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Asymptotics for fractionally integrated Gaussian processes, with a focus on the Gauss-Markov case

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Abstract We investigate the fractional Riemann-Liouville integral of a general continuous Gaussian process, focusing first on the functional weak convergence of suitably rescaled processes. Building on these results and recent theoretical advances, we deduce functional large deviation principles. For small times, the asymptotic behavior depends solely on the covariance of the underlying process at zero, while for large times, appropriate rescaling of the covariance is required and additional regularity assumptions must be imposed. A central aspect of our work is the explicit characterization of the reproducing kernel Hilbert spaces of the fractionally integrated processes, including concrete formulas for the related norms. As a notable special case, when the underlying Gaussian process is Gauss-Markov, the reproducing kernel Hilbert spaces and the covariance structure can be described explicitly, providing precise insights into the corresponding rate functions.

Keywords Large Deviations, reproducing kernel Hilbert spaces, fractional integrals, Gaussian processes
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1 Introduction

Stochastic integrals and their fractional counterparts have become essential tools in modeling complex systems where past states influence the present dynamics. Such models naturally arise in physics, biology, and finance, for example in neuronal activity modeling, rough volatility, and systems with memory ([3, 7]). Fractional integration provides a flexible way to modulate the strength and decay of memory effects over time, allowing one to capture both short-range and long-range dependencies.

In this work, we study the *fractional Riemann-Liouville integral* of a continuous Gaussian process X, defined for $\alpha \in (0,1]$ by

$$\mathcal{I}^{\alpha}(X)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X_s \, ds, \quad t \geq 0,$$

where Γ denotes the Euler Gamma function. This construction generalizes the class of *fractionally integrated Gauss-Markov (FIGM) processes* ([3, 1]), which are widely used in applications due to their explicit covariance structure and representational properties. Fractionally integrated Gauss-Markov processes have recently attracted attention in different asymptotic settings. Besides large deviation questions, asymptotic properties of their extrema have also been investigated. For instance, general results on the maxima of Gaussian processes can be found in Lifshits [13], while the asymptotic behaviour of the maximum of bridges associated with fractionally integrated Gauss-Markov processes has been studied in [2]. The main focus of this paper is on the asymptotic behavior of these fractionally integrated Gaussian processes in two regimes: *small-time* and *large-time*. Our approach proceeds in two steps. First, we establish functional weak convergence results for rescaled versions of $\mathcal{I}^{\alpha}(X)$. Then, using recent theoretical results connecting weak convergence to large deviations for Gaussian measures ([5, 6]), we deduce functional large deviation principles (LDPs).

For small times, the limiting behavior is entirely determined by the covariance of X at zero. If the covariance is non-degenerate, the limit is a deterministic multiple of a power function of time; if it vanishes, the infinitesimal behavior of X near zero dictates the limit. The resulting LDPs have explicit rate functions depending on the fractional order and the covariance structure at zero.

For large times, direct application of standard results is generally precluded by divergent covariances. We therefore consider appropriately rescaled families of processes where the scaling function ensures convergence of the rescaled covariances. Under suitable regularity assumptions, such as convergence or regular variation of the covariance functions, the rescaled processes converge functionally, yielding LDPs with speed and rate functions that depend on the asymptotic behavior of the covariance and the fractional order.

A key aspect of our work is the explicit characterization of the *reproducing kernel Hilbert space* (RKHS) of $\mathcal{I}^{\alpha}(X)$, expressed in terms of the RKHS of the process X. By exploiting general results on the image of RKHS under continuous, injective linear operators ([17, 15]), we obtain concrete formulas for the RKHS and the associated norms. This allows a precise pathwise description of the processes and is essential for deriving the functional LDPs. When X is a Gauss-Markov process, explicit formulas for both RKHS and rate functions are given. The structure of the paper is as follows. In Section 2, we introduce background on Gaussian processes, fractional integration, and RKHS theory. Section 3 identifies the RKHS of fractionally integrated processes. Section 4 establishes weak convergence and large deviation principles for the small-time regime. Section 5 treats weak convergence and large deviation principles for the large-time regime. Finally, Section 6 provides explicit computations for the case where the integrand is a Gauss–Markov process, including covariance, RKHS, and rate function formulas.

2 Preliminary results

2.1 *Large deviations in small-time for continuous Gaussian processes*

In this section, we recall some known facts about large deviations that will be needed in the sequel. Let E be a topological space. A *(good) rate function* on E is a lower semicontinuous mapping $I : E \longrightarrow [0, +\infty]$ such that the level sets $\{I \leq a\}$ are compact for every $a \geq 0$ whereas a *speed function* v is a function $\mathbb{R}^+ \to \mathbb{R}^+$ such that $v(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

Definition 2.1. Let E be a topological space, $\mathscr{B}(E)$ its Borel σ-algebra and $\{\mu_\varepsilon\}_{\varepsilon>0}$ a family of probabilities on $\mathscr{B}(E)$. We say that $\{\mu_\varepsilon\}_{\varepsilon>0}$ satisfies a large deviations principle (LDP) on E with rate function I and speed v if for every closed set $F \subset E$

$$\limsup_{\varepsilon\to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(F) \leq - \inf_{x\in F} I(x)$$

and for every open set $G \subset E$

$$\liminf_{\varepsilon\to 0} \frac{1}{v(\varepsilon)} \log \mu_\varepsilon(G) \geq - \inf_{x\in G} I(x).$$

Definition 2.2. Let T be an arbitrary interval of $\mathbb{R}$ and let $K : T \times T \longrightarrow \mathbb{R}$ be a positive-definite covariance. The *reproducing kernel Hilbert space* associated with the covariance K is the completion $\mathbb{H}$ of the linear space of all functions

$$t \to \sum_{i=1}^{n} \alpha_i K(s_i, t), \quad \alpha_1, \ldots, \alpha_n \in \mathbb{R},\ s_1, \ldots, s_n \in T,\ n \in \mathbb{N},$$

relative to the norm induced by the inner product

$$\Big\langle \sum_{i=1}^{n} \alpha_i K(s_i, \cdot), \sum_{j=1}^{m} \beta_j K(t_j, \cdot) \Big\rangle_{\mathbb{H}} = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_i \beta_j K(s_i, t_j).$$

We notice that if $\phi(t) = \sum_{i=1}^{n} \alpha_i K(s_i, t)$ and $K_s(\cdot) := K(s, \cdot)$, then

$$\phi(t) = \langle \phi, K_t \rangle_{\mathbb{H}}.$$

This is called the *reproducing property*.

RKHSs are described in many textbooks, the reader can refer to Chapter 4 (in particular Section 4.3) in Hida&Hitsuda's book [11] or Chapter 2 (in particular Sections 2.2 and 2.3) in Berlinet&Thomas-Agnan's book [8]. RKHSs play a fundamental role in the theory of large deviations for Gaussian processes.

From now on, we consider *real centered continuous Gaussian processes*, $Z = (Z_t)_{t\in[0,T]}$, which leads to the consideration of Gaussian measures on the Banach space $\mathscr{C} := \mathscr{C}([0,T],\mathbb{R})$ of the continuous paths $[0,T] \to \mathbb{R}$, endowed with the supremum norm $\|\ \|_\infty$. Recent works have shown that under some assumptions, if we rescale a family of Gaussian continuous processes $Z^\varepsilon = (Z_t^\varepsilon)_{t\in[0,T]}$, converging weakly to some Gaussian process $Z^0 = (Z_t^0)_{t\in[0,T]}$, by a factor $v^{-\frac{1}{2}}(\varepsilon)$, where v is a speed function, we get a family of processes satisfying a LDP with speed function v and rate function depending on the limit law Z^0.

The first result in this direction is contained in a work by P. Baldi (see [5]), showing a link between the tightness of a family of Gaussian random variables and the uniform exponential tightness at some speed v of the same family rescaled by a factor $v^{-\frac{1}{2}}$. This correspondence leads to the following result proved by P. Baldi and B. Pacchiarotti (see Theorem 3.3 in [6]).

Theorem 2.3. *For $T > 0$, let $\{Z^\varepsilon\}_{\varepsilon>0}$, be a tight family of continuous centered real Gaussian processes converging in law to some process Z^0.*

Then, for every speed function v, the family $\{v(\varepsilon)^{-\frac{1}{2}} Z^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP in $\mathscr{C}$ at speed v and rate function

$$I(h) = \begin{cases} \frac{1}{2}\|h\|^2_{\mathscr{Z}_0}, & h \in \mathscr{Z}_0 \\ +\infty, & \textit{otherwise}, \end{cases}$$

where $\mathscr{Z}_0$ and $\|\ \|_{\mathscr{Z}_0}$ respectively denote the RKHS and the related norm associated to the limit process Z^0.

2.2 *Fractionally integrated Gaussian processes*

In this section we introduce and study a class of Gaussian processes obtained via fractional integration, which will constitute the fundamental object of this work. We begin by recalling the notion of fractional integral and fractional derivative in the sense of Riemann-Liouville, clarifying some definitions and structural relations that will be used later on. In particular, we focus on fractional Riemann-Liouville integrals applied to continuous Gaussian processes, leading to the notion of fractionally integrated Gaussian (FIG) processes.

After recalling the definition of the fractional Riemann–Liouville integral, we define the associated fractional integral of a continuous Gaussian process and discuss some of its basic properties. These processes remain Gaussian, inherit important structural features from the original process, and exhibit enhanced path regularity. In particular, the FIG processes are Hölder continuous, with a Hölder exponent determined by the order of fractional integration. This regularity property will be crucial in the subsequent analysis, especially in the study of tightness for families of fractionally integrated processes.

Definition 2.4. For $f \in L^1([0,T])$ and $\alpha \in (0,1)$, its *(left) fractional Riemann-Liouville integral* of order α is defined by

$$\mathcal{I}^\alpha(f)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad t \in [0,T].$$

where Γ is the Gamma Euler function, i.e. $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t}\, dt$.

Definition 2.5. For $\alpha \in (0,1)$, a function f admits a *(left) fractional Riemann-Liouville derivative* of order α, if the following expression is well defined:

$$(\mathcal{D}^\alpha f)(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s)\, ds, \qquad t \in [0,T].$$

Alternative definitions of the fractional operators are possible, like the so-called Caputo fractional derivative. An in-depth treatment of fractional operators can be found in [16]. It is easy to see that if $\alpha \to 1^-$, the Riemann-Liouville integral coincides with the ordinary integral, while the fractional Riemann-Liouville derivative of order 1 provides the ordinary derivative. Furthermore, the fractional integral is the left inverse of the fractional derivative of the same order on the image $\mathcal{I}^\alpha(L^1([0,T])$ (see Theorem 2.4 in [16]), i.e.,

$$\mathcal{D}^\alpha \mathcal{I}^\alpha(f)(t) = f(t). \tag{2.1}$$

Having introduced the fractional integral, we are now ready to define the class of fractionally integrated Gaussian processes.

Definition 2.6. (FIG) Let $(X_t)_{t\in[0,T]}$ be a continuous Gaussian process. The fractional integral of order $\alpha \in (0,1]$ of X is the process

$$\mathcal{I}^\alpha(X)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} X_s\, ds\ ,\ t \in [0,T]. \tag{2.2}$$

Remark 2.7. In the special case where X is constant, namely $X_s = X_0$ for every $s \in [0,T]$, where X_0 is a fixed random variable, the fractional integral can be computed explicitly. Indeed,

$$\mathcal{I}^\alpha(X)(t) = \frac{X_0}{\Gamma(\alpha+1)} t^\alpha, \quad t \in [0,T]. \tag{2.3}$$

Thus, in this case, it is a deterministic multiple of X_0.

Note that X is continuous, so the integral process $\mathcal{I}^\alpha(X)$ is well defined and it is adapted in the same probability space as X. The fractionally integrated processes defined in (2.2) (and (2.3)) are Gaussian processes. Under our assumptions, FIG processes are Hölder-continuous. It is well known that the Riemann-Liouville integral maps the Hölder space H^λ $(0 \le \lambda \le 1)$ into $H^{\lambda+\alpha}$, provided that $\alpha + \lambda < 1$ (see Theorem 3.1 in [16]).

3 RKHS of fractionally integrated Gaussian processes

In order to determine the RKHS of the FIG processes, we can make use of a very general result in a work by W. van der Vaart and J.H. van Zanten (see Lemma 7.1 in [17]), which we recall here for completeness,

Lemma 3.1. *Let $T : \mathbb{B} \to \mathbb{B}'$ be a one-to-one, continuous, linear map from a separable Banach space $\mathbb{B}$ into a Banach space $\mathbb{B}'$, and let W be a Borel measurable, zero-mean Gaussian random element in $\mathbb{B}$ with RKHS $\mathbb{H}$. Then the RKHS of the Gaussian random element TW in $\mathbb{B}'$ is equal to $T\mathbb{H}$, and the map*

$$T : \mathbb{H} \to T\mathbb{H}$$

is an isometry with respect to the corresponding RKHS norms.

Thanks to this result, we can explicitly characterize the RKHS of the FIG processes, as shown in the following proposition.

Proposition 3.2. *Let $(X_t)_{t\in[0,T]}$ a Gaussian, centered, continuous process with RKHS $\mathcal{X}$. Then, the RKHS of the process $\mathcal{I}^\alpha(X)$ is the image of $\mathcal{X}$, i.e.*

$$\mathrm{Im}(\mathcal{I}^\alpha(\mathcal{X})) = \left\{\ell \in \mathscr{C} \middle| \, \ell(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} h(u)\, du,\ t \in [0,T],\ h \in \mathcal{X}\right\},$$

equipped with the norm

$$\|\ell\|_{\mathrm{Im}(\mathcal{I}^\alpha(\mathcal{X}))} = \|\mathcal{D}^\alpha \ell\,\|_{\mathcal{X}}.$$

Proof. The result follows directly from the general lemma of van der Vaart and van Zanten discussed above. Indeed, consider the fractional integral operator

$$\mathcal{I}^\alpha : \mathscr{C} \to \mathscr{C}, \quad \mathcal{I}^\alpha(h)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} h(u)\, du.$$

The operator $\mathcal{I}^\alpha$ is linear and continuous from $\mathscr{C}$ to $\mathscr{C}$ because, for all $h \in \mathscr{C}$,

$$\|\mathcal{I}^\alpha h\|_\infty \le \frac{1}{\Gamma(\alpha)} \int_0^T (T-u)^{\alpha-1} |h(u)|\, du \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|h\|_\infty.$$

Furthermore $\mathcal{I}^\alpha$ is one-to-one, thanks to the fractional differentiation formula (2.1), which allows to recover h from $\ell = \mathcal{I}^\alpha h$.

Therefore, the RKHS of $\mathcal{I}^\alpha(X)$ is exactly the image of the RKHS $\mathcal{X}$ of X under $\mathcal{I}^\alpha$, with the RKHS norm preserved. □

Remark 3.3. If the process X is constant, i.e. $K_X(u,v) = K_X(0,0) \neq 0$ for every $u,v \in [0,T]$, then the RKHS of the integrated process can be identified explicitly. It coincides with

$$\mathcal{A} = \{a : [0,T] \to \mathbb{R} \mid a(t) = k\, t^\alpha,\ k \in \mathbb{R}\},$$

endowed with the scalar product

$$\langle f, g\rangle_{\mathcal{A}} = c_f c_g \frac{\Gamma^2(\alpha+1)}{K_X(0,0)}, \qquad f(t) = c_f t^\alpha,\ g(t) = c_g t^\alpha.$$

With this inner product, $\mathcal{A}$ is a one-dimensional Hilbert space. Moreover, it is the RKHS associated with the covariance

$$K_{\mathcal{I}^\alpha(X)}(s,t) = \frac{K_X(0,0)}{\Gamma^2(\alpha+1)}\, s^\alpha t^\alpha,$$

meaning that the covariance sections belong to $\mathcal{A}$ and the reproducing property holds for every $f \in \mathcal{A}$.

4 Weak convergence and LDP for small-time

In this section we prove a functional weak convergence and a large deviations principle for the family of small-time processes under consideration. More precisely we consider the process $Y^{\alpha,\varepsilon}$, for $\varepsilon \to 0$, defined as the fractional Riemann-Liouville integral in small-time

$$Y_t^{\alpha,\varepsilon} = \mathcal{I}^\alpha(X)(\varepsilon t), \qquad t \in [0,T].$$

Our strategy follows the approach outlined in the preliminaries: we first establish the convergence in law of a suitably rescaled version of the process. To prove this functional weak convergence, we will show the tightness of the family and the convergence of the covariance function, which is equivalent to establishing the weak convergence of the finite-dimensional distributions. Once this convergence is established, it follows that the original family satisfies a large deviations principle with a rate function characterized in terms of the RKHS associated with the limiting process.

The choice of the appropriate scaling factor for the rescaled process depends on the behavior of the covariance function at zero. If $K_X(0,0) \neq 0$, the randomness in the small-time limit is dominated by the initial condition, so a simple scaling by $\varepsilon^{-\alpha}$ is sufficient. On the other hand, if $K_X(0,0) = 0$, the order of the standard scaling is not enough to produce a non-trivial limit. By Theorem 3.10 in [6], we obtain precise indications on the correct normalization needed to ensure convergence of the finite-dimensional distributions of the rescaled process to be integrated. In particular, the theorem suggests that such normalization should be asymptotically equivalent to $K_X(\varepsilon,\varepsilon)$ as $\varepsilon \to 0$, which naturally leads to considering the scaling factor $(\varepsilon^{2\alpha}K_X(\varepsilon,\varepsilon))^{-1/2}$. This, in turn, motivates imposing the additional condition on the function $G_\varepsilon(u,v) := \frac{K_X(\varepsilon u,\varepsilon v)}{K_X(\varepsilon,\varepsilon)}$,

$$\lim_{\varepsilon\to 0} G_\varepsilon(u,v) = \hat{K}_X(u,v), \qquad u,v \in [0,T], \tag{4.1}$$

which can be interpreted as a multivariate regular variation property of the covariance function. Under this condition, the limit covariance $\hat{K}_X$ is homogeneous (see Appendix 1 in [9]). Consequently, the limiting integrand is self-similar.

Proposition 4.1. *Let $(X_t)_{t\in[0,T]}$ be a centered Gaussian continuous process with covariance function K_X, then the family of processes $\{\varepsilon^{-\alpha}Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ is tight.*

Moreover, assume that $K_X(0,0) = 0$ and that $|G_\varepsilon| \leq H$ for some $H \in L^p([0,T]^2)$, with $p > \frac{1}{\alpha}$, then the family of processes $\{\varepsilon^{-\alpha}K_X(\varepsilon,\varepsilon)\big)^{-1/2}Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ is also tight.

Proof. We verify that the assumptions of Theorem 23.7 in [12] are satisfied. These assumptions reduce to the Kolmogorov-Chentsov tightness condition. Suppose first $K_X(0,0) = 0$. Since $Y_0^{\alpha,\varepsilon} = 0$ for all $\varepsilon > 0$, to verify the Kolmogorov-Chentsov condition for the family $\{\varepsilon^{-\alpha}K_X(\varepsilon,\varepsilon)^{-1/2}Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$, it is enough to obtain a uniform estimate on the moments of the increments

$$\varepsilon^{-\alpha}K_X(\varepsilon,\varepsilon)^{-1/2}\big(Y_t^{\alpha,\varepsilon} - Y_s^{\alpha,\varepsilon}\big).$$

Without loss of generality, let $0 \leq s \leq t \leq T$. Then

$$\begin{aligned}
&\mathrm{Var}\Big(K_X^{-\frac{1}{2}}(\varepsilon,\varepsilon)\varepsilon^{-\alpha}(Y_t^{\alpha,\varepsilon} - Y_s^{\alpha,\varepsilon})\Big) \\
&\qquad = \varepsilon^{-2\alpha}K_X^{-1}(\varepsilon,\varepsilon)\Big(\mathrm{Var}(Y_t^{\alpha,\varepsilon}) + \mathrm{Var}(Y_s^{\alpha,\varepsilon}) - 2\mathrm{Cov}(Y_s^{\alpha,\varepsilon}, Y_t^{\alpha,\varepsilon})\Big) \\
&\qquad = \frac{1}{\Gamma^2(\alpha)}\Bigg[\int_0^t\int_0^t (t-u)^{\alpha-1}(t-v)^{\alpha-1}G_\varepsilon(u,v)\,du\,dv
\end{aligned}$$

$$+\int_0^s\int_0^s (s-u)^{\alpha-1}(s-v)^{\alpha-1}G_\varepsilon(u,v)\,du\,dv$$
$$\left.-2\int_0^s\int_0^t (s-u)^{\alpha-1}(t-v)^{\alpha-1}G_\varepsilon(u,v)\,du\,dv\right].$$

Using the addition property, we obtain

$$\operatorname{Var}\bigl(\varepsilon^{-\alpha}\,K_X^{-1}(\varepsilon,\varepsilon)(Y_t^{\alpha,\varepsilon}-Y_s^{\alpha,\varepsilon})\bigr)\le\frac{1}{\Gamma^2(\alpha)}(A_\varepsilon+B_\varepsilon+C_\varepsilon+D_\varepsilon),$$

where

$$\begin{aligned}
0\le A_\varepsilon &= \int_0^s\bigl[(s-u)^{\alpha-1}-(t-u)^{\alpha-1}\bigr]du\int_0^t (t-v)^{\alpha-1}|G_\varepsilon(u,v)|dv,\\
0\le B_\varepsilon &= \int_0^s (s-u)^{\alpha-1}du\int_0^s\bigl[(s-v)^{\alpha-1}-(t-v)^{\alpha-1}\bigr]|G_\varepsilon(u,v)|\,dv,\\
0\le C_\varepsilon &= \int_s^t (t-u)^{\alpha-1}du\int_0^t (t-v)^{\alpha-1}|G_\varepsilon(u,v)|\,dv,\\
0\le D_\varepsilon &= \int_0^s (s-u)^{\alpha-1}du\int_s^t (t-v)^{\alpha-1}|G_\varepsilon(u,v)|\,dv.
\end{aligned}$$

We choose $p>\frac{1}{\alpha}$ such that $H\in L^p([0,T]^2)$ and define $M=\|H\|_p$. The conjugate exponent q is such that $\frac{1}{q}=1-\frac{1}{p}>1-\alpha$, therefore $q(1-\alpha)<1$ and by Hölder inequality, we have

$$\begin{aligned}
A_\varepsilon &\le M\left(\int_0^s\bigl[(s-u)^{\alpha-1}-(t-u)^{\alpha-1}\bigr]^q du\int_0^t (t-v)^{q(\alpha-1)}dv\right)^{\frac{1}{q}}\\
&\le M\left(\frac{t^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}}\left(\int_0^s\frac{[(t-u)^{1-\alpha}-(s-u)^{1-\alpha}]^q}{(s-u)^{q(1-\alpha)}(t-u)^{q(1-\alpha)}}du\right)^{\frac{1}{q}}.
\end{aligned}$$

Using the concavity inequality $(t-u)^{1-\alpha}-(s-u)^{1-\alpha}\le(t-s)^{1-\alpha}$, we can conclude

$$A_\varepsilon\le M\frac{T^{\frac{2}{q}+3(\alpha-1)}}{[(q(\alpha-1)+1)(2q(\alpha-1)+1)]^{\frac{1}{q}}}(t-s)^{1-\alpha}.$$

The term B_ε can be bounded from above in a similar way.

$$\begin{aligned}
C_\varepsilon &\le M\left(\int_s^t \bigl(t-u)^{q(\alpha-1)}du\int_0^t (t-v)^{q(\alpha-1)}dv\right)^{\frac{1}{q}}\\
&\le M\left(\frac{t^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}}\left(\frac{(t-s)^{q(\alpha-1)+1}}{q(\alpha-1)+1}\right)^{\frac{1}{q}}\\
&\le\left(\frac{T^{q(\alpha-1)+1}}{(q(\alpha-1)+1)^2}\right)^{\frac{1}{q}}(t-s)^{\alpha-1+\frac{1}{q}},
\end{aligned}$$

The estimation for the term D_ε is analogous. The terms A_ε and B_ε are controlled by a power $(t-s)^{1-\alpha}$, while the other two are controlled by a power $(t-s)^{\alpha-1+\frac{1}{q}}$. Then, summing the four contributions, we obtain for $0 \le s < t \le T$:

$$\mathrm{Var}\left(\varepsilon^{-\alpha} K_X^{-1}(\varepsilon,\varepsilon)(Y_t^{\alpha,\varepsilon} - Y_s^{\alpha,\varepsilon})\right) \le C(\alpha,p,T)\cdot(t-s)^{\beta},$$

where $C(\alpha,p,T)$ is a constant depending on the parameters $\alpha,\ p,\ T$ and

$$\beta = \min\left\{\alpha - \frac{1}{p}, 1-\alpha\right\} \in (0,1).$$

The case $K_X(0,0) \neq 0$ is simpler. Indeed, no normalization is needed and the covariance $K_X(\varepsilon u, \varepsilon v)$ appears directly, replacing $G_\varepsilon(u,v)$. Since K_X is continuous on the compact set $[0,T]^2$, it is uniformly bounded. Therefore, in this case $K_X(\varepsilon u, \varepsilon v)$ is automatically controlled by a constant.

Since the processes are Gaussian, the variance estimates obtained above imply corresponding bounds for moments of arbitrary order. Hence, choosing an integer $m \ge 1$ such that $m\beta > 1$, we obtain

$$\sup_{\varepsilon>0} \mathbb{E}\left[\left|\varepsilon^{-\alpha} K_X(\varepsilon,\varepsilon)^{-1/2}(Y_t^{\alpha,\varepsilon} - Y_s^{\alpha,\varepsilon})\right|^{2m}\right] \le C_m |t-s|^{m\beta},$$

and Kolmogorov-Chentsov's criterion yields tightness in $\mathscr{C}$. The same argument applies in the case $K_X(0,0) \neq 0$.

□

Theorem 4.2. *Let $(X_t)_{t\in[0,T]}$ be a centered Gaussian continuous process with covariance function K_X, then*

1. *if $K_X(0,0) \neq 0$, then as $\varepsilon \to 0$, the rescaled process $\{\varepsilon^{-\alpha} Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ weakly converges, in $\mathscr{C}$ to $\tilde{Y}^{\alpha,0} = \mathcal{I}^\alpha(\tilde{X})$, where $\tilde{X}$ is a constant process equal in law to X_0;*
2. *if $K_X(0,0) = 0$, assume that $|G_\varepsilon| \le H$ for some $H \in L^p([0,T]^2)$, $p > \frac{1}{\alpha}$, and that (4.1) holds. Then, as $\varepsilon \to 0$, the family of rescaled processes $\{\varepsilon^{-\alpha} K_X(\varepsilon,\varepsilon)^{-1/2} Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ weakly converges in $\mathscr{C}$ to $\hat{Y}^{\alpha,0} = \mathcal{I}^\alpha(\hat{X})$ where $\hat{X}$ is a centered Gaussian process with covariance $\hat{K}_X$.*

Proof. The proof proceeds in two steps in both cases. Step 1: convergence of finite-dimensional distributions. For each case, we consider arbitrary times $0 \le s \le t \le T$ and compute the covariance of the rescaled process. If $K_X(0,0) \neq 0$, since K_X is continuous, it is measurable and bounded on compact sets. Hence, the integrand in the covariance representation is dominated by

$$\frac{M}{\Gamma^2(\alpha)}(s-u)^{\alpha-1}(t-v)^{\alpha-1},$$

which is integrable on $[0,s]\times[0,t]$.

The Dominated Convergence Theorem gives

$$\lim_{\varepsilon\to 0} \varepsilon^{-2\alpha}\mathrm{Cov}(Y_t^{\alpha,\varepsilon}, Y_s^{\alpha,\varepsilon}) = \frac{1}{\Gamma^2(\alpha+1)} K_X(0,0)\, s^\alpha t^\alpha,$$

which shows that the finite-dimensional distributions converge to those of $\tilde{Y}^{\alpha,0}$.

If $K_X(0,0) = 0$, note that G_ε and H are measurable and that

$$|G_\varepsilon(u,v)| \le H(u,v), \qquad (u,v) \in [0,T]^2.$$

since H is in $L^p([0,T]^2)$, for some $p > \frac{1}{\alpha}$, Hölder inequality combined with the dominated convergence theorem gives

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{-2\alpha}}{K_X(\varepsilon,\varepsilon)} \mathrm{Cov}(Y_t^{\alpha,\varepsilon}, Y_s^{\alpha,\varepsilon}) = \frac{1}{\Gamma^2(\alpha)} \int_0^s \int_0^t (s-u)^{\alpha-1}(t-v)^{\alpha-1} \hat{K}_X(u,v)\, du\, dv,$$

which shows that the finite-dimensional distributions converge to those of $\hat{Y}^{\alpha,0}$.

Step 2: functional convergence. Proposition 4.1 ensures tightness of the rescaled families. Combined with the convergence of finite-dimensional distributions shown in Step 1, this yields convergence in law in $\mathscr{C}$ to the corresponding limiting processes $\tilde{Y}^{\alpha,0}$ and $\hat{Y}^{\alpha,0}$.

□

Now we have all the tools to establish an LDP in small-time for FIG processes. We will combine the results in [6] with the weak-convergence results that we have just proven. We will also show that it is not necessary to consider centered FIG processes, since the mean of the small-times processes converges uniformly to 0.

Theorem 4.3. *Let $(X_t)_{t\in[0,T]}$ be a Gaussian continuous process with covariance function K_X, then*

1. *if $K_X(0,0) \neq 0$, then the family $\{Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ satisfies a LDP in $\mathscr{C}$ with speed $v(\varepsilon) = \varepsilon^{-2\alpha}$ and rate function*

$$J_0(h) = \begin{cases} \frac{1}{2}\frac{\Gamma^2(\alpha+1)}{K_X(0,0)} \cdot c^2, & \text{if } h(t) = c \cdot t^\alpha,\ c \in \mathbb{R},\ t \in [0,T] \\ +\infty, & \text{otherwise;} \end{cases} \qquad (4.2)$$

2. *if $K_X(0,0) = 0$, assume that $|G_\varepsilon| \le H$ for some $H \in L^p([0,T]^2)$, $p > \frac{1}{\alpha}$, and that (4.1) holds. Then the family $\{Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ satisfies a LDP in $\mathscr{C}$ with speed $v(\varepsilon) = \varepsilon^{-2\alpha} K_X^{-1}(\varepsilon,\varepsilon)$ and rate function*

$$\hat{J}_0(h) = \begin{cases} \frac{1}{2}\|\mathcal{D}^\alpha h\|^2_{\hat{\mathcal{X}}}, & h \in \mathrm{Im}(\mathcal{I}^\alpha(\hat{\mathcal{X}})) \\ +\infty, & \text{otherwise,} \end{cases} \qquad (4.3)$$

where $\hat{\mathcal{X}}$ and $\|\cdot\|_{\hat{\mathcal{X}}}$ respectively denote the RKHS and the related norm associated with the limit covariance $\hat{K}_X$ defined in (4.1).

Proof. First, suppose that X is a centered process. The large deviation principles in both cases follow from Theorem 4.2 together with Theorem 2.3, which provide the small-time LDP for the rescaled Gaussian processes in $\mathscr{C}$ under the stated assumptions. More precisely, Theorem 4.2 identifies the family of processes that admits a non-trivial limit under the considered rescaling, as well as its limiting process, which in turn determines the associated reproducing kernel Hilbert space structure governing the large deviations. Theorem 2.3 then shows that such a rescaled family satisfies a large deviation principle with speed given by the inverse of the square of the scaling parameter. More precisely, since we are interested in a small-time large deviation principle for the original (non-reparametrized) process, the correct normalization is determined by the short-time behaviour of the covariance. In case (1), the assumption $K_X(0,0) \neq 0$ yields the appropriate normalization with speed $v(\varepsilon) = \varepsilon^{-2\alpha}$. In case (2), the additional hypotheses ensure that the rescaled process satisfies a small-time LDP with speed $v(\varepsilon) = \varepsilon^{-2\alpha} K_X(\varepsilon,\varepsilon)^{-1}$.

Finally, the explicit form of the rate functions in (4.2) and (4.3) follows from Remark 3.3 for the first case and Proposition 3.2 for the second. These results characterize the RKHS of the corresponding (limit) Gaussian process and express the rate function in terms of the associated RKHS norm.

If the process is not centered, let us denote $\mathbb{E}[X_t] = m_X(t)$. Then $\mathbb{E}[Y_t^{\alpha,\varepsilon}] = \varepsilon^\alpha \int_0^t (t-u)^{\alpha-1} m_X(\varepsilon u) du$, converges uniformly to 0 as $\varepsilon \to 0$; therefore the two families $\{(Y_t^{\alpha,\varepsilon})_{t\in[0,T]}\}_{\varepsilon>0}$ and $\{(Y_t^{\alpha,\varepsilon} - \mathbb{E}[Y_t^{\alpha,\varepsilon}])_{t\in[0,T]}\}_{\varepsilon>0}$ are exponentially equivalent (at the right speed) and satisfy the same LDP. $\square$

Remark 4.4. It is worth emphasizing that if $\hat{K}_X(u,v) = 1$ for every $u,v \in [0,T]$, namely if $\hat{K}_X$ is homogeneous of degree zero, then the limiting behavior coincides with that of the case $K_X(0,0) \neq 0$, although the normalization is the same as in the case $K_X(0,0) = 0$. More precisely, the family of processes $\{\varepsilon^{-\alpha} K_X(\varepsilon,\varepsilon)^{-1/2} Y^{\alpha,\varepsilon}\}_{\varepsilon>0}$ weakly converges in $\mathscr{C}$ to a Gaussian process of the form described in (2.3), where a standard Gaussian random variable replaces X_0. Furthermore, the family $\{Y^{\alpha,\varepsilon}\}_{\varepsilon\geq 0}$ satisfies a LDP in $\mathscr{C}$ with speed $v(\varepsilon) = \varepsilon^{-2\alpha} K_X^{-1}(\varepsilon,\varepsilon)$ and rate function as in (4.2), with 1 in place of $K_X(0,0)$.

Remark 4.5. An alternative derivation of Theorem 4.3 can be obtained via the *contraction principle*. Indeed, if the family of suitably rescaled small-time processes $\{(v(\varepsilon)^{-1/2} X_{\varepsilon t})_{t\in[0,T]}\}_{\varepsilon>0}$ satisfies a large deviations principle, then, since the fractional integral operator $\mathcal{I}^\alpha$ is continuous, the contraction principle implies that the image family also satisfies a LDP, with a rate function coinciding with those in (4.2) and (4.3).

Compared to this contraction principle approach, the method adopted in this paper requires fewer assumptions on the original process X. In particular, the limiting covariance function may not define a continuous process, yet our approach still applies by directly exploiting the Gaussian structure and the RKHS of the fractional integral.

In both cases, the limiting process can be represented as the fractional integral of a Gaussian process with a well-defined limiting covariance structure. In the first scenario, the limit explicitly retains a stochastic dependence on the initial condition X_0, propagated through the fractional integration. In the second scenario, by contrast, the non-trivial limiting behavior emerges from the rescaled covariance function, giving rise to a new Gaussian structure.

5 Weak convergence and LDP for large time

At this stage, we consider the process $Z^{\alpha,\rho}$, for $\rho \to \infty$, defined as the fractional Riemann-Liouville integral in large-time

$$Z_t^{\alpha,\rho} = \mathcal{I}^\alpha(X)(\rho t), \qquad t \in [0,T],$$

and investigate whether the results obtained in the previous sections can be extended. We consider continuous Gaussian processes $(X_t)_{t\geq 0}$. As noted above, fractional integration amplifies long-range dependence as the fractional order increases. Since the covariance functions of FIG processes involve double integrals, they typically diverge as time grows, which motivates the study of the suitably normalized family of processes.

With the usual change of variables one readily observes that the covariance of the large-time FIG processes scales by a factor $\rho^{2\alpha}$, which diverges as $\rho \to +\infty$. Therefore, the limiting behavior is influenced by the asymptotic properties of the covariance K_X, i.e. the behavior of $K_X(\rho u, \rho v)$ as $\rho \to +\infty$. Several scenarios may occur: it may diverge, converge to a constant, or have no limit. In our analysis, we start with the case where it converges to a non-zero constant.

Proposition 5.1. *Let $(X_t)_{t\geq 0}$ be a centered Gaussian continuous process with covariance function K_X, and suppose that, for every $u, v \in (0, T]$,*

$$\lim_{\rho\to+\infty} K_X(\rho u, \rho v) = K_\infty \neq 0. \tag{5.1}$$

Then,

1. *the family of processes $\{(\rho^{-\alpha} Z^{\alpha,\rho}\}_{\rho>0}$ converges weakly in $\mathscr{C}$ (as $\rho \to +\infty$) to $\tilde{Z}^{\alpha,\infty} = \mathcal{I}^\alpha(\tilde{X}_\infty)$, where X_∞ is a real-valued centered Gaussian random variable with variance K_∞;*

2. *if ψ is an infinitesimal function then the family $\{\psi(\rho)\rho^{-\alpha} Z^{\alpha,\rho}\}_{\rho>0}$ satisfies an LDP in $\mathscr{C}$ with speed $v(\rho) = \psi^{-2}(\rho)$ and rate function*

$$\tilde{J}_\infty(h) = \begin{cases} \frac{1}{2}\frac{\Gamma^2(\alpha+1)}{K_\infty} \cdot c^2, & \text{if } h(t) = c\cdot t^\alpha,\ c\in\mathbb{R},\ t\in[0,T] \\ +\infty, & \text{otherwise.} \end{cases}$$

Proof. The tightness of the rescaled family can be shown exactly as in Proposition 4.1, we just need to observe that K_X is uniformly bounded on $[0, +\infty)^2$. We can still apply the dominated convergence theorem under these assumptions, as $\rho \to +\infty$, in order to get the convergence of the finite-dimensional distributions, which, combined with the tightness of the family, gives the functional convergence in $\mathscr{C}$, as in Theorem 4.2. The LDP follows from Theorem 2.3, while the expression for the rate function can be obtained as in Remark 3.3. □

If (5.1) is not verified we will assume that

$$\lim_{\rho\to+\infty} \frac{K_X(\rho u, \rho v)}{K_X(\rho, \rho)} = \hat{K}_\infty(u, v), \quad u, v \in (0, T], \tag{5.2}$$

for some $\hat{K}_\infty$ not identically zero. This is exactly the same assumption we made in the small-time case. However, in the present setting, the behavior of $K_X(\rho, \rho)$ may either enhance or reduce the speed, since it can tend to zero or to infinity as $\rho \to 0$. The correct scaling factor to prove the tightness and the functional convergence is $K_X^{-\frac{1}{2}}(\rho, \rho)\, \rho^{-\alpha}$.

Proposition 5.2. *Let $(X_t)_{t\geq 0}$ be a centered Gaussian continuous process with covariance function K_X, and let $G_\rho(u,v)$ denote $\frac{K_X(\rho u,\rho v)}{K_X(\rho,\rho).}$ We assume that condition (5.2) holds and that $|G_\rho| \leq H \in L^p([0,T]^2)$, for some $p > \frac{1}{\alpha}$. Then,*

1. *the family of processes $\{\rho^{-\alpha} K_X^{-\frac{1}{2}}(\rho,\rho) Z^{\alpha,\rho}\}_{\rho>0}$ converges weakly in $\mathscr{C}$ (as $\rho \to +\infty$) to $\hat{Z}^{\alpha,\infty} = \mathcal{I}^\alpha(\hat{X}_\infty)$, where $\hat{X}_\infty$ is a the centered Gaussian process with covariance $\hat{K}_\infty$;*

2. *if ψ is an infinitesimal function then the family $\{\psi(\rho)\rho^{-\alpha} K_X^{-\frac{1}{2}}(\rho,\rho) Z^{\alpha,\rho}\}_{\rho>0}$ satisfies an LDP in $\mathscr{C}$ with speed $v(\rho) = \psi^{-2}(\rho)$ and rate function*

$$\hat{J}_\infty(h) = \begin{cases} \frac{1}{2}\|\mathcal{D}^\alpha h\|^2_{\hat{\mathcal{X}}_\infty}, & h \in \mathrm{Im}(\mathcal{I}^\alpha(\hat{\mathcal{X}}_\infty)) \\ \\ +\infty, & \textit{otherwise}, \end{cases}$$

where $\hat{\mathcal{X}}_\infty$ and $\|\cdot\|_{\hat{\mathcal{X}}_\infty}$ respectively denote the RKHS and the related norm associated with the limit covariance $\hat{K}_\infty$ defined in (5.1).

Proof. The proof is the same as in Proposition 5.1, the only difference being the rescaling factor $\rho^{-\alpha} K_X(\rho,\rho)^{-1/2}$. Indeed, with this normalization the covariance structure is rewritten in terms of

$$G_\rho(u,v) = \frac{K_X(\rho u,\rho v)}{K_X(\rho,\rho)},$$

so that all the arguments in Proposition 5.1 apply with $K_X(\rho u,\rho v)$ replaced by $G_\rho(u,v)$. The tightness of the rescaled family is proved exactly as in Proposition 4.1. Furthermore, since $G_\rho(u,v) \to \hat{K}_\infty(u,v)$ pointwise and $|G_\rho| \leq H$ for some $H \in L^p([0,T]^2)$, with $p > \frac{1}{\alpha}$, the dominated convergence theorem can be applied to the covariance integrals defining the finite-dimensional distributions. Therefore, the finite-dimensional distributions converge to those of $\hat{Z}^{\alpha,\infty} = \mathcal{I}^\alpha(\hat{X}_\infty)$. Combining tightness and convergence of finite-dimensional distributions yields weak convergence in $\mathscr{C}$, as in Theorem 4.2. The LDP follows from Theorem 2.3, while the expression of the rate function is obtained as in Proposition 3.2. □

Remark 5.3. If the limiting covariance $\hat{K}_\infty$ is constant, then the same conclusions as in the small-time case apply (see Remark 4.4).

Remark 5.4. The assumption that X is centered is not essential, but it simplifies the exposition. If X is not centered and we denote by $m_X(t) = \mathbb{E}[X_t]$ its mean function, then we have

$$\mathbb{E}\big[Z_t^{\alpha,\rho}\big] = \frac{\rho^\alpha}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(\rho s)\, ds.$$

For the convergence results to hold in the non-centered case, one needs to assume that the normalized mean term above converges uniformly in $t \in [0,T]$ to a continuous function. Under this additional condition, the asymptotic behavior of the centered fluctuations remains unchanged.

6 Fractionally integrated Gauss-Markov processes

In this section, we consider the fractional integral of a special class of continuous Gaussian processes, namely Gauss-Markov processes. Gauss-Markov processes enjoy remarkable representation properties, which make their covariance function particularly easy to characterize. These structural features simplify the analysis of the corresponding integrated processes.

Since Gauss-Markov processes can be viewed as deterministic transformations of Brownian motion, we are able to derive an explicit and general expression for the associated RKHS.

6.1 Definition and properties

Continuous Gauss-Markov processes are defined as follows (see [1]).

Definition 6.1. Let $m,\ h_1,\ h_2 \in \mathscr{C}$ such that $r := \frac{h_1}{h_2}$ is a non-negative, strictly increasing function on $(0,T]$, that admits a continuous extension on $[0,T]$. If $(B_t)_{t\geq 0}$ denotes the standard Brownian motion, we define

$$X_t := m(t) + h_2(t)B_{r(t)},\ t \in [0,T]. \tag{6.1}$$

Since h_1 and h_2 need to have the same sign, we can assume that they are positive, without loss of generality. The process X has factorizable covariance for $s \leq t$:

$$K_X(s,t) = h_2(s)h_2(t)E[B_{r(s)}B_{r(t)}] = h_2(s)h_2(t)r(s) = h_1(s)h_2(t),$$

Remark 6.2. A factorizable covariance function is said to be *triangular*. Let T denote a closed interval in $\mathbb{R}^+$. It can be shown that if $(X_t)_{t\in T}$ is a centered, mean-square continuous Gaussian process (i.e. $\lim_{s\to t} E[|X(t) - X(s)|^2] = 0$ for any $s,\ t \in T$), non singular except possibly at the endpoints of T, then it is a Gauss-Markov process if and only if it has a triangular covariance function. Moreover, every centered Gauss-Markov process X with these properties admits a representation of the form (6.1), where $m = 0,\ \ h_1,\ h_2$ are continuous and r is continuous and strictly increasing in T. Note that *continuous Gauss-Markov processes* on an interval satisfy these assumptions. The above representation of Gauss-Markov processes is also known as Doob's representation, as it was first introduced by Doob in [10]. These results and the relative proofs can be found in [14].

Let X be a process satisfying Definition 6.1. Then we can consider the fractional integral of order $\alpha \in (0,1]$ of a Gauss-Markov process (FIGM),

$$\mathcal{I}^\alpha(X)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(m(s) + h_2(s)B_{r(s)})\, ds\ ,\ t \in [0,T].$$

A recent work by M. Abundo and E. Pirozzi ([3]) describes the covariance function of FIGM processes: its analytic expression is determined, and some particular examples are explored through simulations, such as the fractional integral of Brownian motion, the Ornstein-Uhlenbeck process, and the stationary Ornstein-Uhlenbeck process.

6.2 *RKHS of FIGM processes*

In this section, we will derive the RKHS of FIGM processes by combining two criteria, one is Lemma 3.1, the other is the following Pull-back Theorem (see Theorem 5.7 in [15]) for RKHS of real functions.

Theorem 6.3 (Pull-back Theorem). *Let S and T be interval of $[0,+\infty)$, let $\phi : S \longrightarrow T$ be a function and let $K : T \times T \to \mathbb{R}$ be a covariance function associated with the RHKS $\mathbb{H}(K)$. Then, the RKHS associated to the covariance function $K \circ \phi$ is*

$$\mathbb{H}(K \circ \phi) = \{f \circ \phi : f \in \mathbb{H}(K)\},$$

and, for $u \in \mathbb{H}(K \circ \phi)$, we have that

$$\|u\|_{\mathbb{H}(K\circ\phi)} = \min\{\|f\|_{\mathbb{H}(K)} \mid u = f \circ \phi\}.$$

In particular, if ϕ is a one-to-one correspondence, it induces an isometry between the RKHS associated with K and $K \circ \phi$.

Let X be a centered Gauss-Markov process on $[0,T]$ with parameters $h_2,\ r$ as in Definition 6.1 and let $r_0,\ r_T$ denote $r(0),\ r(T)$ respectively.

We now describe how to construct the RKHS of a centered Gauss-Markov process X on $[0,T]$ with parameters h_2 and r (as in Definition 6.1) by starting from the RKHS of standard Brownian motion. The construction proceeds in three steps:

1. Brownian motion on a interval: Consider a Brownian motion B on a subinterval $[r_0, r_T] \subset \mathbb{R}^+$. Using the decomposition $B_t = B_t - B_{r_0} + B_{r_0}$ for $t \in [r_0, r_T]$, by Theorem 5, §1.4.1, in [8], we can express its RKHS as

$$\mathbb{H}_{[r_0,r_T]} = \{f \in AC[r_0, r_T] : \dot{f} \in L^2([r_0, r_T])\},$$

   ($AC[r_0, r_T]$ is the space of the absolutely continuous functions on $[r_0, r_T]$) equipped with the norm

$$\|f\|^2_{\mathbb{H}_{[r_0,r_T]}} = \frac{(f(r_0))^2}{r_0} + \int_{r_0}^{r_T} |\dot{f}(t)|^2\, dt.$$

2. Time change: Next, consider a strictly increasing function $r : [0,T] \to [r_0, r_T]$ and the time-changed Brownian motion $(B_{r(t)})_{t\in[0,T]}$. By the pull-back theorem (Theorem 6.3), the RKHS of this time-changed process is

$$\mathbb{H}'_{[0,T]} = \{g = f \circ r : f \in \mathbb{H}_{[r_0,r_T]}\},$$

   with the norm $\|g\|_{\mathbb{H}'_{[0,T]}} = \|f\|_{\mathbb{H}_{[r_0,r_T]}}$.

3. Multiplication by h_2: Finally, multiplication by the continuous function $h_2 : [0,T] \to \mathbb{R}$ defines a linear, continuous, one-to-one map

$$T_{h_2} : \mathscr{C} \to \mathscr{C}, \quad g \mapsto h_2 \cdot g.$$

   By Lemma 7.1 in [17], the RKHS of the centered Gauss-Markov process $X_t = h_2(t)B_{r(t)}$ for $t \in [0,T]$, is therefore

$$\mathbb{H}''_{[0,T]} = \{\ell = h_2 \cdot g : g \in \mathbb{H}'_{[0,T]}\},$$

with the norm

$$\|\ell\|_{\mathbb{H}''_{[0,T]}} = \|g\|_{\mathbb{H}'_{[0,T]}} = \left\|\frac{\ell}{h_2} \circ r^{-1}\right\|_{\mathbb{H}_{[r_0,r_T]}}.$$

In this way, starting from the classical Cameron-Martin space of Brownian motion, and applying successively a restriction to a subinterval, a time change, and a multiplication by h_2, we obtain the RKHS of the Gauss-Markov process X.

We now state the following proposition, whose proof follows immediately from Proposition 3.2 and equation (2.1).

Proposition 6.4. *Let $(X_t)_{t\in[0,T]}$ a continuous centered Gauss-Markov process as in(6.1). Then, the RKHS of $\mathcal{I}^\alpha(X)$ is the set*

$$\mathcal{Y} = \left\{ z \in \mathscr{C} \middle| z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} h_2(u) f(r(u)) du,\ t \in [0,T],\ f \in \mathbb{H}_{[r_0,r_T]} \Big) \right\},$$

equipped with the norm

$$\|z\|_{\mathcal{Y}} = \|f\|_{\mathbb{H}_{[r_0,r_T]}} = \left\|\frac{\mathcal{D}^\alpha z}{h_2} \circ r^{-1}\right\|_{\mathbb{H}_{[r_0,r_T]}}.$$

6.3 Small-time asymptotics for FIGM processes

In the previous sections we established a general small-time large deviation principle for fractionally integrated Gaussian processes. The corresponding rate function depends solely on the RKHS of the limit process $\tilde{Y}^{\alpha,0}$ (or $\widehat{Y}^{\alpha,0}$). We now specialize these results to the case where X is a centered Gauss-Markov process with triangular covariance

$$K_X(s,t) = h_1(s)h_2(t), \qquad 0 \le s \le t \le T.$$

In this framework, the small-time behavior is governed by h_1, because h_2 can vanish at 0 only if h_1 does the same, by Definition 6.1. Moreover, both h_1 and h_2 can vanish only for $t = 0$, since r must be well-defined, positive and strictly increasing on $(0, +\infty)$. Let us describe the limit processes appearing in Theorem 4.2.

Case $h_1(0) \neq 0$. This case is identical to the situation $K_X(0,0) \neq 0$ in Theorem 4.2, and the same result follows.

Case $h_1(0) = 0$. In this case, we can assume regular variation at 0 for $h_1,\ h_2$, namely

$$\frac{h_1(\varepsilon u)}{h_1(\varepsilon)} \longrightarrow \hat{h}_1(u) = u^{\gamma_1} \qquad \text{and} \qquad \frac{h_2(\varepsilon v)}{h_2(\varepsilon)} \longrightarrow \hat{h}_2(v) = v^{\gamma_2} \quad u,\ v \in [0,T].$$

By the continuity assumptions on $h_1,\ h_2$, one has $\gamma_1 \ge \gamma_2 \ge 0$. We define $\gamma := \gamma_1 - \gamma_2$ and $\hat{r}(u) := u^\gamma$. Then, by Remark 6.2, the normalized fractionally integrated process converges to the fractionally integrated Gauss-Markov process

$$\hat{Y}_t^{\alpha,0} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} u^{\gamma_2} \cdot B_{u^\gamma}\, du, \qquad t \in [0,T],$$

where B is a standard Brownian motion.

In the general small-time LDP, when $K_X(0,0) = 0$, we required that $|G_\varepsilon| \le H$ for some $H \in L^p([0,T]^2)$, $p > \frac{1}{\alpha}$, and that (4.1) holds. If X is Gauss-Markov with h_1 regularly varying and $h_1(0) = 0$, we have

$$G_\varepsilon(u,v) = \frac{h_1(\varepsilon u)}{h_1(\varepsilon)} \cdot \frac{h_2(\varepsilon v)}{h_2(\varepsilon)}.$$

As $\varepsilon \to 0$, $G_\varepsilon(u,v)$ converges to $u^{\gamma_1} \cdot v^{\gamma_2}$, $0 \le u \le v \le T$ by regular variation. Moreover, by the Potter bounds for regularly varying functions, for every $\delta > 0$ there exists $A > 1$ such that, for ε small enough, $i = 1, 2$,

$$\frac{h_i(\varepsilon u)}{h_i(\varepsilon)} \le A u^{\gamma_i - \delta}$$

Hence $|G_\varepsilon|$ is dominated by a function that is in $L^p([0,T]^2)$ for any $p > 1/\alpha$, provided δ is chosen sufficiently small. The small-time rate functions in Theorem 4.3 then take the explicit forms:

1. Case $h_1(0) \ne 0$. This case is identical to the situation $K_X(0,0) \ne 0$ in Theorem 4.3, and the same result follows.

2. Case $h_1(0) = 0$.

   - If $h_1,\ h_2$ are regularly varying with indices $0 \le \gamma_2 < \gamma_1$, then

     $$\hat{J}_0(z) = \begin{cases} \dfrac{1}{2} \left\| \dfrac{\mathcal{D}^\alpha z}{\hat{h}_2} \circ \hat{r}^{-1} \right\|^2_{\mathbb{H}_{[0,T^\gamma]}}, & z \in \hat{\mathcal{Y}}_0, \\ +\infty, & \text{otherwise}, \end{cases}$$

     where

     $$\hat{\mathcal{Y}}_0 = \left\{ z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} u^{\gamma_2} \cdot f(u^\gamma) du : f \in \mathbb{H}_{[0,T^\gamma]} \right\}.$$

   - If $h_1,\ h_2$ are regularly varying with indices $\gamma_1 = \gamma_2$, we have a borderline case. The limiting process is, for $t \in [0,T]$, the degenerate FIGM process

     $$\hat{Y}_t^{\alpha,0} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} u^{\gamma_1} \cdot B_1\, du = \frac{\Gamma(\gamma_1+1)}{\Gamma(\alpha+\gamma_1+1)} t^{\alpha+\gamma_1} \cdot B_1.$$

     With the same construction used in Remark 3.3, we get

     $$\hat{J}_0(h) = \begin{cases} \frac{1}{2} \frac{\Gamma^2(\alpha+\gamma_1+1)}{\Gamma^2(\gamma_1+1)} \cdot c^2, & \text{if } \ h(t) = c \cdot t^{\alpha+\gamma_1},\ c \in \mathbb{R},\ t \in [0,T] \\ +\infty, & \text{otherwise}. \end{cases}.$$

     We notice that when $\gamma_1 = \gamma_2 = 0$, we are in the situation described in Remark 4.4 and we get exactly the same result.

In conclusion, in the Gauss-Markov setting, the small-time LDP and its rate functions are determined by the behavior of h_1 at zero. All technical assumptions of the general theorem are automatically satisfied by regular variation and the Potter bounds.

6.4 Large-time asymptotics for FIGM processes

In this section, we consider Gauss-Markov processes $(X_t)_{t\geq 0}$ and we want to study their fractional integral for large times. The asymptotic behavior in large times is no longer determined by h_1 since r must be strictly increasing on the whole half-axis, therefore it cannot happen that h_1 approaches 0, if h_2 does not the same. A first simple case is that corresponding to the situation of Proposition 5.1: we assume that

$$\lim_{\rho\to+\infty} h_1(\rho) = h_1^\infty \neq 0 \quad \text{and} \quad \lim_{\rho\to+\infty} h_2(\rho) = h_2^\infty \neq 0. \tag{6.2}$$

The limiting process is

$$\hat{Y}_t^{\alpha,\infty} = X^\infty \frac{t^\alpha}{\Gamma(\alpha+1)},$$

where X^∞ is a Gaussian random variable with variance $h_1^\infty h_2^\infty$. If this is note the case, we assume that h_1 and h_2 are regularly varying.

$$\lim_{\rho\to+\infty} \frac{h_1(\rho u)}{h_1(\rho)} = \hat{h}_1(u) = u^{\gamma_1} \quad \text{and} \quad \lim_{\rho\to+\infty} \frac{h_2(\rho v)}{h_2(\rho)} = \hat{h}_2(v) = v^{\gamma_2},\ u, v \in (0, T]. \tag{6.3}$$

By the continuity assumptions on $h_1,\ h_2$, their orders of regular variation γ_1, γ_2 must be nonnegative, and $\gamma = \gamma_1 - \gamma_2 \geq 0$, otherwise $\frac{h_1}{h_2}$ could not be increasing.

Also for large time all technical assumptions of the general theorem are automatically satisfied by regular variation and the Potter bounds.

In both cases, the limiting processes are still Gauss-Markov, as in the small-time case. We can derive an LDP using Proposition 5.1 and Proposition 5.2 and we can write explicit rate functions. Their expressions are analogous to the small-time rate functions, but we write them for completeness.

1. If (6.2) holds, then

$$J_\infty(h) = \begin{cases} \frac{1}{2}\frac{\Gamma^2(\alpha+1)}{h_1^\infty h_2^\infty}\cdot c^2, & \text{if } h(t) = c\cdot t^\alpha,\ c\in\mathbb{R},\ t\in[0,T] \\ +\infty, & \text{otherwise.} \end{cases}.$$

2. If (6.3) holds, then

   - If $0 \leq \gamma_2 < \gamma_1$,

   $$\hat{J}_\infty(z) = \begin{cases} \frac{1}{2}\cdot\left\|\frac{\mathcal{D}^\alpha z}{\hat{h}_2}\circ\hat{r}^{-1}\right\|^2_{\mathbb{H}_{[0,T^\gamma]}}, & z\in\hat{\mathcal{Y}}_\infty \\ +\infty, & \text{otherwise,} \end{cases}$$

   where

   $$\hat{\mathcal{Y}}_\infty = \left\{ z(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}u^{\gamma_2}\cdot f(u^\gamma)du : f\in\mathbb{H}_{[0,T^\gamma]}\right\}.$$

   - If $\gamma_1 = \gamma_2$,

   $$\hat{J}_\infty(h) = \begin{cases} \frac{1}{2}\frac{\Gamma^2(\alpha+\gamma_1+1)}{\Gamma^2(\gamma_1+1)}\cdot c^2, & \text{if } h(t) = c\cdot t^{\alpha+\gamma_1},\ c\in\mathbb{R},\ t\in[0,T] \\ +\infty, & \text{otherwise.} \end{cases}.$$

6.5 Example: the Ornstein-Uhlenbeck process

A classical example of a continuous Gauss-Markov process is the Ornstein-Uhlenbeck process. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion and let $\theta, \sigma > 0$. Consider the stochastic differential equation

$$dX_t = -\theta X_t \, dt + \sigma \, dB_t, \qquad X_0 = 0.$$

Its unique strong solution is

$$X_t = \sigma e^{-\theta t} \int_0^t e^{\theta s} \, dB_s, \qquad t \geq 0.$$

By Itô's isometry, the martingale

$$M_t := \int_0^t e^{\theta s} \, dB_s$$

has quadratic variation

$$\langle M \rangle_t = \int_0^t e^{2\theta s} \, ds = \frac{e^{2\theta t} - 1}{2\theta}.$$

Therefore (see for example Theorem 8.4 in [4]), there exists a standard Brownian motion W such that

$$M_t = W_{r(t)}, \qquad r(t) := \frac{e^{2\theta t} - 1}{2\theta}.$$

Therefore

$$X_t = \sigma e^{-\theta t} W_{r(t)},$$

which is precisely of the form (6.1) with

$$h_2(t) = \sigma e^{-\theta t}, \qquad r(t) = \frac{e^{2\theta t} - 1}{2\theta}, \qquad h_1(t) = h_2(t) r(t) = \frac{\sigma}{2\theta} \left(e^{\theta t} - e^{-\theta t} \right).$$

The covariance function is thus triangular: for $0 \leq s \leq t$,

$$K_X(s,t) = h_1(s) h_2(t) = \frac{\sigma^2}{2\theta} \left(e^{-\theta(t-s)} - e^{-\theta(t+s)} \right).$$

By Proposition 6.4, the RKHS of the Ornstein-Uhlenbeck process is

$$\mathcal{H}_X = \left\{ h_2(\cdot) \, f(r(\cdot)) : f \in AC([0, r(T)]), \ \dot{f} \in L^2([0, r(T)]) \right\},$$

equipped with the norm

$$\|h_2 f \circ r\|_{\mathcal{H}_X} = \|f\|_{\mathbb{H}_{[0,r(T)]}}.$$

Its fractional integral of order $\alpha \in (0, 1]$ is

$$\mathcal{I}^\alpha(X)(t) = \frac{\sigma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\theta s} W_{r(s)} \, ds.$$

Applying Proposition 6.4, its RKHS is

$$\mathcal{Y} = \left\{ z(t) = \frac{\sigma}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\theta s} f(r(s)) \, ds : f \in AC([0, r(T)]), \ \dot{f} \in L^2([0, r(T)]) \right\},$$

with norm

$$\|z\|_{\mathcal{Y}} = \|f\|_{\mathbb{H}_{[0,r(T)]}}.$$

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References

[1] Abundo, M.: On the representation of an integrated Gauss-Markov process. Scientiae Mathematicae Japonicae Online, 719–723 (2013). MR3330137

[2] Abundo, M.: Asymptotic of the running maximum distribution of a Gaussian bridge. Stoch. Anal. Appl. **41**(6), 1099–1118 (2023). MR4657191. https://doi.org/10.1080/07362994.2022.2123344

[3] Abundo, M., Pirozzi, E.: Fractionally integrated Gauss-Markov processes and applications. Commun. Nonlinear Sci. Numer. Simul. **101**, 105862 (2021). MR4256166. https://doi.org/10.1016/j.cnsns.2021.105862

[4] Baldi, P.: Stochastic Calculus: An Introduction Through Theory and Exercises. Universitext. Springer (2017). https://books.google.it/books?id=fO09DwAAQBAJ MR3726894. https://doi.org/10.1007/978-3-319-62226-2

[5] Baldi, P.: Tightness and exponential tightness of Gaussian probabilities. ESAIM, Probab. Stat. **24**, 113–126 (2020). MR4071318. https://doi.org/10.1051/ps/2020003

[6] Baldi, P., Pacchiarotti, B.: Large deviations of continuous Gaussian processes: from small noise to small time. Stochastics **98**(4), 571–600 (2026). MR5094919. https://doi.org/10.1080/17442508.2025.2593360

[7] Benedetto, E., Sacerdote, L., Zucca, C.: A first passage problem for a bivariate diffusion process: Numerical solution with an application to neuroscience when the process is Gauss-Markov. J. Comput. Appl. Math. **242**, 41–52 (2013). MR2997429. https://doi.org/10.1016/j.cam.2012.10.014

[8] Berlinet, A., Thomas-Agnan, C.: Reproducing Kernel Hilbert Spaces in Probability and Statistics, p. 355. Kluwer Academic Publishers, Boston, MA (2004). https://doi.org/10.1007/978-1-4419-9096-9 MR2239907. https://doi.org/10.1007/978-1-4419-9096-9

[9] Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation. Encyclopedia of Mathematics and its Applications, vol. 1. Cambridge University Press (1989). https://books.google.it/books?id=VCU5D4jMfiEC MR0898871. https://doi.org/10.1017/CBO9780511721434

[10] Doob, J.L.: Heuristic Approach to the Kolmogorov-Smirnov Theorems. Ann. Math. Stat. **20**(3), 393–403 (1949). MR0030732. https://doi.org/10.1214/aoms/1177729991

[11] Hida, T., Hitsuda, M.: Gaussian Processes. Translations of Mathematical Monographs, vol. 120, p. 183. American Mathematical Society, Providence, RI (1993). . https://doi.org/10.1090/mmono/120 MR1216518. https://doi.org/10.1090/mmono/120

[12] Kallenberg, O.: Foundations of Modern Probability, 2nd edn. Probability and its Applications, p. 638. Springer (2002). . http://dx.doi.org/10.1007/978-1-4757-4015-8 MR1876169. https://doi.org/10.1007/978-1-4757-4015-8

[13] Lifshits, M.: Lectures on Gaussian Processes. SpringerBriefs in Mathematics, p. 121. Springer (2012). https://doi.org/10.1007/978-3-642-24939-6 MR3024389. https://doi.org/10.1007/978-3-642-24939-6

[14] Mehr, C.B., McFadden, J.A.: Certain Properties of Gaussian Processes and Their First-Passage Times. J. R. Stat. Soc. Ser. B, Methodol. **27**(3), 505–522 (1965). MR0199885. https://doi.org/https://doi.org/10.1111/j.2517-6161.1965.tb00611.x

[15] Paulsen, V.I., Raghupathi, M.: An Introduction to the Theory of Reproducing Kernel Hilbert Spaces. Cambridge Studies in Advanced Mathematics. Cambridge University Press (2016). MR3526117. https://doi.org/10.1017/CBO9781316219232

[16] Samko, S., Kilbas, A.A., Marichev, O.: Fractional Integrals and Derivatives. Taylor & Francis (1993). https://books.google.it/books?id=SO3FQgAACAAJ MR1347689

[17] Van der Vaart, A.W., van Zanten, J.H.: Reproducing kernel Hilbert spaces of Gaussian priors, pp. 200–222. Institute of Mathematical Statistics (2008). http://dx.doi.org/10.1214/074921708000000156 MR2459226. https://doi.org/10.1214/074921708000000156