambda, x_0)|^2 E_\beta(-\mu t_0^\beta |\lambda|^\alpha) E_\beta(-\mu(t_0 + h)^\beta |\lambda|^\alpha)\, d\lambda \right| dh \\
&\quad = C(\bar{\kappa}, \bar{w}, \bar{A}) \int_{\mathbb{R}} |\widehat{g}(\lambda, x_0)|^2 E_\beta(-\mu t_0^\beta |\lambda|^\alpha) \int_0^{+\infty} E_\beta(-\mu(t_0 + h)^\beta |\lambda|^\alpha) dh\, d\lambda.
\end{aligned} \tag{25}$$

It follows from (5) and $\beta \in (0,1)$ that

$$\int_0^{+\infty} E_\beta(-\mu(t_0 + h)^\beta |\lambda|^\alpha) dh \geq \int_0^{+\infty} \frac{dh}{1 + \Gamma(1-\beta)\mu|\lambda|^\alpha (t_0 + h)^\beta} = +\infty.$$

By the nondegeneracy assumption, the outer integral assigns positive mass to a set on which the inner integral is infinite. Thus, the integral in (25) is divergent and the limit random field is long-range dependent in time at (t_0, x_0).

By Theorem 1, the upper bound in (5), and the Cauchy-Schwarz inequality, it follows that the integral of the spatial covariance function can be bounded as

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\operatorname{Cov}\big(U_0^g(t_0, x_0), U_0^g(t_0, x_0 + h)\big)\right| dh \\
&\quad \leq C(\bar{\kappa}, \bar{w}, \bar{A}) \int_{\mathbb{R}} \int_{\mathbb{R}} |\widehat{g}(\lambda, x_0)||\widehat{g}(\lambda, x_0 + h)| E_\beta^2(-\mu t_0^\beta |\lambda|^\alpha)\, d\lambda dh \\
&\quad = C(\bar{\kappa}, \bar{w}, \bar{A}) \int_{\mathbb{R}} |\widehat{g}(\lambda, x_0)| \int_{\mathbb{R}} |\widehat{g}(\lambda, h)| dh\, E_\beta^2(-\mu t_0^\beta |\lambda|^\alpha)\, d\lambda \\
&\quad \leq C(\bar{\kappa}, \bar{w}, \bar{A}) \int_{\mathbb{R}} |\widehat{g}(\lambda, h)| dh \left(\int_{\mathbb{R}} |\widehat{g}(\lambda, x_0)|^2 d\lambda \int_{\mathbb{R}} (1 + C|\lambda|^\alpha)^{-4}\, d\lambda \right)^{1/2} < +\infty,
\end{aligned}$$

which implies spatial short-range dependence.

The proof for the limit field $\tilde{U}_0^g(t,x)$ follows by the same arguments. □

Remark 4. By the Cauchy-Schwarz inequality,

$$\|\hat{g}(\lambda, \cdot)\|_{L_1(\mathbb{R})} \leq \left(\int_{\mathbb{R}} (1 + |x|^2)^{-s}\, dx \right)^{1/2} \left(\int_{\mathbb{R}} (1 + |x|^2)^s |\hat{g}(\lambda, x)|^2\, dx \right)^{1/2}.$$

Therefore, if $g(\lambda,\cdot) \in H^s(\mathbb{R})$ for some $s > 1/2$, then the Fourier transform $\hat{g}(\lambda,\cdot) \in L_1(\mathbb{R})$, which, under uniformity in λ, guarantees short-range dependence in space of the limit random fields $U_0^g(t,x)$ and $\tilde{U}_0^g(t,x)$ at (x_0,t_0).

Example 5. For the case of Matérn kernels, the Fourier transform $\widehat{g}(\lambda,x)$ is given by (15) and is nondegenerate. Therefore, the corresponding limit fields $U_0^g(t,x)$ and $\widetilde{U}_0^g(t,x)$ are long-range dependent in time for all (x_0,t_0).

Notice that by (15) the function $\widehat{g}(\lambda,\cdot) \notin L_1(\mathbb{R})$. Hence, the second part of Theorem 5 cannot be applied directly in the Matérn case. However, in special cases, one can explicitly compute the covariance function and check condition (24).

Consider the case of $U_0^g(t,x)$. Let $\alpha = \beta = a = 1$. Then, as $E_1(z) = e^z$, by (11) and (15),

$$\mathrm{Cov}\big(U_0^g(t_0,x_0), U_0^g(t_0,x_0+h)\big) = C\int_{\mathbb{R}} e^{-i\lambda h} e^{-2\mu t_0|\lambda|}(1+\lambda^2)^{-(2\nu+1)}\, d\lambda.$$

First, note that since

$$P_{\mu,t_0}(h) := \int_{\mathbb{R}} e^{-i\lambda h} e^{-2\mu t_0|\lambda|}\, d\lambda = \frac{4\mu t_0}{(2\mu t_0)^2 + h^2},$$

then $P_{\mu,t_0}(\cdot) \in L_1(\mathbb{R})$.

Next, by computing the following integral we obtain

$$G_\nu(h) := \int_{\mathbb{R}} e^{-i\lambda h}(1+\lambda^2)^{-(2\nu+1)}\, d\lambda = \frac{2\sqrt{\pi}}{\Gamma(2\nu+1)}\left(\frac{|h|}{2}\right)^{2\nu+\frac{1}{2}} K_{2\nu+\frac{1}{2}}(|h|).$$

To establish the integrability of $G_\nu(\cdot)$, we examine its local behaviour at zero and its asymptotic decay at infinity. Since $K_{2\nu+\frac{1}{2}}(|h|) \sim C|h|^{-2\nu-\frac{1}{2}}$, as $|h| \to 0$, [1, formula 9.6.9], it follows that $G_\nu(h)$ is bounded in a neighbourhood of the origin. Moreover, as $K_{2\nu+\frac{1}{2}}(|h|) \sim C|h|^{-1/2}e^{-|h|}$, when $|h| \to \infty$, [1, formula 9.7.2], we obtain $G_\nu(h) \le C|h|^{2\nu}e^{-|h|}$ for sufficiently large values of h. Hence, $G_\nu(\cdot) \in L_1(\mathbb{R})$.

Because the covariance function $\mathrm{Cov}\big(U_0^g(t_0,x_0), U_0^g(t_0,x_0+h)\big)$ is the Fourier transform of the product $e^{-2\mu t_0|\lambda|}(1+\lambda^2)^{-(2\nu+1)}$, it can be written as the convolution

$$\mathrm{Cov}\left(U_0^g(t_0,x_0), U_0^g(t_0,x_0+h)\right) = C\cdot(P_{\mu,t_0} * G_\nu)(h).$$

Hence, by Young's convolution inequality, we obtain

$$\int_{\mathbb{R}} \left|\mathrm{Cov}\big(U_0^g(t_0,x_0), U_0^g(t_0,x_0+h)\big)\right| dh \le C\|P_{\mu,t_0}\|_{L_1}\|G_\nu\|_{L_1} < +\infty.$$

Therefore, the limit field $U_0^g(t,x)$ is short-range dependent in space at (t_0,x_0).

5 Conclusion

This paper investigated the limit behaviour of solutions to the FRBE with random initial conditions exhibiting both classical and cyclic long-memory structures. The multiscaling limit analysis developed in [5, 3] was extended to a broad class of kernel-smoothing transformations. Properties of the limit fields were studied. In particular, the limit fields are in general non-stationary in both space and time. Their mean-square and sample-path Hölder continuities in temporal and spatial variables were established. The short/long-range dependence properties of the limit fields in space and time were also analysed. It was demonstrated how the limit fields and their properties depend on the smoothing kernels, the parameters of the FRBE, and the locations of the spectral singularities.

It is interesting to note that the limit fields do not depend on the parameter γ, which is consistent with previous results (see, for example, [3, 10, 11]). This can be explained by the fact that the parameter γ acts as a large-scale regularisation, controlling the high-frequency part of the Riesz–Bessel operator, whereas α determines its small-scale behaviour. Therefore, γ disappears in the limit field under zoom-in asymptotics.

To some extent, Theorems 3.1 and 3.2 in [3] can be viewed, informally, as particular cases of Theorems 1 and 2, using the generalised functions, where the spatial kernel function is taken to be the Dirac delta function.

Comparing Theorem 1 with the results for the case of short-range dependent initial conditions in [11], one can note that [11] considered only the case $\beta = 1$, and the results were established using considerably more complex techniques based on the method of moments and diagram formulae. The covariance structure of the limit random field in Theorem 1 matches that in [11], when the spatial kernel function is taken as the Dirac delta function. However, [11] did not provide explicit representations of the limit fields.

Several directions for future research that arise from the present work include

1. Verifying the conjecture that the limit fields are non-degenerate only for the values of the scaling parameters ρ_1, ρ_2, and ρ_3 specified in this paper;
2. Extending the asymptotic framework to settings with regularly varying spectral densities (see, for example, [28, 39, 40]);
3. Generalising the results to subordinated initial conditions (see, for example, [8, 27, 25]);
4. Extending the analysis to multidimensional spatio-temporal random fields (see [9, 11, 41]);
5. Applying the current methodology to other classes of stochastic partial differential equations (for example, [7, 26]).
6. Extending the kernel-smoothing and multiscaling framework to Riesz–Bessel wave equations. Such an extension would require corresponding stochastic integral representations and suitable estimates for their oscillatory Fourier multipliers (see [10, 12]).

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