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Viability for time fractional functional differential equations driven by the fractional Brownian motion

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Abstract Let B^H be a fractional Brownian motion with Hurst index $\frac{1}{2} < H < 1$. In this paper, we consider the time fractional functional differential equation of the form

$$\begin{cases} {}^C D_t^{\gamma} x(t) = f(t, x_t) + G(t, x_t) \frac{d}{dt} B^H(t), & t \in (0, T], \\ x_0(t) = \eta(t), & t \in [-r, 0], \end{cases}$$

where $\frac{3}{2} - H < \gamma < 1$, ${}^C D_t^{\gamma}$ denotes the Caputo derivative, and $x_t \in \mathcal{C}_r = \mathcal{C}([-r, 0], \mathbb{R})$ with $x_t(u) = x(t + u)$, $u \in [-r, 0]$. We prove the global existence and uniqueness of the solution of the equation and study its viability. As an application, we also discuss the existence of positive solutions.

Keywords Fractional Brownian motion, Caputo derivative, stochastic functional differential equation, time delay, viability
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1 Introduction

Over the last decade, there has been considerable interest in studying fractional Brownian motion due to its compact properties such as long/short range dependence,

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self-similarity, stationary increments and Hölder's continuity. Its applications can be found in various scientific areas including telecommunications, turbulence, image processing and finance. Fractional Brownian motion is a generalization of standard Brownian motion and it is a relatively simple stochastic process which possesses the above good properties. Some surveys and complete literatures on fractional Brownian motion could be found in Alós et al. [2], Biagini *et al* [6], Hu [11], Mishura [15], Nourdin [17], Nualart [18], Tudor [21], and the references therein. Recall that a fractional Brownian motion (in short, fBm) $B^H = \{B^H(t), t \geq 0\}$ with Hurst index $H \in (0, 1)$ is a zero mean Gaussian process with the covariance function

$$E\left[B^H(t)B^H(s)\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s \geq 0$. For $H = 1/2$, B^H coincides with the standard Brownian motion B. Since B^H is neither a semimartingale nor a Markov process unless $H = 1/2$, many of the powerful techniques from classical stochastic analysis are not applicable to B^H. However, as a Gaussian process, one can develop the stochastic calculus of variations with respect to B^H.

On the other hand, the viability property has been extensively studied for deterministic differential equations and inclusions, starting with Nagumo's pioneering work in 1943 (see Aubin [3]). Girejko *et al* [9] first considered the viability of the fractional differential equations with the Caputo derivative and Carja *et al* [7] introduced the viability of fractional differential inclusions. Viability theory is a mathematical theory that offers mathematical metaphors of evolution of macrosystems arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory. The characterization of the viability property in a stochastic framework was first given by Aubin and Da Prato [4] in 1990. The key point of their work consists in defining a suitable Bouligand's stochastic tangent cone which generalizes the cone used in the study of the viability property for deterministic systems. Then, Nie-Rascanu [16] established the characterization of viability for some particular sets K. Recently, Melnikov *et al* [14] considered stochastic viability and comparison theorems for mixed stochastic differential equations. Ciotir and Rascanu [8] proved a viability result for multidimensional, time dependent, stochastic differential equations driven by fractional Brownian motion, and moreover, in [22, 23] Xu and Luo extended this to the viability for stochastic functional differential equations associated with fractional Brownian motion. In a more recent development, Li *et al* [13] addressed the problem of a class of coupled multidimensional stochastic differential equations driven by fractional Brownian motion.

Motivated by these results, in this paper, we consider the viability result for the time fractional functional differential equation of the form

$$\begin{cases} {}^C D_t^\gamma x(t) = f(t, x_t) + G(t, x_t)\frac{d}{dt}B^H(t), & t \in (0, T], \\ x_0(t) = \eta(t), & t \in [-r, 0] \end{cases} \tag{1.1}$$

with $\frac{1}{2} < H < 1$ and $\frac{3}{2} - H < \gamma < 1$, where

- $B^H = \{B^H(t), 0 \leq t \leq T\}$ is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$;

- ${}^{C}D_t^{\gamma}$ denotes the Caputo derivative;
- $x_t \in \mathcal{C}_r$ with $x_t(u) = x(t+u)$ for $u \in [-r, 0]$, $\mathcal{C}_r = C([-r, 0])$ is the space of continuous functions f from $[-r, 0]$ to $\mathbb{R}$ endowed by the uniform norm $\|\cdot\|_{\mathcal{C}_r}$;
- $f, G : [0, T] \times \mathcal{C}_r \to \mathbb{R}$ are two Borel functions;
- $\eta : [-r, 0] \to \mathbb{R}$ is a smooth function.

It is important to note that the strongest motivation to study such equations comes for the following reasons. Compared with the classical differential equation, the fractional order models are better to describe the hereditary character of various kinds of materials and processes. The advantages of fractional derivatives become apparent in various fields of science and engineering such as control theory, porous media, viscoelasticity, image and signal processing. (see e.g. [1, 5, 12, 20]). As mentioned before, fractional Brownian motion is a Gaussian process with long-memory properties and a relatively simple structure. Therefore, it is more reasonable to introduce fractional Brownian motion into the noise term of fractional order differential equations to characterize systems with memory.

We know that the equation (1.1) can be written as the following integral form:

$$\begin{cases} x(t) = \eta(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, x_s) ds \\ \qquad\qquad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} G(s, x_s) dB^H(s), \quad t \in (0, T], \\ x_0(t) = \eta(t), \quad t \in [-r, 0], \end{cases} \tag{1.2}$$

where $\Gamma(\cdot)$ denotes the classical Gamma function and the integral with respect to B^H is a pathwise Riemann-Stieltjes (R-S) integral in the sense of Zähle [25, 26]. Our approach is inspired by Nualart and Rascanu [19], while we consider the time fractional cases. This paper is organized as follows. In Section 2, we present the assumptions on the coefficients and briefly review the basic definitions and key results on the fractional integrals and derivatives. In Section 3, we derive some useful estimates for the indefinite integrals and show the existence and uniqueness for the solution of the equation (1.1). In Section 4, we state our main results and obtain the sufficient and necessary condition for a closed subset being a viable domain of the equation (1.1). In Section 5, as an application, we discuss the existence of positive solution when $K = [0, \infty)$.

2 Preliminaries

Throughout this paper we fix a time interval $[0, T]$ and a complete probability space $(\Omega, \mathscr{F}, P)$. For some given real numbers a, b with $a < b$ and $\kappa \in (0, 1]$, we will denote by $\mathcal{C}^{\kappa}([a, b])$ the space of κ-Hölder continuous functions $f : [a, b] \to \mathbb{R}$, endowed with the norm

$$\|f\|_{\kappa;[a,b]} := \|f\| + \sup_{a \le s < t \le b} \frac{|f(t) - f(s)|}{(t-s)^{\kappa}},$$

where $\|f\| = \sup_{t\in[a,b]} |f(t)|$. For any $\lambda \geq 0$, we introduce the following equivalent norm:

$$\|f\|_{\kappa,\lambda;[a,b]} = \sup_{t\in[a,b]} e^{-\lambda t}|f(t)| + \sup_{a\leq s<t\leq b} e^{-\lambda t}\frac{|f(t)-f(s)|}{(t-s)^{\kappa}}.$$

Denote by $\mathcal{C}_r = C([-r,0])$ the space of continuous functions f from $[-r,0]$ to $\mathbb{R}$ endowed by the uniform norm $\|\cdot\|_{\mathcal{C}_r}$.

2.1 *The assumptions on the equation* (1.1)

To study the equation (1.1), we introduce the following assumptions:

(H-f) The function f is local Lipschitz continuous and has linear growth. That is, there exists a constant L_1 and for every $R \geq 0$ there exists $L_R^{(1)}$ such that

$$|f(t,\xi) - f(s,\eta)| \leq L_R^{(1)}\left(|t-s| + \|\xi-\eta\|_{\mathcal{C}_r}\right) (\forall \|\xi\|_{\mathcal{C}_r}, \|\eta\|_{\mathcal{C}_r} \leq R)$$

and

$$|f(t,\xi)| \leq L_1(1+\|\xi\|_{\mathcal{C}_r})$$

for all $\xi,\eta \in \mathcal{C}_r$ and $s,t \in [0,T]$.

(H-G) The function $G(t,x)$ is Fréchet differentiable in x. Moreover, there exist constants $\mu \in (2-H-\gamma, 1]$, L_2 and for every $R \geq 0$ there exists $L_R^{(2)}$ such that

$$|G(t,\xi) - G(s,\eta)| \leq L_2\left(|t-s|^{\mu} + \|\xi-\eta\|_{\mathcal{C}_r}\right)$$

and

$$|DG(t,\xi) - DG(s,\eta)|_{\mathcal{L}(\mathcal{C}_r,\mathbb{R})} \leq L_R^{(2)}\left(|t-s|^{\mu} + \|\xi-\eta\|_{\mathcal{C}_r}\right) (\forall \|\xi\|_{\mathcal{C}_r}, \|\eta\|_{\mathcal{C}_r} \leq R)$$

for all $\xi,\eta \in \mathcal{C}_r$ and $s,t \in [0,T]$.

It is important to note that the assumption (H-G) implies the linear growth property, i.e., there exists a constant $L_3 > 0$ such that

$$|G(t,\xi)| \leq L_3(1+\|\xi\|_{\mathcal{C}_r})$$

for all $\xi \in \mathcal{C}_r$ and $t \in [0,T]$.

2.2 *The fractional derivative*

Now, we recall some definitions and notions of fractional calculus.

Definition 2.1. Let $\alpha > 0$. The fractional integral of order α for a Borel function $f : [0,\infty) \to \mathbb{R}$ is defined as

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds \quad t > 0,$$

provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the classical Gamma function defined by $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$.

Definition 2.2. Let $n > 0$ be an integer number and let $n - 1 < \alpha < n$. The Caputo derivative of order α for a function $f \in \mathcal{C}^n([0, \infty))$ is defined as

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t) \quad t > 0. $$

Based on fractional integrals and derivatives, Zähle [25] has introduced the Riemann-Stieltjes integral. We refer the reader to Young [24], Zähle [25, 26] and Nualart and Rascanu [19] for the general theory of this integral. Fix a parameter $0 < \alpha < \frac{1}{2}$, and denote by $W^{\alpha,1}(0, T; \mathbb{R})$ the space of measurable functions $f : [0, T] \to \mathbb{R}$ such that

$$ \|f\|_{\alpha,1} := \int_0^T \left(\frac{|f(s)|}{s^{\alpha}} + \int_0^s \frac{|f(s) - f(t)|}{(s-t)^{\alpha+1}} dt \right) ds < \infty. $$

We also denote by $W^{1-\alpha,\infty}(0, T; \mathbb{R})$ the space of measurable functions $g : [0, T] \to \mathbb{R}$ such that

$$ \|g\|_{1-\alpha,\infty} := \sup_{0<s<t<T} \frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(u) - g(s)|}{(u-s)^{2-\alpha}} du < \infty. $$

Clearly,

$$ \mathcal{C}^{1-\alpha+\varepsilon}(0, T; \mathbb{R}) \subset W^{1-\alpha,\infty}(0, T; \mathbb{R}) \subset \mathcal{C}^{1-\alpha}(0, T; \mathbb{R}) $$

for any $\varepsilon > 0$. Given two functions $f \in W^{\alpha,1}(0, T; \mathbb{R})$ and $g \in W^{1-\alpha,\infty}(0, T; \mathbb{R})$, the generalized Stieltjes integral $\int_0^T f(s) dg(s)$ is defined by

$$ \int_0^T f(s) dg(s) = (-1)^{\alpha} \int_0^T D_{0+}^{\alpha} f(t) D_{T-}^{1-\alpha} g_{T-}(t) dt, $$

where $(-1)^{\alpha} = e^{\alpha \pi i}$,

$$ D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{t^{\alpha}} + \alpha \int_0^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) $$

and

$$ D_{T-}^{1-\alpha} g_{T-}(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{g(t) - g(T)}{(T-t)^{\alpha}} + \alpha \int_t^T \frac{g(t) - g(s)}{(s-t)^{\alpha+1}} ds \right). $$

Furthermore, we have the estimate

$$ \left\| \int_0^t f dg \right\| \leq \Lambda_{\alpha}(g) \|f\|_{\alpha,1}, \tag{2.1} $$

where

$$ \Lambda_{\alpha}(g) := \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} |D_{t-}^{1-\alpha} g_{t-}(s)| \leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty}. $$

2.3 Fractional Brownian motion

In this subsection, we briefly review some basic results of fBm. As we have pointed out before, fBm $B^H = \{B^H(t), 0 \le t \le T\}$, on the probability space $(\Omega, \mathscr{F}^H, P)$ with Hurst index $H \in (0,1)$ is a central Gaussian processes such that $B^H(0) = 0$ and

$$E\left[B^H(t)B^H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \tag{2.2}$$

for all $s,t \ge 0$. The process is H-self similar and its paths are Hölder continuous of order $\nu \in (0,H)$. Even though it is not a semimartingale, we may define the stochastic integrals with respect to it by using some other methods. Throughout this paper, we assume that $\frac{1}{2} < H < 1$. Thus, we can handle its stochastic calculus and related stochastic differential equations by using Young's integration. By the Hölder continuity of $t \mapsto B_t^H$, we have known that the trajectories of B^H belong to the space $W^{1-\beta,\infty}(0,T;\mathbb{R})$ for all $\beta \in \left(1-H, \frac{1}{2}\right)$. As a consequence, if $u = \{u(t), t \in [0,T]\}$ is a stochastic process whose trajectories belong a.s. to the space $W^{\beta,1}(0,T;\mathbb{R})$ with $\beta \in \left(1-H, \frac{1}{2}\right)$, the Riemann-Stieltjes integral $\int_0^T u(s)dB^H(s)$ exists and

$$\left\|\int_0^T u(s)dB^H(s)\right\| \le \Lambda_\beta(B^H)\|u\|_{\beta,1}.$$

Thus, one study the time fractional functional differential equation (1.2) essentially is equivalent to handle the deterministic time fractional differential equation of the form

$$\begin{cases} x(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s,x_s)ds \\ \qquad\qquad + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} G(s,x_s)dg(s), \quad t \in (0,T] \\ x_0(t) = \eta(t), \quad t \in [-r,0], \end{cases} \tag{2.3}$$

where the coefficients f and G are as in the equation (1.1), $g \in \mathcal{C}^\nu([0,T])$ and $x \in \mathcal{C}^{1-\alpha}([-r,T])$. In fact, we are considering about the above issues in this paper.

3 The existence and uniqueness

In this section, we show the existence and uniqueness of the solution of the equation (1.1). As mentioned above, our analysis starts from the equivalent deterministic equation (2.3) to first establish some basic estimates. Throughout this section we assume that $\frac{1}{2} < \nu < 1$ is arbitrary but fixed and we let

$$\frac{3}{2} - \nu < \gamma < 1, \quad 2 - \nu - \gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}.$$

Remark 3.1. There are several key parameters whose admissible ranges are interdepent. We fix an arbitrary $\nu \in (\frac{1}{2}, 1)$. Given ν, the constant γ is chosen from $\frac{3}{2} - \nu < \gamma < 1$. Then, this choice determines the range of α as $2 - \nu - \gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}$. Furthermore, we can also obtain that these parameters should satisfy $\nu + \gamma > \frac{3}{2}$ and $1 < 2-\nu < \alpha+\gamma < \frac{3}{2}$. Meanwhile, in subsequent proofs, we will also use the auxiliary parameters α_0 and β where $\alpha_0 = \min\{\alpha+\gamma-1, \mu\}$ and $\beta \in (1-\nu, \alpha_0)$. Note that $\alpha_0 < \frac{1}{2}$.

Define the processes $I(x) = \{I(x)(t), 0 \le t \le T\}$ and $J(x) = \{J(x)(t), 0 \le t \le T\}$ as follows

$$I(x)(t) = \int_0^t (t-s)^{\gamma-1} f(s, x_s) ds$$

and

$$J(x)(t) = \int_0^t (t-s)^{\gamma-1} G(s, x_s) dg(s).$$

Unless specified otherwise, we consider the setting where $g \in \mathcal{C}^\nu([0,T])$, and the coefficients f and G satisfy conditions (H-f) and (H-G), respectively. In addition, we simplify the norm $\|x\|_{1-\alpha;[-r,T]}$ as $\|x\|_{1-\alpha}$ when the domain of definition of x is explicit. We will also use M to denote a generic constant which may change from line to line.

Proposition 3.1. *Let $x \in \mathcal{C}^{1-\alpha}([-r,T])$. We have that*

$$J(x) \in \mathcal{C}^{1-\alpha}([0,T]);$$

and there exist some constants $M_1, M_1(\lambda) > 0$ such that

$$\|J(x)\|_{1-\alpha,\lambda} \le M_1 + M_1(\lambda)\|x\|_{1-\alpha,\lambda}.$$

Moreover, M_1 is independent of λ and $M_1(\lambda) \to 0$ as $\lambda \to \infty$.

Proof. For the proof we mimic the ideas from [10]. To prove the first statement, we let $\alpha_0 = \min\{\alpha + \gamma - 1, \mu\}$ and fix $\beta \in (1-\nu, \alpha_0)$. Since $\alpha_0 < \frac{1}{2}$, then $\Lambda_\beta(g) < \infty$. Clearly, we have

$$\begin{aligned} |J(x)(t) - J(x)(s)| \le & \left| \int_s^t (t-u)^{\gamma-1} G(u, x_u) dg(u) \right| \\ & + \left| \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1} \right] G(u, x_u) dg(u) \right| \\ \equiv & J_{11} + J_{12} \end{aligned}$$

for $s, t \in [0,T]$ with $s < t$. From (2.1) we can write

$$\begin{aligned} J_{11} \le \Lambda_\beta(g) \Bigg\{ & \int_s^t \left| (t-u)^{\gamma-1} G(u, x_u) \right| \frac{du}{(u-s)^\beta} \\ & + \int_s^t \int_s^u \left| (t-u)^{\gamma-1} G(u, x_u) - (t-v)^{\gamma-1} G(v, x_v) \right| \frac{dv du}{(u-v)^{\beta+1}} \Bigg\} \end{aligned} \tag{3.1}$$

for $s, t \in [0,T]$ with $s < t$. By the condition (H-G) and $\gamma - \beta > 1 - \alpha$, we get

$$\begin{aligned} \int_s^t \left| (t-u)^{\gamma-1} G(u, x_u) \right| \frac{du}{(u-s)^\beta} & \le \int_s^t L_3 (t-u)^{\gamma-1} \left(1 + \|x_u\|_{\mathcal{C}_r} \right) \frac{du}{(u-s)^\beta} \\ & \le M(t-s)^{\gamma-\beta} \left(1 + \|x\|_{1-\alpha}\right) \le M(t-s)^{1-\alpha} \left(1 + \|x\|_{1-\alpha}\right) \end{aligned} \tag{3.2}$$

for $s, t \in [0,T]$ with $s < t$. Using the condition (H-G) again and the fact

$$\int_s^t \int_s^u \left| (t-u)^{\gamma-1} - (t-v)^{\gamma-1} \right| \frac{dv du}{(u-v)^{\beta+1}} = M_{\gamma,\beta} (t-s)^{\gamma-\beta},$$

where $M_{\gamma,\beta} = \int_0^1 ((1-x)^\gamma - 1 + x^\gamma) \frac{dx}{\gamma x^{\beta+1}} < \infty$, we get that

$$\begin{aligned}
&\int_s^t \int_s^u \left|(t-u)^{\gamma-1} G(u,x_u) - (t-v)^{\gamma-1} G(v,x_v)\right| \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad \le \int_s^t \int_s^u \Big(\left|\left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right] G(v,x_v)\right| \\
&\qquad + \left|(t-u)^{\gamma-1}[G(u,x_u) - G(v,x_v)]\right|\Big) \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad \le \int_s^t \int_s^u L_3 \left|\left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]\right| \left(1 + \|x_v\|_{\mathcal{C}_r}\right) \frac{dvdu}{(u-v)^{\beta+1}} \\
&\qquad + \int_s^t \int_s^u L_2 (t-u)^{\gamma-1} \left[(u-v)^\mu + \|x_u - x_v\|_{\mathcal{C}_r}\right] \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad \le \int_s^t \int_s^u M \left|\left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]\right| (1 + \|x\|_{1-\alpha}) \frac{dvdu}{(u-v)^{\beta+1}} \\
&\qquad + \int_s^t \int_s^u M (t-u)^{\gamma-1} \left[(u-v)^\mu + (u-v)^{1-\alpha} \|x\|_{1-\alpha}\right] \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad \le M(t-s)^{\gamma-\beta} (1 + \|x\|_{1-\alpha}) + M(t-s)^{\gamma+\mu-\beta} + M(t-s)^{\gamma+1-\alpha-\beta} \|x\|_{1-\alpha} \\
&\quad \le M(t-s)^{1-\alpha} (1 + \|x\|_{1-\alpha})
\end{aligned} \tag{3.3}$$

for $s, t \in [0, T]$ with $s < t$. Combining this with (3.1) and (3.2), we see that

$$J_{11} \le M(t-s)^{1-\alpha} (1 + \|x\|_{1-\alpha})$$

for $s, t \in [0, T]$ with $s < t$. For J_{12}, we have

$$\begin{aligned}
J_{12} \le \Lambda_\beta(g) \Bigg\{ &\int_0^s \left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right] G(u,x_u)\right| \frac{du}{u^\beta} \\
&+ \int_0^s \int_0^u \left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right] [G(u,x_u) - G(v,x_v)]\right| \frac{dvdu}{(u-v)^{\beta+1}} \\
&+ \int_0^s \int_0^u \Big|\Big[(t-u)^{\gamma-1} - (t-v)^{\gamma-1} \\
&\qquad -(s-u)^{\gamma-1} + (s-v)^{\gamma-1}\Big] G(v,x_v)\Big| \frac{dvdu}{(u-v)^{\beta+1}} \Bigg\}
\end{aligned} \tag{3.4}$$

by (2.1). We now estimate the three terms in (3.4). For the first term, by the condition (H-G) we get that

$$\begin{aligned}
&\int_0^s \left|\left[(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right] G(u,x_u)\right| \frac{du}{u^\beta} \\
&\quad \le \int_0^s \left|(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right| L_3 (1 + \|x_u\|_{\mathcal{C}_r}) \frac{du}{u^\beta} \\
&\quad \le M (1 + \|x\|_{1-\alpha}) \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] u^{-\beta} du
\end{aligned}$$

$$
\begin{aligned}
&= M\left(1+\|x\|_{1-\alpha}\right)\left\{\int_0^s (s-u)^{\gamma-1}u^{-\beta}du\right.\\
&\qquad\left. -\int_0^t (t-u)^{\gamma-1}u^{-\beta}du+\int_s^t (t-u)^{\gamma-1}u^{-\beta}du\right\}\\
&\le M\left(1+\|x\|_{1-\alpha}\right)\int_s^t (t-u)^{\gamma-1}u^{-\beta}du\\
&\le M\left(1+\|x\|_{1-\alpha}\right)(t-s)^{\gamma-\beta}\le M(t-s)^{1-\alpha}(1+\|x\|_{1-\alpha}). \qquad (3.5)
\end{aligned}
$$

For the second term, we also have that

$$
\begin{aligned}
&\int_0^s\int_0^u \left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right][G(u,x_u)-G(v,x_v)]\right|\frac{dvdu}{(u-v)^{\beta+1}}\\
&\quad\le \int_0^s\int_0^u \left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]L_2\left[(u-v)^{\mu}+\|x_u-x_v\|_{\mathcal{C}_r}\right]\right|\frac{dvdu}{(u-v)^{\beta+1}}\\
&\quad\le M\int_0^s \left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]du\\
&\qquad\qquad \int_0^u \left[(u-v)^{\mu}+(u-v)^{1-\alpha}\|x\|_{1-\alpha}\right]\frac{dv}{(u-v)^{\beta+1}}\\
&\quad\le M[(t-s)^{\gamma+\mu-\beta}+\|x\|_{1-\alpha}(t-s)^{\gamma+1-\alpha-\beta}]\le M(1+\|x\|_{1-\alpha})(t-s)^{1-\alpha}. \qquad (3.6)
\end{aligned}
$$

Finally, some elementary calculations may show that

$$
\begin{aligned}
&\int_0^s\int_0^u \left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]\right|\frac{dvdu}{(u-v)^{\beta+1}}\\
&\qquad=\int_0^s\int_0^u \left[(s-u)^{\gamma-1}-(s-v)^{\gamma-1}\right]\frac{dvdu}{(u-v)^{\beta+1}}\\
&\qquad\quad-\int_0^t\int_0^u \left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\frac{dvdu}{(u-v)^{\beta+1}}\\
&\qquad\quad+\int_s^t\int_0^u \left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\frac{dvdu}{(u-v)^{\beta+1}}\\
&\qquad\le\left(\int_s^t\int_0^s+\int_s^t\int_s^u\right)\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right]\frac{dvdu}{(u-v)^{\beta+1}}\\
&\qquad\le\int_s^t\int_0^s\frac{(t-u)^{\gamma-1}}{(u-v)^{\beta+1}}dvdu+M(t-s)^{\gamma-\beta}\le M(t-s)^{\gamma-\beta}\le M(t-s)^{1-\alpha} \qquad (3.7)
\end{aligned}
$$

for $s,t\in[0,T]$ with $s<t$. It follows from the condition (H-G) that

$$
\begin{aligned}
&\int_0^s\int_0^u \left|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]G(v,x_v)\right|\frac{dvdu}{(u-v)^{\beta+1}}\\
&\le\int_0^s\int_0^u \Big|\left[(t-u)^{\gamma-1}-(t-v)^{\gamma-1}\right.\\
&\qquad\qquad\left.-(s-u)^{\gamma-1}+(s-v)^{\gamma-1}\right]L_3\left(1+\|x_v\|_{\mathcal{C}_r}\right)\Big|\frac{dvdu}{(u-v)^{\beta+1}}
\end{aligned}
$$

$$\leq M\left(1+\|x\|_{1-\alpha}\right)(t-s)^{1-\alpha} \tag{3.8}$$

for $s, t \in [0, T]$ with $s < t$. Combining this with (3.4), (3.5), (3.6) and (3.8), we get that

$$J_{12} \leq M(t-s)^{1-\alpha}\left(1+\|x\|_{1-\alpha}\right)$$

for $s, t \in [0, T]$ with $s < t$. Thus, we have showed that

$$|J(x)(t)-J(x)(s)| = J_{11}+J_{12} \leq M(t-s)^{1-\alpha}\left(1+\|x\|_{1-\alpha}\right)$$

for $s, t \in [0, T]$ with $s < t$, which implies that $J(x) \in \mathcal{C}^{1-\alpha}([0, T])$.

For the second assertion, fix $\beta \in (1-\nu, \alpha_0)$ as above. Then, by (2.1) we have that

$$\begin{aligned}
&|J(x)(t)-J(x)(s)| \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \\
&\leq \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left|\int_0^s\left[(t-u)^{\gamma-1} G(u, x_u)-(s-u)^{\gamma-1} G(u, x_u)\right] dg(u)\right| \\
&\quad+\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left|\int_s^t(t-u)^{\gamma-1} G(u, x_u) dg(u)\right| \\
&\leq \frac{\Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_0^s\left|(t-u)^{\gamma-1} G(u, x_u)-(s-u)^{\gamma-1} G(u, x_u)\right| u^{-\beta} du\right. \\
&\quad+\int_0^s \int_0^u \left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right.\right. \\
&\qquad\qquad \left.\left.-(t-v)^{\gamma-1}+(s-v)^{\gamma-1}\right] G(v, x_v)\right| \frac{dv du}{(u-v)^{\beta+1}} \\
&\quad+\int_0^s \int_0^u\left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right]\left[G(u, x_u)-G(v, x_v)\right]\right| \frac{du dv}{(u-v)^{\beta+1}} \\
&\quad+\int_s^t\left|(t-u)^{\gamma-1} G(u, x_u)\right| \frac{du}{(u-s)^\beta} \\
&\quad\left.+\int_s^t \int_s^u\left|(t-u)^{\gamma-1} G(u, x_u)-(t-v)^{\gamma-1} G(v, x_v)\right| \frac{dv du}{(u-v)^{\beta+1}}\right\} \\
&\equiv \sum_{i=1}^5 J_{2i}
\end{aligned} \tag{3.9}$$

for $t, s \in [0, T]$ with $s < t$. We need to estimate $J_{2i}, i = 1, 2, \ldots, 5$. From the condition (H-G), we get

$$\begin{aligned}
J_{24} &\leq \frac{M \Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_s^t \frac{\left|(t-u)^{\gamma-1}\left(1+\|x_u\|_{\mathcal{C}_r}\right)\right|}{(u-s)^\beta} du \\
&\leq M \Lambda_\beta(g) T^{\gamma-\beta+\alpha-1}+\frac{M \Lambda_\beta(g)\|x\|_{1-\alpha, \lambda}}{(t-s)^{1-\alpha}} \int_s^t \frac{(t-u)^{\gamma-1} e^{-\lambda(t-u)}}{(u-s)^\beta} du \\
&\leq M \Lambda_\beta(g) T^{\gamma-\beta+\alpha-1}+M \Lambda_\beta(g)\|x\|_{1-\alpha, \lambda} \int_s^t \frac{e^{-\lambda(t-u)}}{(t-u)^{2-\alpha-\gamma}(u-s)^\beta} du \\
&\leq M \Lambda_\beta(g) T^{\gamma-\beta+\alpha-1}+M \frac{\Lambda_\beta(g)\|x\|_{1-\alpha, \lambda}}{\lambda^{\gamma-1+\alpha-\beta}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz
\end{aligned} \tag{3.10}$$

and

$$
\begin{aligned}
J_{25} \le & \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_s^t \int_s^u \left| \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1} \right] G(v, x_v) \right| \frac{dvdu}{(u-v)^{\beta+1}} \right. \\
& \left. + \int_s^t \int_s^u \left| (t-u)^{\gamma-1} \left[G(u, x_u) - G(v, x_v) \right] \right| \frac{dvdu}{(u-v)^{\beta+1}} \right\} \\
\le & \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_s^t \int_s^u \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1} \right] \left(1 + e^{\lambda v} \|x\|_{1-\alpha,\lambda} \right) \frac{dvdu}{(u-v)^{\beta+1}} \right. \\
& + \int_s^t \int_s^u (t-u)^{\gamma-1} (u-v)^\mu \frac{dvdu}{(u-v)^{\beta+1}} \\
& \left. + \int_s^t \int_s^u (t-u)^{\gamma-1} e^{\lambda u} \|x\|_{1-\alpha,\lambda} \frac{dvdu}{(u-v)^{\beta+\alpha}} \right\} \\
\le & M\Lambda_\beta(g) \left[T^{\gamma-\beta+\alpha-1} + T^{\gamma-\beta+\alpha-1+\mu} \right] \\
& + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \left\{ \int_s^t \int_s^u \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1} \right] e^{-\lambda(t-v)} \frac{dvdu}{(u-v)^{\beta+1}} \right. \\
& \left. + \int_s^t \int_s^u (t-u)^{\gamma-1} e^{-\lambda(t-u)} \frac{dvdu}{(u-v)^{\beta+\alpha}} \right\} \\
\le & M\Lambda_\beta(g) \left\{ 1 + \frac{\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \int_0^{t-s} \int_0^{t-s-w} \frac{\left[r^{\gamma-1} - (r+w)^{\gamma-1} \right] e^{-\lambda(r+w)}}{w^{\beta+1}} drdw \right. \\
& \left. + \frac{\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \int_s^t (t-u)^{\gamma-1} e^{-\lambda(t-u)} (u-s)^{1-\beta-\alpha} du \right\} \\
\le & M\Lambda_\beta(g) \left\{ 1 + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-\beta-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \int_0^\infty z^{\alpha+\gamma-2} e^{-z} dz \right\}
\end{aligned} \tag{3.11}
$$

for $t, s \in [0, T]$ with $s < t$. For the term J_{21}, we have

$$
\begin{aligned}
& \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \frac{\left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1} \right] e^{\lambda u}}{u^\beta} du \\
& \quad = \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_0^s \frac{(s-u)^{\gamma-1} e^{\lambda u}}{u^\beta} du - \left(\int_0^t - \int_s^t \right) \frac{(t-u)^{\gamma-1} e^{\lambda u}}{u^\beta} du \right\} \\
& \quad \le \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_s^t \frac{(t-u)^{\gamma-1} e^{\lambda u}}{u^\beta} du \le \frac{1}{\lambda^{\gamma-1+\alpha-\beta}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz
\end{aligned}
$$

for all $s, t \in [0, T]$ with $s < t$. It follows that

$$
\begin{aligned}
J_{21} \le & \frac{\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1} \right] L_3 \left(1 + \|x_u\|_{\mathcal{C}_r} \right) u^{-\beta} du \\
& \le \frac{M\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1} \right] \left(1 + e^{\lambda u} \|x\|_{1-\alpha,\lambda} \right) u^{-\beta} du \\
& \le M\Lambda_\beta(g) T^{\gamma-1+\alpha-\beta} + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda} e^{-\lambda t}}{(t-s)^{1-\alpha}}
\end{aligned}
$$

$$\cdot \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] e^{\lambda u} u^{-\beta} du$$
$$\leq M\Lambda_\beta(g) T^{\gamma-1+\alpha-\beta} + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{\lambda^{\gamma-1+\alpha-\beta}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz \quad (3.12)$$

for $t, s \in [0, T]$ with $s < t$. For the term J_{22}, we have

$$\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \Big\{ \left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right]$$
$$- \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right] \Big\} e^{\lambda u} \frac{dv du}{(u-v)^{\beta+1}}$$
$$= \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_0^s \int_0^u \left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right] e^{\lambda u} \frac{dv du}{(u-v)^{\beta+1}} \right.$$
$$\left. - \left(\int_0^t \int_0^u - \int_s^t \int_0^u \right) \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right] e^{\lambda u} \frac{dv du}{(u-v)^{\beta+1}} \right\}$$
$$\leq \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left(\int_s^t \int_s^u + \int_s^t \int_0^s \right) \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right] e^{\lambda u} \frac{dv du}{(u-v)^{\beta+1}}$$
$$\leq \frac{M}{\lambda^{\alpha+\gamma-\beta-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz + \frac{M}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz$$

for $t, s \in [0, T]$ with $s < t$, which implies that

$$J_{22} \leq \frac{M\Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}}$$
$$\cdot \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right] - \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]}{(u-v)^{\beta+1}} \left(1 + \|x_v\|_{\mathcal{C}_r}\right) dv du$$
$$\leq M\Lambda_\beta(g) T^{\gamma-\beta+\alpha-1} + \frac{M\Lambda_\beta(g) e^{-\lambda t} \|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}}$$
$$\cdot \int_0^s \int_0^u \frac{\left\{\left[(s-u)^{\gamma-1} - (s-v)^{\gamma-1}\right] - \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right]\right\} e^{\lambda v}}{(u-v)^{\beta+1}} dv du$$
$$\leq M\Lambda_\beta(g) \left\{ 1 + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-\beta-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz \right.$$
$$\left. + \frac{\|x\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz \right\} \quad (3.13)$$

for $t, s \in [0, T]$ with $s < t$. For the last term J_{23}, we have

$$J_{23} \leq \frac{\Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}}$$
$$\cdot \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] L_2 \left[(u-v)^\mu + \|x_u - x_v\|_{\mathcal{C}_r}\right]}{(u-v)^{\beta+1}} dv du$$
$$\leq \frac{M\Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}}$$

$$\cdot \int_0^s \int_0^u \frac{\left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right]\left[(u-v)^{\mu}+e^{\lambda u}(u-v)^{1-\alpha}\|x\|_{1-\alpha,\lambda}\right]}{(u-v)^{\beta+1}} dvdu$$

$$\leq M\Lambda_\beta(g)T^{\mu-\beta+\gamma-1+\alpha} + \frac{M\Lambda_\beta(g)e^{-\lambda t}\|x\|_{1-\alpha,\lambda}}{(t-s)^{1-\alpha}} \cdot \int_0^s \left[(s-u)^{\gamma-1}-(t-u)^{\gamma-1}\right] e^{\lambda u}u^{1-\alpha-\beta}du$$

$$\leq M\Lambda_\beta(g)T^{\mu-\beta+\gamma-1+\alpha} + \frac{M\Lambda_\beta(g)\|x\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \int_0^\infty e^{-z}z^{\alpha+\gamma-2}dz \tag{3.14}$$

for $t, s \in [0,T]$ with $s < t$. Thus, we have showed that there exist some constants $M_1, M_1(\lambda) > 0$ such that

$$\|J(x)\|_{1-\alpha,\lambda} \leq M_1 + M_1(\lambda)\|x\|_{1-\alpha,\lambda},$$

where M_1 is independent of λ and $M_1(\lambda) \to 0$ as $\lambda \to \infty$. This completes the proof. □

Proposition 3.2. *Let $x, y \in \mathcal{C}^{1-\alpha}([-r,T])$ with $\|x\| \leq R$ and $\|y\| \leq R$. There exists $M_R^{(1)}(\lambda) > 0$ such that $M_R^{(1)}(\lambda) \to 0$ as $\lambda \to \infty$, and*

$$\|J(x)-J(y)\|_{1-\alpha,\lambda} \leq M_R^{(1)}(\lambda)\left(1+\|x\|_{1-\alpha}+\|y\|_{1-\alpha}\right)\|x-y\|_{1-\alpha,\lambda}.$$

Proof. Let $\beta \in (1-\nu, \alpha_0)$ as in the proof of Proposition 3.1. We have

$$\begin{aligned}
&\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\,|J(x)(t)-J(y)(t)-J(x)(s)+J(y)(s)| \\
&\quad\leq \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\left|\int_0^s \left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right][G(u,x_u)-G(u,y_u)]\,dg(u)\right|\right. \\
&\qquad\left. + \left|\int_s^t (t-u)^{\gamma-1}[G(u,x_u)-G(u,y_u)]\,dg(u)\right|\right\} \\
&\quad\equiv J_{31}+J_{32}
\end{aligned}$$

for $t, s \in [0,T]$ with $s < t$. From (2.1), we have

$$\begin{aligned}
J_{31} \leq{}& \frac{\Lambda_\beta(g)e^{-\lambda t}}{(t-s)^{1-\alpha}}\left\{\int_0^s \left|\left[(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right][G(u,x_u)-G(u,y_u)]\right|\frac{du}{u^\beta}\right. \\
&+ \int_0^s\int_0^u \Big|(t-u)^{\gamma-1}-(s-u)^{\gamma-1} \\
&\qquad - (t-v)^{\gamma-1}+(s-v)^{\gamma-1}\Big|\,|G(v,x_v)-G(v,y_v)|\frac{dvdu}{(u-v)^{\beta+1}} \\
&+ \int_0^s\int_0^u \left|(t-u)^{\gamma-1}-(s-u)^{\gamma-1}\right| \\
&\qquad\left. \cdot\,|G(u,x_u)-G(v,x_v)-G(u,y_u)+G(v,y_v)|\frac{dvdu}{(u-v)^{\beta+1}}\right\} \\
\equiv{}& \Lambda_\beta(g)\sum_{i=1}^3 J_{31}(i)
\end{aligned} \tag{3.15}$$

for $t, s \in [0, T]$ with $s < t$. Then, we have

$$\begin{aligned} J_{31}(1) \le &\frac{L_2 e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] \|x_u - y_u\|_{\mathcal{C}_r} \frac{du}{u^\beta} \\ \le &\frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz, \quad \text{(Similar to (3.12))}, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} J_{31}(2) \le &\frac{L_2 e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \Big|(t-u)^{\gamma-1} - (s-u)^{\gamma-1} \\ &- (t-v)^{\gamma-1} + (s-v)^{\gamma-1}\Big| \|x_v - y_v\|_{\mathcal{C}_r} \frac{dvdu}{(u-v)^{\beta+1}} \\ \le &\frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \left\{ \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz \right. \\ &\left. + \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz \right\}, \quad \text{(Similar to (3.13))}. \end{aligned} \tag{3.17}$$

By the mean value theorem and the condition (H-G), we have for all $u, v \in [0, T]$, $\|x\| \le R$ and $\|y\| \le R$,

$$\begin{aligned} &|G(u, x_u) - G(u, y_u) - G(v, x_v) + G(v, y_v)| \\ &= \left| \int_0^1 DG(v, rx_v + (1-r)y_v)(x_u - y_u) dr \right. \\ &\quad - \int_0^1 DG(v, rx_v + (1-r)y_v)(x_v - y_v) dr \\ &\quad + \int_0^1 DG(u, rx_u + (1-r)y_u)(x_u - y_u) dr \\ &\quad \left. - \int_0^1 DG(v, rx_v + (1-r)y_v)(x_u - y_u) dr \right| \\ &\le \left| \int_0^1 DG(v, rx_v + (1-r)y_v)(x_u - y_u - x_v + y_v) dr \right| \\ &\quad + \left| \int_0^1 [DG(u, rx_u + (1-r)y_u) - DG(v, rx_v + (1-r)y_v)](x_u - y_u) dr \right| \\ &\le L_2 \|x_u - y_u - x_v + y_v\|_{\mathcal{C}_r} \\ &\quad + L_R^{(2)} \left[\|x_u - x_v\|_{\mathcal{C}_r} + \|y_u - y_v\|_{\mathcal{C}_r} + (u-v)^\mu \right] \|x_u - y_u\|_{\mathcal{C}_r}. \end{aligned}$$

It follows that

$$\begin{aligned} J_{31}(3) \le &\frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_0^s \int_0^u \left|(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right| \right. \\ &\cdot L_2 \|x_u - y_u - x_v + y_v\|_{\mathcal{C}_r} \frac{L_2 dvdu}{(u-v)^{\beta+1}} \\ &+ \int_0^s \int_0^u \left|(t-u)^{\gamma-1} - (s-u)^{\gamma-1}\right| \end{aligned}$$

$$
\begin{aligned}
&\cdot L_R^{(2)} \left[\|x_u - x_v\|_{\mathcal{C}_r} + \|y_u - y_v\|_{\mathcal{C}_r}\right] \|x_u - y_u\|_{\mathcal{C}_r} \frac{L_R^{(2)} dv du}{(u-v)^{\beta+1}} \\
&+ \int_0^s \int_0^u \left|(t-u)^{\gamma-1} - (s-u)^{\gamma-1} L_R^{(2)}\right| (u-v)^{\mu} \|x_u - y_u\|_{\mathcal{C}_r} \frac{L_R^{(2)} dv du}{(u-v)^{\beta+1}} \Bigg\} \\
&\equiv J_{31}(3,1) + J_{31}(3,2) + J_{31}(3,3)
\end{aligned} \tag{3.18}
$$

for $t, s \in [0, T]$ with $s < t$. Some elementary calculations, such as those shown in (3.14), can illustrate that

$$
\begin{aligned}
J_{31}(3,1) &\le \frac{M e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] \|x - y\|_{1-\alpha,\lambda} \frac{e^{\lambda u} dv du}{(u-v)^{\beta+\alpha}} \\
&\le M \frac{\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-2} dz,
\end{aligned}
$$

$$
\begin{aligned}
J_{31}(3,2) &\le \frac{M_R e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] \\
&\qquad \cdot \|x_u - y_u\|_{\mathcal{C}_r} \left(\|x\|_{1-\alpha} + \|y\|_{1-\alpha}\right) \frac{e^{\lambda u} dv du}{(u-v)^{\beta+\alpha}} \\
&\le \frac{M_R \|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \left(\|x\|_{1-\alpha} + \|y\|_{1-\alpha}\right) \int_0^\infty e^{-z} z^{\alpha+\gamma-2} dz,
\end{aligned}
$$

and

$$
\begin{aligned}
J_{31}(3,3) \le &\frac{M_R e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_0^s \int_0^u \left[(s-u)^{\gamma-1} - (t-u)^{\gamma-1}\right] \\
&\cdot \|x-y\|_{1-\alpha,\lambda} \frac{e^{\lambda u} dv du}{(u-v)^{\beta+1-\mu}} \le M_R \frac{\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-2} dz
\end{aligned}
$$

for all $s, t \in [0, T]$ with $s < t$. Combining this with inequalities (3.15)-(3.18), we get

$$
J_{31} \le M_R(\lambda)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}) \|x-y\|_{1-\alpha,\lambda} \tag{3.19}
$$

for all $s, t \in [0, T]$ with $s < t$ and $\lim_{\lambda \to \infty} M_R(\lambda) = 0$.

Now, let us estimate the term J_{32}. From (2.1), we have that

$$
\begin{aligned}
J_{32} \le &\frac{\Lambda_\beta(g) e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_s^t \left|(t-u)^{\gamma-1} \left[G(u, x_u) - G(u, y_u)\right]\right| \frac{du}{(u-s)^\beta} \right. \\
&+ \int_s^t \int_s^u \left|(t-u)^{\gamma-1} - (t-v)^{\gamma-1}\right| |G(v, x_v) - G(v, y_v)| \frac{dv du}{(u-v)^{\beta+1}} \\
&\left. + \int_s^t \int_s^u (t-u)^{\gamma-1} |G(u, x_u) - G(v, x_v) - G(u, y_u) + G(v, y_v)| \frac{dv du}{(u-v)^{\beta+1}} \right\} \\
\equiv &\Lambda_\beta(g) \sum_{i=1}^3 J_{32}(i)
\end{aligned} \tag{3.20}
$$

for all $s, t \in [0, T]$ with $s < t$. Obviously, we have

$$\begin{aligned} J_{32}(1) \leq & \frac{L_2 e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_s^t (t-u)^{\gamma-1} \|x_u - y_u\|_{\mathcal{C}_r} \frac{du}{(u-s)^\beta} \\ \leq & \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \sup_{k>0} \int_0^k \frac{e^{-z}}{z^{2-\alpha-\gamma}(k-z)^\beta} dz, \quad \text{(Similar to (3.10))}, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} J_{32}(2) \leq & \frac{L_2 e^{-\lambda t}}{(t-s)^{1-\alpha}} \int_s^t \int_s^u \left[(t-u)^{\gamma-1} - (t-v)^{\gamma-1} \right] \|x_v - y_v\|_{\mathcal{C}_r} \frac{dv du}{(u-v)^{\beta+1}} \\ \leq & \frac{M\|x-y\|_{1-\alpha,\lambda}}{\lambda^{\gamma-\beta+\alpha-1}} \int_0^\infty e^{-z} z^{\alpha+\gamma-\beta-2} dz, \quad \text{(Similar to (3.11))}, \end{aligned} \tag{3.22}$$

for all $s, t \in [0, T]$ with $s < t$. In the same arguments as in the term $J_{31}(3)$, we decompose it into three separate terms and evaluate each through direct calculation. Then, we get

$$\begin{aligned} J_{32}(3) \leq & \frac{e^{-\lambda t}}{(t-s)^{1-\alpha}} \left\{ \int_s^t \int_s^u (t-u)^{\gamma-1} L_2 \|x_u - y_u - x_v + y_v\|_{\mathcal{C}_r} \frac{L_2 dv du}{(u-v)^{\beta+1}} \right. \\ & + \int_s^t \int_s^u (t-u)^{\gamma-1} L_R^{(2)} \left[\|x_u - x_v\|_{\mathcal{C}_r} + \|y_u - y_v\|_{\mathcal{C}_r} + (u-v)^\mu \right] \\ & \left. \cdot \|x_u - y_u\|_{\mathcal{C}_r} \frac{dv du}{(u-v)^{\beta+1}} \right\} \\ \leq & \frac{M_R \|x-y\|_{1-\alpha,\lambda}}{\lambda^{\alpha+\gamma-1}} (1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}) \int_0^\infty z^{\alpha+\gamma-2} e^{-z} dz \end{aligned} \tag{3.23}$$

for all $s, t \in [0, T]$ with $s < t$. It follows from (3.20)-(3.23) that

$$J_{32} \leq M_R(\lambda)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}) \|x-y\|_{1-\alpha,\lambda} \tag{3.24}$$

for all $s, t \in [0, T]$ with $s < t$ and $\lim_{\lambda\to\infty} M_R(\lambda) = 0$. Thus, we have showed that the desired estimate

$$\|J(x) - J(y)\| \leq M_R^{(1)}(\lambda) (1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}) \|x-y\|_{1-\alpha,\lambda}$$

holds and the constant $M_R^{(1)}(\lambda)$ satisfied $\lim\limits_{\lambda\to\infty} M_R^{(1)}(\lambda) = 0$. □

We can also get similar estimates for $I(x)$.

Proposition 3.3. *Let $x \in \mathcal{C}^{1-\alpha}([-r, T])$. We have that $I(x) \in \mathcal{C}^{1-\alpha}([0, T])$ and there exist some constants $M_2, M_2(\lambda) > 0$ such that*

$$\|I(x)\|_{1-\alpha,\lambda} \leq M_2 + M_2(\lambda)\|x\|_{1-\alpha,\lambda}.$$

If $x, y \in \mathcal{C}^{1-\alpha}([-r, T])$ with $\|x\| \leq R$ and $\|y\| \leq R$, there exists $M_R^{(2)}(\lambda) > 0$ such that

$$\|I(x) - I(y)\|_{1-\alpha,\lambda} \leq M_R^{(2)}(\lambda)\|x-y\|_{1-\alpha,\lambda}$$

Moreover, M_2 is independent of λ and $M_2(\lambda), M_R^{(2)}(\lambda) \to 0$ as $\lambda \to \infty$.

Define the operator

$$\Psi : \mathcal{C}^{1-\alpha}([-r,T]) \to \mathcal{C}^{1-\alpha}([-r,T])$$

by $\Psi(x)(t) = x(t)$ for $t \in [-r,0]$ and

$$\Psi(x)(t) = x(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,x_s)ds + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} G(s,x_s)dg(s)$$

for $t \in [0,T]$, where the coefficients f and G are as in the equation (1.1), $g \in \mathcal{C}^{\nu}([0,T])$ and $x \in \mathcal{C}^{1-\alpha}([-r,T])$. According to Proposition 3.2 and Proposition 3.3, we get the following result.

Corollary 3.1. *Let $x, y \in \mathcal{C}^{1-\alpha}([-r,T])$ with $\|x\| \le R$ and $\|y\| \le R$. There exist some constants $M_3, M_3(\lambda), M_R^{(3)}(\lambda) > 0$ such that $M_3(\lambda), M_R^{(3)}(\lambda) \to 0$ as $\lambda \to \infty$, and*

$$\|\Psi(x)\|_{1-\alpha,\lambda} \le M_3 + \|x(0)\|_{1-\alpha,\lambda} + M_3(\lambda)\|x\|_{1-\alpha,\lambda}$$

and

$$\begin{aligned}\|\Psi(x) - \Psi(y)\|_{1-\alpha,\lambda} \le & \|x(0) - y(0)\|_{1-\alpha,\lambda} \\ & + M_R^{(3)}(\lambda)\left(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha}\right)\|x-y\|_{1-\alpha,\lambda}.\end{aligned}$$

Consider the following form of the deterministic time fractional differential equation (2.3):

$$\begin{cases} x(t) = \eta(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,x_s)ds \\ \qquad\qquad + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} G(s,x_s)dg(s), \quad t \in (0,T] \\ x(t) = \eta(t), \quad t \in [-r,0], \end{cases}$$

where the coefficients f and G are as in the equation (1.1), $g \in \mathcal{C}^{\nu}([0,T])$ and $x \in \mathcal{C}^{1-\alpha}([-r,T])$.

Theorem 3.1. *If $\eta \in \mathcal{C}^{1-\alpha}([-r,0])$, then the equation (2.3) admits a unique solution in $\mathcal{C}^{1-\alpha}([-r,T])$.*

Proof. We first prove the existence of the solution. Let

$$\mathcal{H}^{1-\alpha}([-r,T],\eta) = \left\{ x \in \mathcal{C}^{1-\alpha}([-r,T]) \,\middle|\, x = \eta \text{ on } [-r,0] \right\}$$

and define the operator

$$\Phi : \mathcal{H}^{1-\alpha}([-r,T],\eta) \to \mathcal{H}^{1-\alpha}([-r,T],\eta)$$

by $\Phi(x)(t) = \eta(t)$ with $t \in [-r,0]$ and

$$\Phi(x)(t) = \eta(0) + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s,x_s)ds + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} G(s,x_s)dg(s)$$

with $t \in [0,T]$ for $g \in \mathcal{C}^{\nu}([0,T])$ and $x \in \mathcal{C}^{1-\alpha}([-r,T])$. Choose R such that $\|x\| \le R$. Then, Corollary 3.1 implies that

$$\|\Phi(x)\|_{1-\alpha,\lambda} \le M_3 + \|\eta\|_{1-\alpha,\lambda} + M_3(\lambda)\|x\|_{1-\alpha,\lambda},$$

where M_3 is a constant and $\lim_{\lambda\to\infty} M_3(\lambda) = 0$. Let λ_0 be sufficiently large such that $M_3(\lambda_0) \le \frac{1}{2}$ and denote $M_0 = 2(\|\eta\|_{1-\alpha,\lambda_0} + M_3)$ and

$$B_{\lambda_0} = \left\{x \in \mathcal{H}^{1-\alpha}([-r,T],\eta) \;\middle|\; \|x\|_{1-\alpha,\lambda_0} \le M_0\right\}.$$

We then have that Φ maps B_{λ_0} into itself. Thus, to give the existence we need to show that there exists $\lambda > \lambda_0$ such that Φ is a contraction on B_{λ_0} under the norm $\|\cdot\|_{1-\alpha,\lambda}$. Choose R such that $\|x\| \le R$ and $\|y\| \le R$. In fact, by Corollary 3.1, we see that

$$\|\Phi(x) - \Phi(y)\|_{1-\alpha,\lambda} \le M_R^{(3)}(\lambda)(1 + \|x\|_{1-\alpha} + \|y\|_{1-\alpha})\|x - y\|_{1-\alpha,\lambda}$$

for all $x, y \in \mathcal{H}^{1-\alpha}([-r,T],\eta)$. It follows that

$$\|\Phi(x) - \Phi(y)\|_{1-\alpha,\lambda} \le M_R^{(3)}(\lambda)(1 + 2R_0)\|x - y\|_{1-\alpha,\lambda}$$

for all $x, y \in B_{\lambda_0}$, where

$$R_0 = \sup_{x\in B_{\lambda_0}} \|x\|_{1-\alpha}.$$

Taking $\lambda > \lambda_0$ with $M_R^{(3)}(\lambda)(1 + 2R_0) < 1/2$ to lead

$$\|\Phi(x) - \Phi(y)\|_{1-\alpha,\lambda} \le \frac{1}{2}\|x - y\|_{1-\alpha,\lambda}$$

for $x, y \in B_{\lambda_0}$. This means that Φ is a contraction on the B_λ of the complete metric space $\mathcal{C}^{1-\alpha}([-r,T])$, which implies that Φ has a fixed point x in B_{λ_0}. From the definition of Φ, the fixed point x is a solution of (2.3) in $\mathcal{C}^{1-\alpha}([-r,T])$.

We now prove the uniqueness. Let x an y be two solutions of (2.3) in $\mathcal{C}^{1-\alpha}([-r,T])$, we can choose R such that $\|x\| \le R$ and $\|y\| \le R$. Then Corollary 3.1 implies that

$$\|x - y\|_{1-\alpha,\lambda} \le \frac{1}{2}\|x - y\|_{1-\alpha,\lambda}$$

for λ large enough, which shows that $x = y$. This completes the proof. □

By the properties of fractional Brownian motion, as a simple consequence of the above facts, we have expounded and proved the existence and uniqueness of the solution of (1.1).

Theorem 3.2. *Let $\frac{1}{2} < H < 1$ and $\frac{3}{2} - H < \gamma < 1$. Assume that the coefficients f and G satisfy the assumptions* (H-f) *and* (H-G)*, respectively. If $2-H-\gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}$ and $\eta \in \mathcal{C}^{1-\alpha}([-r,0])$ almost surely, then there exists a unique solution x of* (1.1) *with paths in $\mathcal{C}^{1-\alpha}([-r,T])$ almost surely.*

Remark 3.2. In [10], Han-Yan showed that the existence and uniqueness of the solution for (1.1) with the coefficients f and G are independent of t and under global Lipschitz conditions. But here we obtain the results for more general equations under local Lipschitz conditions.

4 Viability

Keep the notations in Section 3. For $g \in \mathcal{C}^{\nu}([0,T])$, we consider the deterministic time fractional differential equation of the form

$$x^{s\eta}(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} f(r, x_r^{s\eta})dr + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} G(r, x_r^{s\eta})dg(r), \quad t \in (s,T] \tag{4.1}$$

with the initial condition $x_s = \eta \in \mathcal{C}_r$, where $0 \le s \le T$.

Lemma 4.1. *If $x^{s\eta}$ is a solution of (4.1), then $x^{s\eta}$ is $(1-\alpha)$-Hölder continuous and*

$$\|x^{s\eta}\|_{1-\alpha;[s,T]} \le M(1 + \|\eta\|_{\mathcal{C}_r})$$

for all $0 \le s \le T$.

Proof. By Proposition 3.1 and 3.3, we have

$$\begin{aligned}\|x^{s\eta}\|_{1-\alpha,\lambda;[s,T]} &\le \|\eta\|_{\mathcal{C}_r} + \frac{1}{\Gamma(\gamma)}\left\|\int_s^t (t-r)^{\gamma-1} f(r, x_r^{s\eta})dr\right\|_{1-\alpha,\lambda;[s,T]} \\ &\quad + \frac{1}{\Gamma(\gamma)}\left\|\int_s^t (t-r)^{\gamma-1} G(r, x_r^{s\eta})dg(r)\right\|_{1-\alpha,\lambda;[s,T]} \\ &\le M_0(1 + \|\eta\|_{\mathcal{C}_r}) + M(\lambda)\|x^{s\eta}\|_{1-\alpha,\lambda;[s,T]}\end{aligned}$$

for all $\lambda \ge 1$ and $0 \le s < T$, where the constant $M_0 > 0$ is independent of λ, and $M(\lambda)$ converge to zero, as $\lambda \to \infty$. Thus, we can choose $\overline{\lambda}$ sufficiently large such that

$$M(\lambda) \le \frac{1}{2}$$

for all $\lambda > \overline{\lambda}$. It follows that

$$\|x^{s\eta}\|_{1-\alpha,\lambda;[s,T]} \le 2M_0(1 + \|\eta\|_{\mathcal{C}_r}).$$

and

$$\|x^{s\eta}\|_{1-\alpha;[s,T]} \le 2e^{\overline{\lambda}T} M_0(1 + \|\eta\|_{\mathcal{C}_r}) = M(1 + \|\eta\|_{\mathcal{C}_r})$$

for all $0 \le s < T$ and $\lambda \ge \overline{\lambda}$. This completes the proof. □

Lemma 4.2. *If X is a Hölder continuous function with $\|X\|_{1-\alpha;[s,T]} \le R$, then there exist some positive constants $M_R^i, i = 1,2,3,4$ such that*

$$\left|\int_\tau^t (t-r)^{\gamma-1}[f(r,X_r) - f(s,X_s)]dr\right| \le M_R^1 (t-\tau)^{\gamma}(t-s)^{1-\alpha}, \tag{4.2}$$

$$\left|\int_\tau^t (t-r)^{\gamma-1}[G(r,X_r) - G(s,X_s)]dg(r)\right| \le M_R^2 (t-\tau)^{1-\alpha}(t-s)^{\min\{\mu,1-\alpha\}} \tag{4.3}$$

with $0 \le s \le \tau \le t \le T$, *and*

$$\left|\int_s^{\tau_1} [(\tau_2 - r)^{\gamma-1} - (t-r)^{\gamma-1}][f(r, X_r) - f(\tau_1, X_{\tau_1})]dr\right| \le M_R^3(\tau_2 - t)^\gamma, \tag{4.4}$$

$$\left|\int_s^{\tau_1} [(\tau_2 - r)^{\gamma-1} - (t-r)^{\gamma-1}][G(r, X_r) - G(\tau_1, X_{\tau_1})]dg(r)\right| \le M_R^4(\tau_2 - t)^{1-\alpha} \tag{4.5}$$

with $0 \le s \le \tau_1 \le t \le \tau_2 \le T$.

Proof. By condition (H-f), we have that

$$\begin{aligned}
\left|\int_\tau^t (t-r)^{\gamma-1}[f(r, X_r) - f(s, X_s)]dr\right| &\le \int_\tau^t (t-r)^{\gamma-1}\,|f(r, X_r) - f(s, X_s)|\,dr \\
&\le \int_\tau^t (t-r)^{\gamma-1} dr \sup_{r\in[\tau,t]} |f(r, X_r) - f(s, X_s)| \\
&\le \frac{(t-\tau)^\gamma}{\gamma} L_R^{(1)} \sup_{r\in[\tau,t]} \left(|r-s| + \|X_r - X_s\|_{\mathcal{C}_r}\right) \\
&\le \frac{L_R^{(1)}(t-\tau)^\gamma}{\gamma} \sup_{r\in[\tau,t]} [(T^\alpha + R)(r-s)^{1-\alpha}] \le M_R^1 (t-\tau)^\gamma (t-s)^{1-\alpha}
\end{aligned}$$

which gives (4.2). Noting the condition (H-G) implies that

$$|G(r, X_r) - G(s, X_s)| \le L_2(|r-s|^\mu + \|X_r - X_s\|_{\mathcal{C}_r}) \le L_2(|r-s|^\mu + R|r-s|^{1-\alpha}),$$

fix $\beta \in (1-\nu, \alpha_0)$ and we get that

$$\begin{aligned}
&\left|\int_\tau^t (t-r)^{\gamma-1}[G(r, X_r) - G(s, X_s)]dg(r)\right| \\
&\le \Lambda_\beta(g)\Bigg\{\int_\tau^t \left|(t-r)^{\gamma-1}[G(r, X_r) - G(s, X_s)]\right| \frac{dr}{(r-\tau)^\beta} \\
&\quad + \int_\tau^t \int_\tau^r \Big|(t-r)^{\gamma-1}[G(r, X_r) - G(s, X_s)] \\
&\qquad\qquad - (t-u)^{\gamma-1}[G(u, X_u) - G(s, X_s)]\Big| \frac{du dr}{(r-u)^{\beta+1}}\Bigg\} \\
&\le \Lambda_\beta(g)\left\{\int_\tau^t (t-r)^{\gamma-1}|G(r, X_r) - G(s, X_s)| \frac{dr}{(r-\tau)^\beta}\right. \\
&\quad + \int_\tau^t \int_\tau^r \Big[\left|(t-r)^{\gamma-1} - (t-u)^{\gamma-1}\right| |G(u, X_u) - G(s, X_s)| \\
&\qquad\qquad \left. + (t-r)^{\gamma-1}|G(r, X_r) - G(u, X_u)|\Big] \frac{du dr}{(r-u)^{\beta+1}}\right\} \\
&\le \Lambda_\beta(g)\left\{\int_\tau^t (t-r)^{\gamma-1} L_2[(r-s)^\mu + R(r-s)^{1-\alpha}] \frac{dr}{(r-\tau)^\beta}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\tau}^{t}\int_{\tau}^{r}\Big\{\big|(t-r)^{\gamma-1}-(t-u)^{\gamma-1}\big|L_2[(u-s)^{\mu}+R(u-s)^{1-\alpha}] \\
&\qquad +(t-r)^{\gamma-1}L_2[(r-u)^{\mu}+R(r-u)^{1-\alpha}]\Big\}\frac{dudr}{(r-u)^{\beta+1}}\Big\} \\
&\le M_R\Big[(t-\tau)^{\gamma-\beta}(t-s)^{\min\{\mu,1-\alpha\}} \\
&\quad +(t-\tau)^{\gamma-\beta}(t-s)^{\min\{\mu,1-\alpha\}}+(t-\tau)^{\gamma-\beta+\min\{\mu,1-\alpha\}}\Big] \\
&\le M_R^2(t-\tau)^{1-\alpha}(t-s)^{\min\{\mu,1-\alpha\}},
\end{aligned}
$$

and (4.3) follows. Moreover, the inequality (4.4) follows from the next estimates:

$$
\begin{aligned}
&\left|\int_{s}^{\tau_1}\left[(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right][f(r,X_r)-f(\tau_1,X_{\tau_1})]dr\right| \\
&\quad\le\int_{s}^{\tau_1}\left[(t-r)^{\gamma-1}-(\tau_2-r)^{\gamma-1}\right]\left|f(r,X_r)-f(\tau_1,X_{\tau_1})\right|dr \\
&\quad\le\int_{s}^{\tau_1}\left[(t-r)^{\gamma-1}-(\tau_2-r)^{\gamma-1}\right]dr\sup_{r\in[s,\tau_1]}\left|f(r,X_r)-f(\tau_1,X_{\tau_1})\right| \\
&\quad\le\frac{(\tau_2-t)^{\gamma}}{\gamma}L_1\left(2+\|X_r\|_{\mathcal{C}_r}+\|X_{\tau_1}\|_{\mathcal{C}_r}\right)\le M_R^3(\tau_2-t)^{\gamma}.
\end{aligned}
$$

For the inequality (4.5) with $0\le s\le\tau_1\le t\le\tau_2\le T$, fix $\beta\in(1-\nu,\alpha_0)$. Since

$$
\begin{aligned}
&\int_{s}^{\tau_1}\left[(t-r)^{\gamma-1}-(\tau_2-r)^{\gamma-1}\right](r-s)^{-\beta}dr \\
&=\left(\int_{s}^{t}-\int_{\tau_1}^{t}\right)(t-r)^{\gamma-1}(r-s)^{-\beta}dr-\left(\int_{s}^{\tau_2}-\int_{\tau_1}^{\tau_2}\right)(\tau_2-r)^{\gamma-1}(r-s)^{-\beta}dr \\
&\le-\int_{\tau_1}^{t}(\tau_2-r)^{\gamma-1}(r-s)^{-\beta}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\gamma-1}(r-s)^{-\beta}dr \\
&=\int_{t}^{\tau_2}(\tau_2-r)^{\gamma-1}(r-s)^{-\beta}dr
\end{aligned}
$$

and similarly

$$
\begin{aligned}
&\int_{s}^{\tau_1}\int_{s}^{r}\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}+(t-u)^{\gamma-1}\right|\frac{dudr}{(r-u)^{\beta+1}} \\
&=\left(\int_{s}^{t}\int_{s}^{r}-\int_{\tau_1}^{t}\int_{s}^{r}\right)[(t-r)^{\gamma-1}-(t-u)^{\gamma-1}]\frac{dudr}{(r-u)^{\beta+1}} \\
&\qquad-\left(\int_{s}^{\tau_2}\int_{s}^{r}-\int_{\tau_1}^{\tau_2}\int_{s}^{r}\right)[(\tau_2-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}]\frac{dudr}{(r-u)^{\beta+1}} \\
&\le\int_{t}^{\tau_2}\int_{s}^{r}\left[(\tau_2-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}\right]\frac{dudr}{(r-u)^{\beta+1}},
\end{aligned}
$$

we also have that

$$
\left|\int_{s}^{\tau_1}[(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}][G(r,X_r)-G(\tau_1,X_{\tau_1})]dg(r)\right|
$$

$$\begin{aligned}
&\leq \Lambda_\beta(g)\left\{\int_s^{\tau_1}\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right||G(r,X_r)-G(\tau_1,X_{\tau_1})|\frac{dr}{(r-s)^\beta}\right.\\
&\quad+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}+(t-u)^{\gamma-1}\right|\\
&\qquad\cdot|G(u,X_u)-G(\tau_1,X_{\tau_1})|\frac{dudr}{(r-u)^{\beta+1}}\\
&\quad\left.+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right||G(u,X_u)-G(r,X_r)|\frac{dudr}{(r-u)^{\beta+1}}\right\}\\
&\leq M\Lambda_\beta(g)\left\{\int_s^{\tau_1}[(t-r)^{\gamma-1}-(\tau_2-r)^{\gamma-1}][(\tau_1-r)^\mu+R(\tau_1-r)^{1-\alpha}]\frac{dr}{(r-s)^\beta}\right.\\
&\quad+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}+(t-u)^{\gamma-1}\right|\\
&\qquad\cdot\left[(\tau_1-u)^\mu+R(\tau_1-u)^{1-\alpha}\right]\frac{dudr}{(r-u)^{\beta+1}}\\
&\quad\left.+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right|\left[(r-u)^\mu+R(r-u)^{1-\alpha}\right]\frac{dudr}{(r-u)^{\beta+1}}\right\}\\
&\leq M\Lambda_\beta(g)\left\{\int_s^{\tau_1}\left[(t-r)^{\gamma-1}-(\tau_2-r)^{\gamma-1}\right]\left[T^\mu+RT^{1-\alpha}\right]\frac{dr}{(r-s)^\beta}\right.\\
&\quad+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}+(t-u)^{\gamma-1}\right|\\
&\qquad\cdot\left[T^\mu+RT^{1-\alpha}\right]\frac{dudr}{(r-u)^{\beta+1}}\\
&\quad\left.+\int_s^{\tau_1}\int_s^r\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right|\left[\frac{dudr}{(r-u)^{\beta+1-\mu}}+\frac{Rdudr}{(r-u)^{\alpha+\beta}}\right]\right\}\\
&\leq M\Lambda_\beta(g)\left\{\int_t^{\tau_2}(\tau_2-r)^{\gamma-1}\frac{dr}{(r-s)^\beta}\right.\\
&\quad+\int_t^{\tau_2}\int_s^r\left[(\tau_2-r)^{\gamma-1}-(\tau_2-u)^{\gamma-1}\right]\frac{dudr}{(r-u)^{\beta+1}}\\
&\quad\left.+\int_s^{\tau_1}\left|(\tau_2-r)^{\gamma-1}-(t-r)^{\gamma-1}\right|\left[T^{\mu-\beta}+RT^{1-\alpha-\beta}\right]dr\right\}\\
&\leq M_R\left[(\tau_2-t)^{\gamma-\beta}+(\tau_2-t)^{\gamma-\beta}+(\tau_2-t)^\gamma\right]\\
&\leq M_R^4(\tau_2-t)^{1-\alpha}.
\end{aligned}$$

Thus, we have completed the proof. □

Lemma 4.3. *If X is a Hölder continuous function with $\|X\|_{1-\alpha;[s,T]}\leq R$, then there exist some positive constants $\overline{M}_R$, $\underline{M}_R$ such that*

$$\left|\int_s^\tau[(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}]f(u,X_u)du\right|\leq\overline{M}_R(t-\tau)^\gamma, \tag{4.6}$$

$$\left|\int_s^\tau[(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}]G(u,X_u)dg(u)\right|\leq\underline{M}_R(t-\tau)^{1-\alpha} \tag{4.7}$$

with $0 \le s \le \tau \le t \le T$.

Proof. The proof of (4.6) is easy, so we omit it here. For (4.7), fix $\beta \in (1-\nu, \alpha_0)$ and we have

$$
\begin{aligned}
&\left|\int_s^\tau [(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}]G(u,X_u)dg(u)\right| \\
&\le \Lambda_\beta(g)\Big\{\int_s^\tau \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| |G(u,X_u)| \frac{du}{(u-s)^\beta} \\
&\quad + \int_s^\tau \int_s^u \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| |G(u,X_u)-G(v,X_v)| \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad + \int_s^\tau \int_s^u \Big|(\tau-u)^{\gamma-1}-(\tau-v)^{\gamma-1} \\
&\qquad\qquad -(t-u)^{\gamma-1}+(t-v)^{\gamma-1}\Big| |G(v,X_v)| \frac{dvdu}{(u-v)^{\beta+1}}\Big\} \\
&\le \Lambda_\beta(g)\Big\{\int_s^\tau \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| L_3[1+\|X_u\|_{\mathcal{C}_r}] \frac{du}{(u-s)^\beta} \\
&\quad + \int_s^\tau \int_s^u \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| L_2[(u-v)^\mu+\|X_u-X_v\|_{\mathcal{C}_r}] \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad + \int_s^\tau \int_s^u \Big|(\tau-u)^{\gamma-1}-(\tau-v)^{\gamma-1} \\
&\qquad\qquad -(t-u)^{\gamma-1}+(t-v)^{\gamma-1}\Big| L_3[1+\|X_v\|_{\mathcal{C}_r}] \frac{dvdu}{(u-v)^{\beta+1}}\Big\} \\
&\le M\Lambda_\beta(g)\Big\{\int_s^\tau \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| (1+\|X\|_{1-\alpha;[s,T]}) \frac{du}{(u-s)^\beta} \\
&\quad + \int_s^\tau \int_s^u \left|(\tau-u)^{\gamma-1}-(t-u)^{\gamma-1}\right| \\
&\qquad\qquad \cdot \left[(u-v)^\mu+(u-v)^{1-\alpha}\|X\|_{1-\alpha;[s,T]}\right] \frac{dvdu}{(u-v)^{\beta+1}} \\
&\quad + \int_s^\tau \int_s^u \Big|(\tau-u)^{\gamma-1}-(\tau-v)^{\gamma-1} \\
&\qquad\qquad -(t-u)^{\gamma-1}+(t-v)^{\gamma-1}\Big| (1+\|X\|_{1-\alpha;[s,T]}) \frac{dvdu}{(u-v)^{\beta+1}}\Big\} \\
&\le M_R[(t-\tau)^{\gamma-\beta}+(t-\tau)^{\gamma-\beta+\min\{\mu,1-\alpha\}}+(t-\tau)^{\gamma-\beta}] \\
&\le M_R(t-\tau)^{\gamma-\beta} \le \underline{M}_R(t-\tau)^{1-\alpha}
\end{aligned}
$$

for $0 \le s \le \tau \le t \le T$ and the lemma follows. □

Definition 4.1. Let $\mathcal{K} = \{K(t) : t \in [0,T]\}$ be a family of subsets of $\mathbb{R}$. We say that $\mathcal{K}$ is viable for (4.1) if for each $s \in [0,T]$ and each $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$, there exists at least one mild solution $x^{s\eta} = \{x^{s\eta}(t), t \in [s-r,T]\}$ such that $x^{s\eta}(t) \in K(t)$ for all $t \in [s,T]$.

Definition 4.2. The family $\mathcal{K}$ is said to be invariant for (4.1) if for each $s \in [0, T]$ and each $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$, all mild solutions $\{x^{s\eta}\}$ of (4.1) have the property

$$x^{s\eta}(t) \in K(t)$$

for all $t \in [s, T]$.

Note that when the equation has a unique mild solution, viability is equivalent with invariance. That is, if the conditions (H-f) and (H-G) hold, viability and invariance for (4.1) is coincident.

Definition 4.3 (Contingency Set)**.** Let $s \in [0, T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$. The $(1-\alpha)-$fractional $g-$contingent set to $K(s)$ in (s, η), denoted by $C_{K(s)}(s, \eta)$, is the set of pairs $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}$ such that there exist $\bar{h} > 0$, a function $Q : [s, s + \bar{h}] \longrightarrow \mathbb{R}$, and moreover for every $R > 0$ satisfying $\|\eta\|_{\mathcal{C}_r} \leq R$ there exist some constants $\omega = \omega_R \in (0, 1)$, $\underline{G}_R > 0$ and $\overline{G}_R > 0$ independent of $(s, \bar{h})$ such that

$$|Q(\rho) - Q(\tau)| \leq \underline{G}_R |\rho - \tau|^{1-\alpha}, \qquad |Q(\tau)| \leq \overline{G}_R |\tau - s|^{1+\omega},$$

for all $\rho, \tau \in [s, s + \bar{h}]$ and

$$\eta(0) + \frac{(t-s)^\gamma}{\Gamma(\gamma+1)}\xi + \frac{\zeta}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} dg(r) + Q(t) \in K(t)$$

for all $t \in [s, s + \bar{h}]$.

Definition 4.4 (Tangency Set)**.** Let $s \in [0, T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$. The $(1-\alpha)-$fractional $g-$tangent set to $K(s)$ in (s, η), denoted by $T_{K(s)}(s, \eta)$, is the set of pairs $(\xi, \zeta) \in \mathbb{R} \times \mathbb{R}$, such that there exist $\tilde{h} > 0$, and two functions $U : [s, s + \tilde{h}] \longrightarrow \mathbb{R}$ and $V : [s, s + \tilde{h}] \longrightarrow \mathbb{R}$ with $U(s) = 0$ and $V(s) = 0$, and moreover for every $R > 0$ satisfying $\|\eta\|_{\mathcal{C}_r} \leq R$ there exist two constants $\underline{D}_R$, $\overline{D}_R$ independent of $(s, \tilde{h})$ such that

$$|U(\rho) - U(\tau)| \leq \underline{D}_R |\rho - \tau|^{1-\alpha}, \qquad |V(\rho) - V(\tau)| \leq \overline{D}_R |\rho - \tau|^{min\{\mu, 1-\alpha\}}$$

for all $\rho, \tau \in [s, s + \tilde{h}]$ and

$$\eta(0) + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [\xi + U(r)] dr + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [\zeta + V(r)] dg(r) \in K(t)$$

for all $t \in [s, s + \tilde{h}]$.

Lemma 4.4. *Given two stochastic process U, V satisfying the conditions in Definition 4.4. Then for all $s \leq \tau \leq t \leq s + \tilde{h}$, there exist constants $\underline{M}_R$, $\overline{M}_R$ such that*

$$\left| \int_\tau^t (t-r)^{\gamma-1} U(r) dr \right| \leq \underline{M}_R (t-s)^{1-\alpha} (t-\tau)^\gamma, \tag{4.8}$$

$$\left| \int_\tau^t (t-r)^{\gamma-1} V(r) dg(r) \right| \leq \overline{M}_R (t-s)^{\min\{\mu, 1-\alpha\}} (t-\tau)^{1-\alpha}. \tag{4.9}$$

Proof. For (4.8),

$$\left|\int_\tau^t (t-r)^{\gamma-1}U(r)dr\right| = \left|\int_\tau^t (t-r)^{\gamma-1}[U(r)-U(s)]dr\right|$$
$$\leq \underline{D}_R \int_\tau^t (t-r)^{\gamma-1}(r-s)^{1-\alpha}dr \leq \underline{M}_R(t-s)^{1-\alpha}(t-\tau)^{\gamma}.$$

Indeed, fix $\beta \in (1-\nu, \alpha_0)$ and we have

$$\left|\int_\tau^t (t-r)^{\gamma-1}V(r)dg(r)\right|$$
$$\leq \Lambda_\beta(g)\int_\tau^t \left\{\frac{|(t-r)^{\gamma-1}[V(r)-V(s)]|}{(r-\tau)^\beta}\right.$$
$$\left.+\int_\tau^r \frac{|(t-r)^{\gamma-1}V(r)-(t-u)^{\gamma-1}V(u)|}{(r-u)^{\beta+1}}du\right\}dr$$
$$\leq \Lambda_\beta(g)\int_\tau^t \left\{\frac{|(t-r)^{\gamma-1}[V(r)-V(s)]|}{(r-\tau)^\beta}\right.$$
$$\left.+\int_\tau^r \frac{|[(t-r)^{\gamma-1}-(t-u)^{\gamma-1}][V(u)-V(s)]|+|(t-r)^{\gamma-1}[V(r)-V(u)]|}{(r-u)^{\beta+1}}du\right\}dr$$
$$\leq M_R(t-s)^{\min\{\mu,1-\alpha\}}(t-\tau)^{\gamma-\beta} \leq \overline{M}_R(t-s)^{\min\{\mu,1-\alpha\}}(t-\tau)^{1-\alpha},$$

which gives (4.9). □

We can now state the main result in this section.

Theorem 4.1. *Let $\mathcal{K} = \{K(t) : t \in [0,T]\}$ be a family of nonempty closed subsets of $\mathbb{R}$. Assume (H-f) and (H-G) are satisfied and*

$$2-\nu-\gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}.$$

Then the following assertions are equivalent:

(i) $\mathcal{K}$ is $\mathcal{C}^{1-\alpha}$–viable for the fractional differential equation (4.1);

(ii) For $t \in [0,T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(t)$, $(f(t,\eta), G(t,\eta)) \in T_{K(t)}(t,\eta)$;

(iii) For $t \in [0,T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(t)$, $(f(t,\eta), G(t,\eta)) \in C_{K(t)}(t,\eta)$.

Proof. We first show that (i) implies (ii). Let $s \in [0,T]$, $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$ and $x^{s\eta} \in \mathcal{C}^{1-\alpha}([s,T];\mathbb{R})$ be a solution of (4.1) such that $x^{s\eta}(t) \in K(t)$ for all $t \in [s,T]$. Set $R > 0$ such that $\|\eta_s\|_{\mathcal{C}_r} \leq R$. Then, Lemma 4.1 implies that

$$\|x^{s\eta}\|_{1-\alpha;[s,T]} \leq R_0 = M(1+R).$$

Taking $\tilde{h} = \min\{T-t, 1\}$, we then have

$$x^{s\eta}(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1}f(r,x_r^{s\eta})dr + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1}G(r,x_r^{s\eta})dg(r)$$

for all $t \in [s, s+\tilde{h}]$. Clearly, for all $t \in [s, s+\tilde{h}]$

$$x^{s\eta}(t) = \eta(0) + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} \left[f(s,\eta) + U(r)\right] dr \\ + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} \left[G(s,\eta) + V(r)\right] dg(r),$$

where $U(r) = f(r, x_r^{s\eta}) - f(s,\eta)$ and $V(r) = G(r, x_r^{s\eta}) - G(s,\eta)$. Obviously, U and V satisfy the conditions described in Definition 4.4, and the assertion (ii) follows.

We nextly show that (ii) implies that (iii). Let $\|\eta\|_{\mathcal{C}_r} \le R$. We need to verify that

$$Q(t) = \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} U(r) dr + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} V(r) dg(r)$$

satisfies the Hölder condition from the definition of the contingency property. According to Lemma 4.4(the inequalities (4.8) and (4.9) with $\tau = s$), we have

$$|Q(t)| \le \tilde{G}_R (t-s)^{1+\omega}.$$

Let $s \le \tau \le t \le s + \bar{h}$. Fix $\beta \in (1-\nu, \alpha_0)$ and some elementary calculation may show that

$$\begin{aligned}
&\left| \int_s^\tau [(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1}] V(r) dg(r) \right| \\
&\le \frac{\Lambda_\beta(g)}{\Gamma(\gamma)} \Bigg\{ \int_s^\tau \frac{|[(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1}][V(r) - V(s)]|}{(r-s)^\beta} dr \\
&+ \int_s^\tau \int_s^r \frac{|[(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1}]V(r) - [(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}]V(u)|}{(r-u)^{\beta+1}} dudr \Bigg\} \\
&\le \frac{\Lambda_\beta(g)}{\Gamma(\gamma)} \Bigg\{ \int_s^\tau \frac{\underline{D}_R |[(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1}](r-s)^{\min\{\mu, 1-\alpha\}}|}{(r-s)^\beta} dr \\
&+ \int_s^\tau \int_s^r \frac{\left|[(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1} - (t-u)^{\gamma-1} + (\tau-u)^{\gamma-1}][V(r) - V(s)]\right|}{(r-u)^{\beta+1}} dudr \\
&+ \int_s^\tau \int_s^r \frac{|[(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}][V(r) - V(u)]|}{(r-u)^{\beta+1}} dudr \Bigg\} \\
&\le M(t-\tau)^{\gamma-\beta+\min\{\mu,1-\alpha\}} \le M(t-\tau)^{1-\alpha}.
\end{aligned}$$

Combining this with Lemma 4.4, we obtain that

$$\begin{aligned}
|Q(t) - Q(\tau)| =& \frac{1}{\Gamma(\gamma)} \left| \int_s^t (t-r)^{\gamma-1} U(r) dr + \int_s^t (t-r)^{\gamma-1} V(r) dg(r) \right. \\
&\left. - \int_s^\tau (\tau-r)^{\gamma-1} U(r) dr - \int_s^\tau (\tau-r)^{\gamma-1} V(r) dg(r) \right| \\
\le& \frac{1}{\Gamma(\gamma)} \Bigg\{ \left| \int_s^\tau \left[(t-r)^{\gamma-1} - (\tau-r)^{\gamma-1}\right] [U(r) - U(s)] dr \right| \\
&+ \left| \int_\tau^t (t-r)^{\gamma-1} [U(r) - U(s)] dr \right|
\end{aligned}$$

$$+\left|\int_s^\tau \left[(t-r)^{\gamma-1}-(\tau-r)^{\gamma-1}\right] V(r)dg(r)\right|$$
$$+\left|\int_\tau^t (t-r)^{\gamma-1}V(r)dg(r)\right|\Bigg\}$$
$$\leq \overline{G}_R(t-\tau)^{1-\alpha}.$$

This shows that Q is $(1-\alpha)$-Hölder continuous on $[s, s+\bar{h}]$.

We finally show that the assertion (iii) implies (i). Fix $s \in [0,T]$, $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$ and $0 < \varepsilon \leq 1$. Let $R > 0$ satisfy $\|\eta\|_{\mathcal{C}_r} \leq R$ and denote by $\mathcal{A}_\varepsilon(s,\eta)$ the set of pairs (T_x, x), where $T_x \in [0,T]$ and $x : [s, T_x] \to \mathbb{R}$ is a Hölder continuous function satisfying the following conditions:

(1) $x_s = \eta$, $x(t) \in K(t)$ for all $t \in [s, T_x]$ and there exists a positive constant $M_0 \geq R$ depending only on R, T, α, γ and independent of ε, such that

$$\|x(\cdot)\|_{1-\alpha;[s,T_x]} \leq M_0;$$

(2) The error function $\xi : [s, T_x] \to \mathbb{R}$ defined by

$$\xi(t) = x(t) - \eta(0) - \frac{1}{\Gamma(\gamma)}\int_s^t (t-u)^{\gamma-1} f(u, x_u)du - \frac{1}{\Gamma(\gamma)}\int_s^t (t-u)^{\gamma-1} G(u, x_u)dg(u)$$

satisfies

(i) $|\xi(t)| \leq \varepsilon(t-s)^{\gamma-\alpha}$ for all $0 \leq s < t \leq T_x$,

(ii) $|\xi(t) - \xi(\tau)| \leq M_1|t-\tau|^{1-\alpha}$ for all $0 \leq s < \tau < t \leq T_x$, where the constant M_1 depending only on R, T, α, γ and independent of ε.

It is easy to check that the set $\mathcal{A}_\varepsilon(s,\eta)$ is non-empty. Define

$$T_{x^\varepsilon} := \sup\{\theta \in [s,T] : \exists(\theta, x) \in \mathcal{A}_\varepsilon(s,\eta)\},$$

and construct a trajectory x^ε on the maximal interval $[s, T_{x^\varepsilon})$ by concatenating admissible trajectories on subintervals converging to T_{x^ε}. The uniform Hölder bound in condition (1) ensures that x^ε is uniformly continuous on $[s, T_{x^\varepsilon})$, which implies that $\lim_{r\to T_{x^\varepsilon}^-} x^\varepsilon(r)$ exists. Then, x^ε can be continuously extended to the closed interval $[s, T_{x^\varepsilon}]$. The extended function still satisfies conditions (1)–(2), hence $(T_{x^\varepsilon}, x^\varepsilon) \in \mathcal{A}_\varepsilon(s,\eta)$. We shall prove that $T_{x^\varepsilon} = T$ by reductio ad absurdum.

Assume that $T_{x^\varepsilon} < T$. Denote $x^\varepsilon_{T_{x^\varepsilon}} = \eta^\varepsilon$. We have $\|x^\varepsilon\|_{1-\alpha;[s,T_{x^\varepsilon}]} \leq M_0$ and in particular

$$\|\eta^\varepsilon\|_{\mathcal{C}_r} \leq M_0.$$

From the hypotheses we have known that

$$(f(T_{x^\varepsilon}, \eta^\varepsilon), G(T_{x^\varepsilon}, \eta^\varepsilon))$$

is $(1-\alpha)$-fractional g–contingent to $\mathcal{K}$ in $(T_{x^\varepsilon}, \eta^\varepsilon)$, i.e. there exist $\overline{h}_\varepsilon > 0$ sufficiently small (for the moment $0 < \overline{h}_\varepsilon < \min\{T - T_{x^\varepsilon}, 1\}$), and a Borel function

$$Q^\varepsilon : [T_{x^\varepsilon}, T_{x^\varepsilon} + \overline{h}_\varepsilon] \to \mathbb{R},$$

and some positive constants $\omega = \omega_{M_0} \in (0,1)$, $\overline{G}_0 = \overline{G}_{M_0}$, $\underline{G}_0 = \underline{G}_{M_0} > 0$ independent of $(T_{x^\varepsilon}, \overline{h}_\varepsilon)$ such that

$$|Q^\varepsilon(s) - Q^\varepsilon(\tau)| \le \overline{G}_0 |s-\tau|^{1-\alpha} \quad \text{and} \quad |Q^\varepsilon(s)| \le \underline{G}_0 |s - T_{x^\varepsilon}|^{1+\omega}$$

for all $s, \tau \in [T_{x^\varepsilon}, T_{x^\varepsilon} + \overline{h}_\varepsilon]$, and

$$\eta^\varepsilon(0) + \frac{(t - T_{x^\varepsilon})^\gamma}{\Gamma(\gamma+1)} f(T_{x^\varepsilon}, \eta^\varepsilon) + \frac{G(T_{x^\varepsilon}, \eta^\varepsilon)}{\Gamma(\gamma)} \int_{T_{x^\varepsilon}}^t (t-r)^{\gamma-1} dg(r) + Q^\varepsilon(t) \in K(t)$$

for all $t \in [T_{x^\varepsilon}, T_{x^\varepsilon} + \overline{h}_\varepsilon]$. We set $T^\varepsilon = T_{x^\varepsilon} + \overline{h}_\varepsilon$ and define $\hat{x}^\varepsilon : [s, T^\varepsilon] \to \mathbb{R}$ as an extension of x^ε by

$$\hat{x}^\varepsilon(t) = \begin{cases} x^\varepsilon(t), & t \in [s, T_{x^\varepsilon}] \\ \eta^\varepsilon(0) + \frac{(t-T_{x^\varepsilon})^\gamma}{\Gamma(\gamma+1)} f(T_{x^\varepsilon}, \eta^\varepsilon) & \\ \qquad + \frac{G(T_{x^\varepsilon}, \eta^\varepsilon)}{\Gamma(\gamma)} \int_{T_{x^\varepsilon}}^t (t-r)^{\gamma-1} dg(r) + Q^\varepsilon(t), & t \in (T_{x^\varepsilon}, T^\varepsilon]. \end{cases}$$

We will prove that the extension $(T^\varepsilon, \hat{x}^\varepsilon) \in \mathcal{A}_\varepsilon(s, \eta)$ in two steps.

Step I. Clearly $\hat{x}^\varepsilon_s = \eta^\varepsilon$ and $\hat{x}^\varepsilon(t) \in K(t)$ for all $t \in [s, T^\varepsilon]$. Now let we show that

$$\|\hat{x}^\varepsilon\|_{1-\alpha;[s,T^\varepsilon]} \le M_0^{(1)},$$

where $M_0^{(1)} \ge R$ and $M_0^{(1)}$ depends only on T, α, γ. For $T_{x^\varepsilon} \le \tau \le t \le T^\varepsilon$, we have

$$\begin{aligned} |\hat{x}^\varepsilon(t) - \hat{x}^\varepsilon(\tau)| = &\left| \frac{(t - T_{x^\varepsilon})^\gamma}{\Gamma(\gamma+1)} f(T_{x^\varepsilon}, \eta^\varepsilon) - \frac{(\tau - T_{x^\varepsilon})^\gamma}{\Gamma(\gamma+1)} f(T_{x^\varepsilon}, \eta^\varepsilon) \right| \\ &+ \frac{|G(T_{x^\varepsilon}, \eta^\varepsilon)|}{\Gamma(\gamma)} \left| \int_{T_{x^\varepsilon}}^t (t-r)^{\gamma-1} dg(r) - \int_{T_{x^\varepsilon}}^\tau (\tau-r)^{\gamma-1} dg(r) \right| \\ &+ |Q^\varepsilon(t) - Q^\varepsilon(\tau)| \\ \le &\frac{|t-\tau|^\gamma}{\Gamma(\gamma+1)} (1 + \|\eta^\varepsilon\|_{\mathcal{C}_r}) + \frac{M|t-\tau|^{1-\alpha}}{\Gamma(\gamma)} (1 + \|\eta^\varepsilon\|_{\mathcal{C}_r}) + M|t-\tau|^{1-\alpha} \\ \le &M_R |t-\tau|^{1-\alpha}, \end{aligned}$$

and

$$|\hat{x}^\varepsilon(t)| \le |\hat{x}^\varepsilon(t) - \hat{x}^\varepsilon(T_{x^\varepsilon})| + \|\eta^\varepsilon\|_{\mathcal{C}_r} \le M_R T^{1-\alpha} + M_0.$$

Hence,

$$\begin{aligned} \|\hat{x}^\varepsilon(\cdot)\|_{1-\alpha;[s,T^\varepsilon]} \le &2\|\hat{x}^\varepsilon(\cdot)\|_{1-\alpha;[s,T_{x^\varepsilon}]} + 2\|\hat{x}^\varepsilon(\cdot)\|_{1-\alpha;[T_{x^\varepsilon},T^\varepsilon]} \\ &\le 2M_0 + 2M_R T^{1-\alpha} + 2M_0 + 2M_R := M_0^{(1)}. \end{aligned}$$

Step II. Let the error functions $\xi^{\varepsilon} : [s, T_{x^{\varepsilon}}] \to \mathbb{R}$ and $\hat{\xi}^{\varepsilon} : [s, T^{\varepsilon}] \to \mathbb{R}$ as follows

$$
\begin{aligned}
\xi^{\varepsilon} &= x^{\varepsilon}(t) - \eta(0) - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} f(u, x_u^{\varepsilon}) du \\
&\qquad - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} G(u, x_u^{\varepsilon}) dg(u), \\
\hat{\xi}^{\varepsilon} &= \hat{x}^{\varepsilon}(t) - \eta(0) - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} f(u, \hat{x}_u^{\varepsilon}) du \\
&\qquad - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} G(u, \hat{x}_u^{\varepsilon}) dg(u).
\end{aligned}
$$

Clearly, $|\xi^{\varepsilon}(t)| = |\hat{\xi}^{\varepsilon}(t)| \le \varepsilon(t-s)^{\gamma-\alpha}$ for all $t \in [s, T_{x^{\varepsilon}}]$. Let $t \in [T_{x^{\varepsilon}}, T^{\varepsilon}]$. Using Lemma 4.2(the inequalities (4.2),(4.3) with $\tau = s = T_{x^{\varepsilon}}$, $X_s = \eta^{\varepsilon}$, $X_u = \hat{x}_u^{\varepsilon}$) and Lemma 4.3(the inequalities (4.6),(4.7) with $\tau = T_{x^{\varepsilon}}$, $X_u = x_u^{\varepsilon}$), we have

$$
\begin{aligned}
|\hat{\xi}^{\varepsilon}(t)| \le & \left| \eta^{\varepsilon}(0) - \eta(0) - \frac{1}{\Gamma(\gamma)} \int_s^{T_{x^{\varepsilon}}} (T_{x^{\varepsilon}} - u)^{\gamma-1} f(u, x_u^{\varepsilon}) du \right. \\
& \left. - \frac{1}{\Gamma(\gamma)} \int_s^{T_{x^{\varepsilon}}} (T_{x^{\varepsilon}} - u)^{\gamma-1} G(u, x_u^{\varepsilon}) dg(u) \right| \\
& + \left| \frac{(t - T_{x^{\varepsilon}})^{\gamma}}{\Gamma(\gamma+1)} f(T_{x^{\varepsilon}}, \eta^{\varepsilon}) + \frac{G(T_{x^{\varepsilon}}, \eta^{\varepsilon})}{\Gamma(\gamma)} \int_{T_{x^{\varepsilon}}}^t (t-r)^{\gamma-1} dg(r) \right. \\
& - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} f(u, \hat{x}_u^{\varepsilon}) du - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} G(u, \hat{x}_u^{\varepsilon}) dg(u) \\
& + \frac{1}{\Gamma(\gamma)} \int_s^{T_{x^{\varepsilon}}} (T_{x^{\varepsilon}} - u)^{\gamma-1} f(u, x_u^{\varepsilon}) du \\
& \left. + \frac{1}{\Gamma(\gamma)} \int_s^{T_{x^{\varepsilon}}} (T_{x^{\varepsilon}} - u)^{\gamma-1} G(u, x_u^{\varepsilon}) dg(u) \right| + |Q^{\varepsilon}(t)| \\
& \le \varepsilon (T_{x^{\varepsilon}} - s)^{\gamma-\alpha} + \underline{G}_0 (t - T_{x^{\varepsilon}})^{1+\omega} \\
& + \frac{1}{\Gamma(\gamma)} \left| \int_{T_{x^{\varepsilon}}}^t (t-u)^{\gamma-1} \left[f(T_{x^{\varepsilon}}, \eta^{\varepsilon}) - f(u, \hat{x}_u^{\varepsilon}) \right] du \right. \\
& \qquad \left. + \int_{T_{x^{\varepsilon}}}^t (t-u)^{\gamma-1} \left[G(T_{x^{\varepsilon}}, \eta^{\varepsilon}) - G(u, \hat{x}_u^{\varepsilon}) \right] dg(u) \right| \\
& + \frac{1}{\Gamma(\gamma)} \left| \int_s^{T_{x^{\varepsilon}}} [(T_{x^{\varepsilon}} - u)^{\gamma-1} - (t-u)^{\gamma-1}] f(u, x_u^{\varepsilon}) du \right. \\
& \qquad \left. + \int_s^{T_{x^{\varepsilon}}} [(T_{x^{\varepsilon}} - u)^{\gamma-1} - (t-u)^{\gamma-1}] G(u, x_u^{\varepsilon}) dg(u) \right| \\
& \le \varepsilon (T_{x^{\varepsilon}} - s)^{\gamma-\alpha} + \underline{G}_0 (t - T_{x^{\varepsilon}})^{1+\omega} + M_R^1 (t - T_{x^{\varepsilon}})^{1+\gamma-\alpha} \\
& \quad + M_R^2 (t - T_{x^{\varepsilon}})^{1-\alpha+\min\{\mu, 1-\alpha\}} + \overline{M}_R (t - T_{x^{\varepsilon}})^{\gamma} + \underline{M}_R (t - T_{x^{\varepsilon}})^{1-\alpha} \\
& \le \varepsilon (t-s)^{\gamma-\alpha}
\end{aligned}
$$

for $\overline{h}_\varepsilon$ sufficiently small. Hence

$$|\hat{\xi}^\varepsilon(t)| \le \varepsilon(t-s)^{\gamma-\alpha} \qquad \text{for all} \quad t \in [s, T^\varepsilon].$$

Let $T_{x^\varepsilon} \le \tau \le t \le T^\varepsilon$. Using Lemma 4.2 again (the inequalities (4.2),(4.3) with $s = T_{x^\varepsilon}$, $X_s = \eta^\varepsilon$, $X_u = \hat{x}^\varepsilon_u$, and the inequalities (4.4),(4.5) with $\tau_1 = T_{x^\varepsilon}$, $X_{\tau_1} = \eta^\varepsilon$, $X_u = \hat{x}^\varepsilon_u$, $T_{x^\varepsilon} \le \tau \le t \le T^\varepsilon$), we have

$$\begin{aligned}
|\hat{\xi}^\varepsilon(t) - \hat{\xi}^\varepsilon(\tau)| \le \frac{1}{\Gamma(\gamma)} \Bigg\{ & \Bigg| \int_\tau^t (t-u)^{\gamma-1}[f(T_{x^\varepsilon}, \eta^\varepsilon) - f(u, \hat{x}^\varepsilon_u)]du \\
& + \int_\tau^t (t-u)^{\gamma-1}[G(T_{x^\varepsilon}, \eta^\varepsilon) - G(u, \hat{x}^\varepsilon_u)]dg(u) \Bigg| \\
& + \Bigg| \int_{T_{x^\varepsilon}}^\tau [(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}] \left[f(T_{x^\varepsilon}, \eta^\varepsilon) - f(u, \hat{x}^\varepsilon_u)\right] du \\
& + \int_{T_{x^\varepsilon}}^\tau [(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}] \left[G(T_{x^\varepsilon}, \eta^\varepsilon) - G(u, \hat{x}^\varepsilon_u)\right] dg(u) \Bigg| \\
& + \Bigg| \int_\tau^{T_{x^\varepsilon}} [(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}] \left[f(T_{x^\varepsilon}, \eta^\varepsilon) - f(u, \hat{x}^\varepsilon_u)\right] du \\
& + \int_\tau^{T_{x^\varepsilon}} [(t-u)^{\gamma-1} - (\tau-u)^{\gamma-1}]\left[G(T_{x^\varepsilon}, \eta^\varepsilon) - G(u, \hat{x}^\varepsilon_u)\right]dg(u) \Bigg| \Bigg\} \\
& + |Q^\varepsilon(t) - Q^\varepsilon(t)| \\
\le\ & M_R(t-\tau)^{1-\alpha}.
\end{aligned}$$

From the definition of $\mathcal{A}_\varepsilon(s, \eta)$ it follows that

$$|\hat{\xi}^\varepsilon(t) - \hat{\xi}^\varepsilon(\tau)| = |\xi^\varepsilon(t) - \xi^\varepsilon(\tau)| \le M_1(t-\tau)^{1-\alpha}.$$

for $s \le \tau < t \le T_{x^\varepsilon}$.

On the other hand, we also have

$$\begin{aligned}
|\hat{\xi}^\varepsilon(t) - \hat{\xi}^\varepsilon(\tau)| &\le |\hat{\xi}^\varepsilon(t) - \hat{\xi}^\varepsilon(T_{x^\varepsilon})| + |\hat{\xi}^\varepsilon(T_{x^\varepsilon}) - \hat{\xi}^\varepsilon(\tau)| \\
&\le M_R(t - T_{x^\varepsilon})^{1-\alpha} + M_1(T_{x^\varepsilon} - \tau)^{1-\alpha} \\
&\le (M_R \vee M_1)(t-\tau)^{1-\alpha}
\end{aligned}$$

for $s \le \tau < T_{x^\varepsilon} < t \le T^\varepsilon$. Therefore, we have constructed an extension $(T^\varepsilon, \hat{x}^\varepsilon)$ with $T_{x^\varepsilon} < T^\varepsilon \le T$ such that $(T^\varepsilon, \hat{x}^\varepsilon) \in \mathcal{A}_\varepsilon(s, \eta)$. This contradicts the definition of T_{x^ε} as the supremum of admissible existence times. So, we have $T_{x^\varepsilon} = T$.

Let x^ε be the solution defined on $[s, T]$ of $\mathcal{A}_\varepsilon(s, \eta)$. Then from the definition of $\mathcal{A}_\varepsilon(s, \eta)$, we see that $x^\varepsilon_s = \eta$, $x^\varepsilon(t) \in K(t)$ for all $t \in [s, T]$, and there exists a positive constant $M_0 \le R$ depending only on α, γ, T, R, such that

$$\|x^\varepsilon(\cdot)\|_{1-\alpha;[s,T]} \le M_0.$$

Consider the error function $\xi^\varepsilon : [s, T] \to \mathbb{R}$ defined by

$$\xi^\varepsilon = x^\varepsilon(t) - \eta(0) - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} f(u, x^\varepsilon_u)du - \frac{1}{\Gamma(\gamma)} \int_s^t (t-u)^{\gamma-1} G(u, x^\varepsilon_u)dg(u)$$

such that

- $|\xi^\varepsilon(t)| \le \varepsilon(t-s)^{\gamma-\alpha}$ for all $0 \le s < t \le T$,
- $|\xi^\varepsilon(t) - \xi^\varepsilon(\tau)| \le M_1|t-\tau|^{1-\alpha}$ for all $0 \le s < \tau < t \le T$,

where the constant M_1 depends only on α, γ, T, R independent of ε.

To show that (1) is true, we need to estimate $\|\xi^\varepsilon\|_{\alpha,\lambda;[s,T]}$ and $\|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]}$. We have

$$\begin{aligned}\|\xi^\varepsilon\|_{\alpha,\lambda;[s,T]} \le \|\xi^\varepsilon\|_{\alpha;[s,T]} &= \sup_{t\in[s,T]} |\xi^\varepsilon(t)| + \sup_{s\le\tau<t\le T} \frac{|\xi^\varepsilon(t)-\xi^\varepsilon(\tau)|}{(t-\tau)^\alpha} \\ &\le \varepsilon(T-s)^{\gamma-\alpha} + \sup_{s\le\tau<t\le T} |\xi^\varepsilon(t)-\xi^\varepsilon(\tau)|^{\frac{1}{2}-\alpha} \frac{|\xi^\varepsilon(t)-\xi^\varepsilon(\tau)|^{\frac{1}{2}+\alpha}}{(t-\tau)^\alpha} \\ &\le \varepsilon(T-s)^{\gamma-\alpha} + [2\varepsilon(T-s)^{\gamma-\alpha}]^{\frac{1}{2}-\alpha} M_1^{\frac{1}{2}+\alpha}(T-s)^{(\frac{1}{2}-\alpha)(1+\alpha)} \\ &\le M_R\varepsilon^{\frac{1}{2}-\alpha}\end{aligned}$$

for any $\lambda \ge 0$. It remains to prove that the limit of the sequence x^ε exists as $\varepsilon \to 0$ and this limit is a solution to the equation (4.1). Let $0 < \varepsilon, \kappa \le 1$. Using the Proposition 3.2 and Proposition 3.3, we get

$$\begin{aligned}&\|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]} \\ &\quad\le \|I(x^\varepsilon) - I(x^\kappa)\|_{\alpha,\lambda;[s,T]} + \|J(x^\varepsilon) - J(x^\kappa)\|_{\alpha,\lambda;[s,T]} + \|\xi^\varepsilon - \xi^\kappa\|_{\alpha,\lambda;[s,T]} \\ &\quad\le M(\lambda)\|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]} + M_R(\lambda)(1 + \|x^\varepsilon\|_{\alpha,\lambda;[s,T]} + \|x^\varepsilon\|_{\alpha,\lambda;[s,T]}) \\ &\qquad \cdot \|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]} + M_R\varepsilon^{\frac{1}{2}-\alpha} + M_R\kappa^{\frac{1}{2}-\alpha} \\ &\quad\le [M(\lambda) + M_R(\lambda)(1+2M_0)]\|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]} + M_R\varepsilon^{\frac{1}{2}-\alpha} + M_R\kappa^{\frac{1}{2}-\alpha}.\end{aligned}$$

Let $\lambda = \bar{\lambda}$ sufficiently large such that

$$M(\lambda) + M_R(\lambda)(1+2M_0) \le \frac{1}{2}.$$

Then

$$\|x^\varepsilon - x^\kappa\|_{\infty;[s,T]} \le e^{\bar{\lambda}T}\|x^\varepsilon - x^\kappa\|_{\alpha,\lambda;[s,T]} \le 2M_R e^{\bar{\lambda}T}\left[\varepsilon^{\frac{1}{2}-\alpha} + \kappa^{\frac{1}{2}-\alpha}\right].$$

Hence there exists $x^{s\eta}$ such that $x^\varepsilon \to x^{s\eta}$ in $C([s,T];\mathbb{R})$ and $x^\varepsilon \to x^{s\eta}$ in $W^{\alpha,\infty}([s,T];\mathbb{R})$. Noting that

$$\|x^\varepsilon\|_{\infty;[s,T]} + \frac{|x^\varepsilon(t) - x^\varepsilon(\tau)|}{|t-\tau|^{1-\alpha}} \le M_0$$

for all $t, \tau \in [s,T]$, we obtain

$$\|x^{s\eta}(\cdot)\|_{1-\alpha;[s,T]} \le M_0$$

by setting $\varepsilon \to 0$. By $x^\varepsilon(t) \in K(t)$ for all $t \in [s,T]$ it follows that $x^{s\eta}(t) \in K(t)$ for all $t \in [s,T]$, which implies that $x^{s\eta}(t)$ is a solution of (4.1). Thus, we have prove that the assertion (1) is true and the theorem follows. □

Theorem 4.2. *Let $\mathcal{K} = \{K(t) : t \in [0,T]\}$ be a family of nonempty closed subsets of $\mathbb{R}$. Assume (H-f) and (H-G) are satisfied and*

$$2 - H - \gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}.$$

Then the following assertions are equivalent:

- *$\mathcal{K}$ is $\mathcal{C}^{1-\alpha}$-viable for the fractional differential equation (1.2), i.e. for each $s \in [0,T]$ and each $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$, there exists a mild solution $x^{s\eta}(\omega,\cdot) \in C^{1-\alpha}(s,T)$ of the equation*

$$\begin{cases} x^{s\eta}(t) = \eta_s(0) + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} f(r, x_r^{s\eta})dr \\ \qquad\qquad + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} G(r, x_r^{s\eta})dB^H(r), \quad t \in (s,T], a.s.\omega \in \Omega \\ x_s = \eta \in \mathcal{C}_r, \end{cases}$$

  and $x^{s\eta}(t) \in K(t)$, for all $t \in [s,T]$;

- *for each $s \in [0,T]$ and each $\eta \in \mathcal{C}_r$ with $\eta(0) \in K(s)$, $(f(s,\eta), G(s,\eta))$ is $(1-\alpha)$-fractional B^H-contingent to $K(s)$ in (s,η).*

Proof. As mentioned before, the result follows directly from the deterministic Theorem 4.1. □

Moreover, it follows:

Corollary 4.1. *Assume (H-f) and (H-G) are satisfied and*

$$2 - \nu - \gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}.$$

If $K = [a,b] \subset \mathbb{R}$ is independent of t, the following assertions are equivalent:

(j) K is $\mathcal{C}^{1-\alpha}$–viable for the fractional differential equation (4.1);

(jj) For $t \in [0,T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) = a$ or $\eta(0) = b$, $(f(t,\eta), G(t,\eta)) \in T_K(t,\eta)$;

(jjj) For $t \in [0,T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) = a$ or $\eta(0) = b$, $(f(t,\eta), G(t,\eta)) \in C_K(t,\eta)$.

Proof. Since K is independent of t, from Theorem 4.1, it is obvious that $(j) \Rightarrow (jj) \Rightarrow (jjj)$. It is sufficient to prove $(jjj) \Rightarrow (j)$. Let $t \in [0,T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) \in (a,b)$ be chosen arbitrary. Since $x^{s\eta}(t)$ is continuous, there exists a random viable $\tilde{h}$ such that for all $t \in [s, s+\tilde{h}]$,,

$$x^{s\eta}(t) = \eta(0) + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} f(r, x_r^{s\eta})dr + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1} G(r, x_r^{s\eta})dg(r) \in K,$$

which can be written as

$$x^{s\eta}(t) = \eta(0)+\frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1}[f(s,\eta)+U(r)]dr + \frac{1}{\Gamma(\gamma)}\int_s^t (t-r)^{\gamma-1}[G(s,\eta)+V(r)]dg(r) \in K,$$

where $U(r) = f(r,x_r^{s\eta}) - f(s,\eta)$ and $V(r) = G(r,x_r^{s\eta}) - G(s,\eta)$. Obviously $(f(t,\eta),G(t,\eta)) \in T_K(t,\eta)$. By (jjj) and taking into account that K is independent of t, we can get (iii) of Theorem 4.1. Then the corollary follows. □

5 Positive solution

An application of viability is the existence of positive solution. Firstly we prove some preliminary results.

Lemma 5.1. *For every $s \in [0,+\infty)$ and $1-H<\gamma$, we have*

$$\limsup_{t\downarrow s}\left|\frac{1}{(t-s)^\gamma}\int_s^t (t-r)^{\gamma-1}dB^H(r)\right| = \infty \tag{5.1}$$

with probability one.

Proof. Given $s \geq 0$ and consider Gaussian random variables

$$X_t(s) = \frac{\int_s^t (t-r)^{\gamma-1}dB^H(r)}{(t-s)^{\gamma+H-1}}, \quad t \geq s.$$

Notice that

$$\begin{aligned} E&\left|\frac{\int_s^t (t-r)^{\gamma-1}dB^H(r)}{(t-s)^{\gamma+H-1}}\right|^2 \\ &= \frac{H(2H-1)}{(t-s)^{2\gamma+2H-2}}\int_s^t\int_s^t (t-r_1)^{\gamma-1}(t-r_2)^{\gamma-1}|r_1-r_2|^{2H-2}dr_1dr_2 \\ &= \frac{H(2H-1)}{(t-s)^{2\gamma+2H-2}}\int_0^{t-s}\int_0^{t-s} x^{\gamma-1}y^{\gamma-1}|x-y|^{2H-2}dxdy \\ &= \frac{H(2H-1)}{\gamma+H-1}\mathbf{B}(\gamma,2H-1) \equiv \sigma^2, \end{aligned}$$

where $\mathbf{B}(\cdot,\cdot)$ is the classical Beta function. We see that

$$X_t(s) = \sigma\zeta$$

in distribution for any $0 \leq s < t$, where $\zeta \sim N(0,1)$. It remains to prove that the convergence (5.1) holds with probability one. To end this, for the non-negative sequence $\{\delta_n\}$ with $\delta_n \downarrow 0$ $(n\to\infty)$ we define the events

$$A_s(\delta_n) = \left\{\sup_{0\leq t-s\leq\delta_n}\left|\frac{1}{(t-s)^\gamma}\int_s^t (t-r)^{\gamma-1}dB^H(r)\right| > d\right\}$$

$$= \left\{ \sup_{0 \le t-s \le \delta_n} \left| \frac{1}{(t-s)^{1-H}} X_t(s) \right| > d \right\}, \qquad n = 1, 2, \ldots$$

for any $d > 0$ and $s \geq 0$. Then, $A_s(\delta_n) \supset A_s(\delta_{n+1})$ for all $n \geq 1$ and

$$\begin{aligned} P\left(A_s(\delta_n)\right) &\geq P\left(\left| \frac{1}{\delta_n{}^{1-H}} X_{\delta_n+s}(s) \right| > d \right) \\ &= P\left(|\zeta| > \frac{d}{\sigma} \delta_n^{1-H} \right) \longrightarrow P\left(|\zeta| > 0\right) = 1 \end{aligned}$$

for any $d > 0$ and $s \geq 0$, as n tends to infinity. This shows that

$$P\left(A_s(\delta_n)\right) \longrightarrow 1,$$

as n tends to infinity. By the Borel-Cantelli lemma and the arbitrariness of d, the convergence (5.1) follows. □

Remark 5.1. Since $\frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r)$ is centered Gaussian and symmetric in distribution, the events

$$\begin{aligned} A &= \left\{ \limsup_{t \downarrow s} \frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r) = \infty \right\}, \\ B &= \left\{ \liminf_{t \downarrow s} \frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r) = -\infty \right\} \end{aligned}$$

have the same probability. As $t \to \infty$, the variance of $\frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r)$ diverges, and the trajectories will almost surely be unbounded and therefore cross any fixed threshold. Consequently, these events are not disjoint. Lemma 5.1 implies that $P(A \cup B) = 1$. Then, the following convergence hold:

$$P\left(\limsup_{t \downarrow s} \frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r) = \infty \right) = 1$$

and

$$P\left(\liminf_{t \downarrow s} \frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r) = -\infty \right) = 1.$$

Lemma 5.2. *Assume (H-f) and (H-G) are satisfied and $2 - H - \gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}$. Let $K = [a, b]$ independent of t. Then for $t \in [0, T]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) = a$,*

$$(f(t, \eta), G(t, \eta)) \in T_K(t, \eta) \Longleftrightarrow f(t, a) \geq 0, G(t, a) = 0,$$

and $\eta \in \mathcal{C}_r$ with $\eta(0) = b$,

$$(f(t, \eta), G(t, \eta)) \in T_K(t, \eta) \Longleftrightarrow f(t, b) \leq 0, G(t, b) = 0.$$

Proof. It is convenient to prove only for the case: $\eta \in \mathcal{C}_r$ with $\eta(0) = a$, the other case is similar.

To prove the necessity. If $f(t,a) \geq 0$, $G(t,a) = 0$, taking $U(r) \equiv 0$, $V(r) \equiv 0$, we can find $\bar{h}$ small enough, such that for all $t \in [s, s+\bar{h}]$,

$$a \leq \eta(0) + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [f(s,a) + U(r)] dr + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [G(s,a) + V(r)] dB^H(r) \leq b.$$

This means that $(f(t,\eta), G(t,\eta)) \in T_K(t,\eta)$.

To prove the sufficiency. Since $(f(t,\eta), G(t,\eta)) \in T_K(t,\eta)$, there exist $\tilde{h} > 0$ and two functions U and V satisfying the conditions in Definition 4.4, and for all $t \in [s, s+\tilde{h}]$ and $\eta \in \mathcal{C}_r$ with $\eta(0) = a$,

$$a \leq \eta(0) + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [f(s,a) + U(r)] dr + \frac{1}{\Gamma(\gamma)} \int_s^t (t-r)^{\gamma-1} [G(s,a) + V(r)] dB^H(r) \leq b.$$

Then, we have

$$\frac{(t-s)^\gamma}{\gamma} f(s,a) + G(s,a) \int_s^t (t-r)^{\gamma-1} dB^H(r) + \int_s^t (t-r)^{\gamma-1} U(r) dr + \int_s^t (t-r)^{\gamma-1} V(r) dB^H(r) \geq 0. \tag{5.2}$$

By Remark 5.1, for P-almost every $\omega \in \Omega$, (5.2) is satisfied and there exists a sequence $s \leq t_n = t_n(\omega) \leq s + \bar{h}(\omega)$, $t_n \downarrow s$, such that

$$\lim_{t_n \downarrow s} \frac{1}{(t_n - s)^\gamma} \int_s^{t_n} (t_n - r)^{\gamma-1} dB^H(r,\omega) = +\infty.$$

According to Lemma 4.4

$$\frac{\int_s^{t_n} (t_n - s)^{\gamma-1} U(r) dr + \int_s^{t_n} (t_n - s)^{\gamma-1} V(r) dB^H(r,\omega)}{(t_n - s)^\gamma} \to 0,$$

we have $G(s,a) \leq 0$.

Similarly we can prove that $G(s,a) \geq 0$ by

$$P\left(\liminf_{t \downarrow s} \frac{1}{(t-s)^\gamma} \int_s^t (t-r)^{\gamma-1} dB^H(r) = -\infty \right) = 1.$$

Consequently,

$$G(s,a) = 0.$$

Then by (5.2), we deduce that

$$f(s,a) + \frac{\int_s^t (t-r)^{\gamma-1} U(r) dr + \int_s^t (t-r)^{\gamma-1} V(r) dB^H(r)}{(t-s)^\gamma} \geq 0.$$

Via Lemma 4.4, it follows that

$$f(s,a) \geq 0$$

by taking $t \to s$. □

Corollary 5.1. *Assume (H-f) and (H-G) are satisfied and* $2-H-\gamma < \alpha < \min\left\{\frac{1}{2}, \mu\right\}$. *Then, for any* $t \in [0,T]$ *and* $\eta \in \mathcal{C}_r$ *with* $\eta(0) \geq 0$, *the fractional equation* (1.1) *has a positive solution if and only if*

$$f(t,0) \geq 0, \quad G(t,0) = 0$$

for all $t \in [0,T]$.

Proof. Taking $K = [0,+\infty)$. By Lemma 5.2, we have that $(f(t,0), G(t,0)) \in T_K(t,0)$ if and only if $f(t,0) \geq 0$ and $G(t,0) = 0$. Taking into account of Corollary 4.1, the result follows. □

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