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$\mathbb{L}^p$-solution of generalized BSDEs in a general filtration with stochastic monotone coefficients

Badr Elmansouri*, Mohamed El Otmani

Laboratory of Analysis and Applied Mathematics (LAMA), Faculty of sciences Agadir, Ibn Zohr University, 80000, Agadir, Morocco

badr.elmansouri@edu.uiz.ac.ma (B. Elmansouri), m.elotmani@uiz.ac.ma (M. El Otmani)

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Abstract Multidimensional generalized backward stochastic differential equations (GBSDEs) are studied within a general filtration that supports a Brownian motion under weak assumptions on the associated data. The existence and uniqueness of solutions in $\mathbb{L}^p$ for $p \in (1, 2)$ are established. The results apply to generators that are stochastic monotone in the y-variable, stochastic Lipschitz in the z-variable, and satisfy a general stochastic linear growth condition.

Keywords Generalized backward stochastic differential equation, stochastic monotone generators, stochastic Lipschitz generators, $\mathbb{L}^p$-solution, general filtration

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1 Introduction

The theory of backward stochastic differential equations (BSDEs) has been thoroughly studied and shown to have a wide range of applications in various mathematical domains, including partial differential equations (PDEs) [43], stochastic control and differential games [29, 30], mathematical finance [17, 54], and other related fields. Bismut [5] originally introduced the concept as the adjoint equations related to stochastic Pontryagin maximum principles in stochastic control theory. Pardoux and Peng [46] were the first to study the general case of nonlinear multidimensional BSDEs. Roughly

*Corresponding author.

speaking, for a finite horizon time $T \in (0, +\infty)$, a solution of such equations, associated with a terminal value ξ and a generator (coefficient or driver) $f(\omega, t, y, z)$, is a pair of stochastic processes $(Y_t, Z_t)_{t \leq T}$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad t \in [0, T], \tag{1}$$

where $W = (W_t)_{t \leq T}$ is a standard Brownian motion, and the solution process $(Y_t, Z_t)_{t \leq T}$ is adapted to the natural filtration of W. Under a uniform Lipschitz condition on the driver f and a square integrability condition on ξ and the process $(f(\omega, t, 0, 0))_{t \leq T}$, the authors of [46] demonstrated the existence and uniqueness of a solution.

Following this work, many researchers have aimed to weaken the uniform Lipschitz continuity constraint on the generator to address more interesting problems. In this context, significant research has been conducted on the existence, uniqueness, and comparison theorems for $\mathbb{L}^2$-solutions of the BSDE (1) with square-integrable parameters and a condition weaker than the Lipschitz one considered in [46]; see, e.g., [2, 26, 28], among others. However, in some practical applications, even when considering an appropriate condition on the generator f weaker than the Lipschitz one, the terminal condition ξ and the driver process $(f(\omega, t, 0, 0))_{t \leq T}$ of the BSDE (1) are not necessarily assumed to be square-integrable. Consequently, considering BSDEs with $\mathbb{L}^p$-integrable data and $\mathbb{L}^p$-solutions for $p \geq 1$ has attracted significant interest over the last decade. Briand et al. [6] demonstrated the existence and uniqueness of $\mathbb{L}^p$-solutions for $p \in (1, 2)$ of the BSDE (1) when the generator f is monotonic in y and Lipschitz continuous in z. For additional relevant works, see [10, 13, 24, 25, 53] and the references therein.

Using a new class of BSDEs that involves the integral with respect to a continuous nondecreasing process interpreted as the local time of a diffusion process on the boundary, Pardoux and Zhang [48] provided a probabilistic representation for a solution of a system of parabolic and elliptic semilinear PDEs with the Neumann boundary conditions. This new type of BSDE is called Generalized BSDEs (GBSDEs). A solution of such equations is a pair of adapted processes $(Y_t, Z_t)_{t \leq T}$ that satisfies the equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s)d\kappa_s - \int_t^T Z_s dW_s, \quad t \in [0, T]. \tag{2}$$

Here:

- ξ is an $\mathbb{R}^d$-valued $\mathcal{F}_T$-measurable random variable,
- $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times k} \to \mathbb{R}^d$ is an $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbb{R}^{d \times k})$-measurable random function such that for any $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d \times k}$, the process $(\omega, t) \mapsto f(\omega, t, y, z)$ is progressively measurable,
- $g : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is an $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable random function such that for any $y \in \mathbb{R}^d$, the process $(\omega, t) \mapsto g(\omega, t, y)$ is progressively measurable,

- $\kappa = (\kappa_t)_{t\leq T}$ is an $\mathbb{R}$-valued adapted, continuous, nondecreasing process on $[0, T]$.

Under a monotonicity assumption on the drivers f and g, and appropriate $\mathbb{L}^2$-integrability conditions on the data, the authors established the existence and uniqueness of a solution using a convolution approximation. Extending this framework, Pardoux [45] addressed the discontinuous case by incorporating a jump term into (2), represented by an independent Poisson random measure. More recently, Elmansouri and El Otmani [20, 21] demonstrated existence and uniqueness results for GBSDEs in a general filtration under similar or more general assumptions compared to those in [45, 48].

In contrast to the standard BSDE formulation (1), the introduction of the Stieltjes–Lebesgue integral with respect to κ in (2) necessitates a refinement of the integrability conditions on the data (ξ, f, g). These conditions are stronger than those commonly considered in the literature for (1), which typically involve only (ξ, f). This adjustment reflects the deeper connection between each class of BSDEs and their associated PDEs. For the classical BSDE (1), various authors have provided probabilistic representations of solutions to systems of semilinear PDEs, both parabolic and elliptic (see, e.g., [43, 44, 49]). Elliptic equations with Dirichlet boundary conditions based on (1) have been studied in [11], while homogeneous linear Neumann boundary conditions have been addressed in [31]. In all these works, the underlying Markovian process is either a classical diffusion driven by a Brownian motion, as in [11, 44, 49], or a reflecting Brownian motion involving its boundary local time, as in [31]. However, in the presence of a nonlinear Neumann boundary condition, a probabilistic interpretation of the viscosity solution to a system of elliptic PDEs cannot be obtained via the standard BSDE (1). Instead, the generalized BSDE (5) must be employed, wherein the nonlinear term g appears in the boundary condition, leading to the incorporation of the boundary local time process κ. This change shifts the diffusion process from the classical to the reflected setting. Consequently, establishing a probabilistic representation for solutions of parabolic PDEs with nonlinear Neumann boundary conditions requires the generalized BSDE (5), together with integrability assumptions on the data involving the process κ, in order to ensure existence and uniqueness of a solution, as shown in [18, 45, 48], among others.

However, all the aforementioned works, concerning BSDE (1) or GBSDE (2), deal with square or $\mathbb{L}^p$-integrable parameters and different weak conditions on the drivers only in a Brownian framework. In such a case, it is well known that the predictable representation property holds for every local martingale (see, e.g., Theorem 43 in [51, p. 186]). However, this is no longer valid for more general filtrations (see Section III.4 in [32]), and the description of a solution must include an extra martingale term orthogonal to W. More precisely, for a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\leq T}$ carrying (or supporting) the Brownian motion W, every right-continuous with left limits (RCLL) local martingale $(\mathcal{N}_t)_{t\leq T}$ can be represented as (see, e.g., Lemma 4.24 in [32, p. 185])

$$\mathcal{N}_t = \int_0^t Z_s dW_s + M_t, \quad t \in [0, T], \tag{3}$$

for some predictable process $(Z_t)_{t\leq T}$ such that $\int_0^T \|Z_s\|^2 ds < +\infty$ a.s. and an RCLL local martingale $(M_t)_{t\leq T}$ such that $[M, W] = 0$ a.s., where $[M, W]$ denotes the covariation process of M and W. Therefore, in the general filtration setting $\mathbb{F}$, the standard BSDE takes the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T dM_s, \quad t \in [0, T]. \tag{4}$$

This type of BSDE (4) has been studied by Klimsiak and Rozkosz [35] in an arbitrary complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a complete right-continuous filtration $\mathbb{F}$, in the case where the driver f depends only on the state variable y, and the noise source is given by a general RCLL martingale. Additionally, a probabilistic definition of solutions to semilinear elliptic equations with operators associated with regular Dirichlet forms is provided. In the case of a stochastic basis supporting a Brownian motion W, Liang et al. [41] studied BSDEs of the form (4), where a connection with nonlinear PDEs involving integral operators is established. For RCLL martingales in a general filtration, this approach was developed in the groundbreaking works of Carbone et al. [7] and El Karoui and Huang [15] for classical BSDEs, and later by Elmansouri and El Otmani [21] for GBSDEs in a more general framework. In the same filtration context, but within the $\mathbb{L}^p$-setup, Kruse and Popier [36, 37] studied the problem of $\mathbb{L}^p$-solutions ($p > 1$) for BSDEs in a general filtration supporting a Brownian motion and an independent Poisson random measure. The authors proved the existence and uniqueness of a solution when the driver f is monotone with respect to y and uniformly Lipschitz with respect to z.

The general representation (3) in the filtration $\mathbb{F}$ arises in several important cases, particularly in financial applications. In this context, Kusuoka [38] showed that the representation (3) holds when $\mathbb{F}$ is the progressive enlargement of a Brownian filtration $\mathbb{G}$ by a default time τ, which is not necessarily a $\mathbb{G}$-stopping time. More precisely, one has $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}^0_{t+\varepsilon}$ where $\mathcal{F}^0_t := \mathcal{G}_t \vee \sigma(\min(\tau, t))$. In such cases, the orthogonal martingale M is given explicitly by

$$M_t = \int_0^t U_s(dD_s - \gamma_s ds),$$

where $D_t := \mathbb{1}_{\{\tau \le t\}}$ and $(\gamma_t)_{t\le T}$ is an $\mathbb{F}$-predictable intensity of τ, while $(U_t)_{t\le T}$ is an $\mathbb{F}$-predictable process such that $\int_0^T |U_s|^2 \gamma_s ds < +\infty$ a.s. These representations have been widely used in the BSDE literature (see, e.g., [12, 50]). For further studies, we refer to Section 3.3 in Bielecki et al. [4], Theorem 2.1 in Jeanblanc and Le Cam [34], and [33]. Besides Lemma 4.24 in [32, p. 185], this structure (3) is also known as the *Kunita–Watanabe decomposition*, explicitly developed in [1] and used in [16] for pricing contingent claims under general information flows (i.e., not necessarily generated by a Brownian motion). This type of decomposition has also been analyzed in the restricted information setting by Ceci et al. [9], and applied to risk minimization via BSDE techniques in [8].

Compared to the existing literature, and to the best of our knowledge, the study of $\mathbb{L}^p$-solutions for $p > 1$ in a filtration generated by a Brownian motion W has been primarily addressed in [47, Chapter 5], which also explores applications to PDEs, while GBSDEs in a more general filtration have been studied in the $\mathbb{L}^2$-case by Elmansouri and El Otmani [20, 21]. However, for $p \in (1, 2)$, $\mathbb{L}^p$-solutions for GBSDEs in a general filtration $\mathbb{F}$ have not been extensively investigated. More precisely, we consider the generalized formulation of the BSDE (4) given by the multidimensional GBSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) d\kappa_s - \int_t^T Z_s dW_s - \int_t^T dM_s, \quad t \in [0, T]. \tag{5}$$

The main idea here for the existence results is to study $\mathbb{L}^2$-solution for GBSDE (5), using and extending the general result from [21].

Motivated by the works mentioned above, it is naturally interesting to investigate the existence and uniqueness results for $\mathbb{L}^p$-solutions ($p \in (1, 2)$) of multidimensional GBSDEs (5) under suitable, more general conditions on the data. To this end, under a stochastic monotonicity condition on f and g with respect to the y-variable, a stochastic Lipschitz condition on f with respect to the z-variable, a general stochastic linear growth condition, and an appropriate $\mathbb{L}^p$-integrability condition on the data, we aim to establish a general existence and uniqueness result for $\mathbb{L}^p$-solutions of the multidimensional GBSDEs (5) for $p \in (1, 2)$.

In the sequel, we present several publications that studied the uniqueness and existence of $\mathbb{L}^p$-solutions for BSDEs (1) with time-varying or stochastic monotonic conditions on the coefficient f, since results for the GBSDE (5) have not been examined previously. Xiao et al. [52] consider the case where the driver g satisfies a time-varying monotonicity condition in y, meaning that there exists a deterministic integrable function $[0, T] \ni t \mapsto \alpha_t \in \mathbb{R}_+$ such that for each $y, y' \in \mathbb{R}^d$ and $z \in \mathbb{R}^{d \times k}$,

$$(y - y') \left(f(\omega, t, y, z) - f(\omega, t, y', z) \right) \leq \alpha_t \left| y - y' \right|^2, \quad d\mathbb{P} \otimes dt\text{-a.e.},$$

and a time-varying Lipschitz continuity condition on z, meaning that there exists a deterministic square-integrable function $[0, T] \ni t \mapsto \eta_t \in \mathbb{R}_+$ such that, for any $z, z' \in \mathbb{R}^{d \times k}$,

$$\left| f(\omega, t, y, z) - f(\omega, t, y, z') \right| \leq \eta_t \left\| z - z' \right\|, \quad d\mathbb{P} \otimes dt\text{-a.e.}$$

Under these conditions and appropriate $\mathbb{L}^p$-integrability conditions on the data ξ and the process $(f(t, 0, 0))_{t \leq T}$, the authors in [52] prove the existence and uniqueness of $\mathbb{L}^p$-solutions for $p > 1$ using the method of convolution and weak convergence. Pardoux and Răşcanu [47] also study existence and uniqueness results for $\mathbb{L}^p$-solutions ($p > 1$) for multidimensional BSDEs of the form (1) and (2) (see [47, Chapter 5]) under different growth conditions, including the case where $(\alpha_t)_{t \leq T}$, $(\eta_t)_{t \leq T}$ are deterministic functions and where α takes values in $\mathbb{R}$ as stochastic processes. They also provide the connection with semilinear PDEs and parabolic variational inequalities with a mixed nonlinear multivalued Neumann–Dirichlet boundary condition. Very recently, Li et al. [39] established the existence and uniqueness of $\mathbb{L}^p$-solutions for the BSDE (1) under a stochastic monotonicity condition on the driver f with respect to (y, z). Specifically, for two given positive progressively measurable processes $\Omega \times [0, T] \ni (\omega, t) \mapsto (\alpha_t(\omega), \eta_t(\omega)) \in \mathbb{R}_+ \times \mathbb{R}_+$, as a direct extension of the earlier work by Li et al. [40] in the $\mathbb{L}^2$ case, the following conditions are assumed:

$$(y - y') \left(f(\omega, t, y, z) - f(\omega, t, y', z) \right) \leq \alpha_t(\omega) \left| y - y' \right|^2, \quad d\mathbb{P} \otimes dt\text{-a.e.},$$

and

$$\left| f(\omega, t, y, z) - f(\omega, t, y, z') \right| \leq \eta_t(\omega) \left\| z - z' \right\|, \quad d\mathbb{P} \otimes dt\text{-a.e.},$$

for each $y, y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}^{d\times k}$.

In this paper, under the above-mentioned stochastic monotonicity (where we drop the positivity assumption on the process $(\alpha_t)_{t\le t}$) and Lipschitz conditions on the drivers f and g, we establish a general existence and uniqueness result for $\mathbb{L}^p$-solutions ($p \in (1,2)$) of multidimensional GBSDEs in a general filtration carrying a Brownian motion. The first part of this paper is devoted to establishing essential a priori $\mathbb{L}^p$-estimates for the solutions to the GBSDE (5) for $p \in (1,2)$. It is worth noting that, compared to the Brownian case with stochastic coefficients and $\mathbb{L}^p$-solutions ($p > 1$) treated in [47], we work within a general filtration setup. Therefore, our state process $(Y_t)_{t\le T}$ is not necessarily continuous, but only RCLL, which introduces additional challenges in our work. Specifically, compared to (1) or (2), our GBSDE (5) includes a jump term represented by the orthogonal martingale M, complicating the proof since the bracket process involving the quadratic jumps of M (or the state process Y) must be carefully handled. Afterwards, using these results, we study the existence and uniqueness of $\mathbb{L}^p$-solutions for $p \in (1,2)$ when the generators f and g are stochastically monotonic with respect to y, and f is stochastically Lipschitz with respect to z, along with a general stochastic linear growth condition in y. By using the result from the $\mathbb{L}^2$ case (i.e., for $p = 2$) established in the paper by Elmansouri and El Otmani [21], we derive the existence and uniqueness of $\mathbb{L}^p$-solutions for $p \in (1,2)$ by constructing an appropriate sequence of GBSDEs of the form (5).

Finally, it is worth mentioning that the case $p \in (2,+\infty)$ can also be treated within our framework without additional complexity compared to the case $p \in (1,2)$. This is due to the fact that for $p > 2$, the function $\mathbb{R}^d \ni x \mapsto |x|^p$ is sufficiently smooth, which allows for the direct application of Itô's formula and other classical arguments (see [13] for a related study). On the other hand, the case $p \in (1,2)$ is less regular and requires alternative representation formulas. In view of this, our results extend and improve upon the works of Briand et al. [6], Pardoux and Zhang [48], Xiao [52], the aforementioned contributions [2, 3, 10, 24–26, 46–48], and the recent studies [39, 40], among others.

The rest of this paper is organized as follows. Section 2 introduces some notations, definitions, and results used in the paper. Section 3 establishes some important a priori $\mathbb{L}^p$-estimates ($p \in (1,2)$) for solutions of the GBSDE (5). In Section 4, we prove the existence and uniqueness result for the $\mathbb{L}^p$-solutions for $p \in (1,2)$.

2 Preliminaries

Let $T > 0$ be a fixed deterministic horizon time, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\le T}$, carrying a k-dimensional Brownian motion $(W_t)_{t\le T}$. The filtration $\mathbb{F}$ is assumed to satisfy the usual conditions of right-continuity and completeness. The initial σ-field $\mathcal{F}_0$ is assumed to be trivial, and $\mathcal{F}$ is assumed to be $\mathcal{F}_T$. Unless explicitly stated, all stochastic processes are considered on the time interval $[0,T]$, and the measurability properties of stochastic processes (such as adaptedness, predictability, progressive measurability) are taken with respect to $\mathbb{F}$.

The bracket process (or quadratic variation) of any given $\mathbb{R}^d$-valued RCLL local martingale M is defined by $[M]$. The notation $[M]^c$ denotes the continuous part of the quadratic variation $[M]$. Specifically, the bracket process $[M]$ of M is defined for every $t \in [0,T]$ as follows:

$$[M]_t = \sum_{i=1}^{d} \left[M^i\right]_t ,$$

where M^i is the ith component of the vector $M = \left(M^1, M^2, \ldots, M^d\right)$.

For another given RCLL local martingale N, $[M,N]$ denotes the matrix-valued quadratic covariation process of M and N, defined for every $t \in [0,T]$ as

$$[M,N]_t = \left([M^i, N^j]_t\right)_{1\le i,j\le d} .$$

Note that our notation $[M]_t$ represents the trace of the matrix-valued process $[M,M]_t= \left([M^i, M^j]_t\right)_{1\le i,j\le d}$ and should not be confused with the full matrix-valued quadratic covariation.

The Euclidean norm of a vector $y \in \mathbb{R}^d$ is defined by $|y|^2 = \sum_{i=1}^{d} |y_i|^2$, and for a given matrix $z \in \mathbb{R}^{k\times d}$, we set $\|z\|^2 = \mathrm{Trace}(zz^*)$, where z^* denotes the transpose of z.

For an RCLL process $(\mathcal{X}_t)_{t\le T}$, $\mathcal{X}_{t-} := \lim\limits_{s\nearrow t} \mathcal{X}_s$ denotes the left limit of $\mathcal{X}$ at t. We set $\mathcal{X}_{0-} = \mathcal{X}_0$ by convention. The process $\mathcal{X}_- = (\mathcal{X}_{t-})_{t\in[0,T]}$ is called the left-limited process, and $\Delta\mathcal{X} = \mathcal{X} - \mathcal{X}_-$ is the jump process associated with $\mathcal{X}$. More precisely, for any $t \in [0,T]$, we have $\Delta\mathcal{X}_t = \mathcal{X}_t - \mathcal{X}_{t-}$, which is the jump of $\mathcal{X}$ at time t.

For an adapted process with finite variation $\mathcal{V} = (\mathcal{V}_t)_{t\le T}$, we denote by $\|\mathcal{V}\| = (\|\mathcal{V}\|_t)_{t\le T}$ the total variation process on $[0,T]$.

To simplify the notation, we omit any dependence on ω of a given process or random function. By convention, all brackets and stochastic integrals are assumed to be zero at time zero.

Let $\beta, \mu \ge 0$, $p > 1$ and $(a_t)_{t\le T}$ be a progressively measurable positive process. We consider the nondecreasing continuous process $A_t := \int_0^t a_s^2 ds$ for $t \in [0,T]$.

To define the $\mathbb{L}^p$-solution of our GBSDE (5) for $p > 1$, we need to introduce the following spaces.

- $\mathcal{D}^p_{\beta,\mu}$ is the space of $\mathbb{R}^d$-valued and $\mathbb{F}$-adapted RCLL processes $(Y_t)_{t\le T}$ such that

$$\|Y\|_{\mathcal{D}^p_{\beta,\mu}} = \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t + \frac{p}{2}\mu\kappa_t}|Y_t|^p\right]\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{S}^{p,A}_{\beta,\mu}$ is the space of $\mathbb{R}^d$-valued and $\mathbb{F}$-adapted RCLL processes $(Y_t)_{t\le T}$ such that

$$\|Y\|_{\mathcal{S}^{p,A}_{\beta,\mu}} = \left(\mathbb{E}\left[\int_0^T e^{\frac{p}{2}\beta A_s + \frac{p}{2}\mu\kappa_s}|Y_s|^p dA_s\right]\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{S}^{p,\kappa}_{\beta,\mu}$ is the space of $\mathbb{R}^d$-valued and $\mathbb{F}$-adapted RCLL processes $(Y_t)_{t\le T}$ such that

$$\|Y\|_{\mathcal{S}^{p,\kappa}_{\beta,\mu}} = \left(\mathbb{E}\left[\int_0^T e^{\frac{p}{2}\beta A_s + \frac{p}{2}\mu\kappa_s}|Y_s|^p d\kappa_s\right]\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{H}^p_{\beta,\mu}$ is the space of $\mathbb{R}^{d\times k}$-valued and $\mathbb{F}$-predictable processes $(Z_t)_{t\leq T}$ such that

$$\|Z\|_{\mathcal{H}^p_{\beta,\mu}} = \left(\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s}\|Z_s\|^2 ds\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{M}^p_{\beta,\mu}$ is the space of all $\mathbb{F}$-martingales $(M_t)_{t\leq T}$ orthogonal to W such that

$$\|M\|_{\mathcal{M}^p_{\beta,\mu}} = \left(\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s} d\left[M\right]_s\right)^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} < +\infty.$$

Finally, we define the spaces:

- $\mathfrak{B}^p_{\beta,\mu} := \mathcal{D}^p_{\beta,\mu} \cap \mathcal{S}^{p,A}_{\beta,\mu} \cap \mathcal{S}^{p,\kappa}_{\beta,\mu}$ endowed with the norm $\|Y\|_{\mathfrak{B}^p_{\beta,\mu}} = \|Y\|^p_{\mathcal{D}^p_{\beta,\mu}} + \|Y\|^p_{\mathcal{S}^{p,A}_{\beta,\mu}} + \|Y\|^p_{\mathcal{S}^{p,\kappa}_{\beta,\mu}}$,
- $\mathcal{E}^p_{\beta,\mu} := \mathfrak{B}^p_{\beta,\mu} \times \mathcal{H}^p_{\beta,\mu} \times \mathcal{M}^p_{\beta,\mu}$ endowed with the norm $\|(Y,Z,M)\|^p_{\mathcal{E}^p_{\beta,\mu}} := \|Y\|^p_{\mathfrak{B}^p_{\beta,\mu}} + \|Z\|^p_{\mathcal{H}^p_{\beta,\mu}} + \|M\|^p_{\mathcal{M}^p_{\beta,\mu}}$.

Let $p > 1$. Throughout the rest of this paper, the following conditions on the terminal value ξ and the generator f are denoted by $\mathbf{(H\text{-}M)_p}$.

Conditions on the data (ξ, f, g, κ). For some $\beta, \mu > 0$, we assume the following.

(H1) For all $(t,z) \in [0,T]\times\mathbb{R}^{d\times k}$, the mappings $y \mapsto f(t,y,z)$ and $y \mapsto g(t,y)$ are continuous, $d\mathbb{P}\otimes dt$-a.e. and $d\mathbb{P}\otimes d\kappa_t$-a.e., respectively.

(H2) There exists an $\mathbb{F}$-progressively measurable process $\alpha : \Omega\times[0,T] \to \mathbb{R}$ such that for all $t \in [0,T]$, $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times k}$, $d\mathbb{P}\otimes dt$-a.e.,

$$(y-y')\left(f(t,y,z) - f(t,y',z)\right) \leq \alpha_t \left|y-y'\right|^2.$$

(H3) There exists an $\mathbb{F}$-progressively measurable process $\theta : \Omega\times[0,T] \to \mathbb{R}^*_-$ such that for all $t \in [0,T]$, $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times k}$, $d\mathbb{P}\otimes dt$-a.e.,

$$(y-y')\left(g(t,y) - g(t,y')\right) \leq \theta_t \left|y-y'\right|^2.$$

(H4) There exists an $\mathbb{F}$-progressively measurable process $\eta : \Omega\times[0,T] \to \mathbb{R}^*_+$ such that for all $t \in [0,T]$, $y, \in \mathbb{R}^d$, $z, z' \in \mathbb{R}^{d\times k}$, $d\mathbb{P}\otimes dt$-a.e.,

$$\left|f(t,y,z) - f(t,y,z')\right| \leq \eta_t \left\|z-z'\right\|.$$

(H5) There exists a constant $\epsilon > 0$ such that $a_s^2 := \phi_s^2 + \eta_s^2 \geq \epsilon$ for any $s \in [0,T]$.[1]

[1] Note that the choice of η_s is not unique. In particular, if the condition $a_s^2 \geq \epsilon$ is not satisfied, one can replace η_s by $\eta_s + \epsilon$ for some constant $\epsilon > 0$.

(H6) The process $\kappa = (\kappa_t)_{t\leq T}$ is a real-valued, $\mathbb{F}$-adapted, continuous, and nondecreasing process, and the terminal condition ξ is an $\mathcal{F}_T$-measurable random variable satisfying

$$\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}|\xi|^p\right]<+\infty.$$

(H7) There exist three progressively measurable processes $\varphi,\psi:\Omega\times[0,T]\to[1,+\infty)$, $\phi:\Omega\times[0,T]\to(0,+\infty)$, $\zeta:\Omega\times[0,T]\to(0,\Gamma]$ for some $\Gamma>0$ such that $|f(t,y,0)|\leq\varphi_t+\phi_t|y|$, $|g(t,y)|\leq\psi_t+\zeta_t|y|$ and

$$\mathbb{E}\int_0^T e^{\beta A_s+\mu\kappa_s}\left(|\varphi_s|^p\,ds+|\psi_s|^p\,d\kappa_s\right)<+\infty.$$

We now provide the definition of an $\mathbb{L}^p$-solution for the GBSDE (5).

Definition 1. Let $p\in(1,2]$. A triplet $(Y_t,Z_t,M_t)_{t\leq T}$ is called an $\mathbb{L}^p$-solution of the GBSDE (5) if the following conditions are satisfied:

- (Y,Z,M) satisfies (5),
- (Y,Z,M) belongs to $\mathcal{E}^p_{\beta,\mu}$ for some $\beta,\mu>0$.

Sometimes, we shall refer to the triplet $(Y_t,Z_t,M_t)_{t\leq T}$ as a solution in $\mathcal{E}^p_{\beta,\mu}$ if (Y,Z,M) satisfies (5) and belongs to $\mathcal{E}^p_{\beta,\mu}$ for some $\beta,\mu\geq 0$.

Let us point out that compared to the existing literature on $\mathbb{L}^p$-solutions ($p>1$) for classical BSDEs with monotonic drivers and deterministic parameters, the standard integrability condition on the terminal condition ξ as in [6, 22, 36, 37], given by $\mathbb{E}\left[|\xi|^p\right]<+\infty$, suffices to derive the desired a priori estimates and, consequently, the existence and uniqueness results using approximation techniques.

For BSDEs with stochastic Lipschitz drivers in the case $p=2$, as in [15], the condition $\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right]<+\infty$ is imposed to derive analogous results. In the same $\mathbb{L}^2$-setting, but for the class of generalized BSDEs involving the local time of a diffusion process on the boundary κ, the assumption $\mathbb{E}\left[e^{\mu\kappa_T}|\xi|^2\right]<+\infty$ is used in [45, 48] to establish existence and uniqueness.

Thus, in our work which combines both stochastic monotonicity (in y) and stochastic Lipschitz continuity (in z), the integrability condition (H6) is a natural continuation and extension of this framework. In particular, in the case where the stochastic processes $(A_t)_{t\leq T}$ and $(\kappa_t)_{t\leq T}$ are bounded by a constant $\mathfrak{c}>0$, i.e., $A_T+\kappa_T\leq\mathfrak{c}$, the weighted integrability condition (H6) and the standard one $\mathbb{E}\left[|\xi|^p\right]<+\infty$ are equivalent. Indeed, using the monotonicity of A and κ, we have

$$\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}|\xi|^p\right]\leq e^{\frac{p}{2}\mathfrak{c}(\beta+\mu)}\mathbb{E}\left[|\xi|^p\right]\leq e^{\frac{p}{2}\mathfrak{c}(\beta+\mu)}\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}|\xi|^p\right].$$

Therefore, in this particular situation, the weighted integrability condition (H6) and the standard $\mathbb{L}^p$ condition are equivalent. A similar analysis holds when A is a deterministic function and κ is bounded. Furthermore, when A is bounded (e.g., when $\phi_t=\phi$ and $\eta_t=\eta$ with $\phi,\eta>0$ constants) and $p=2$, we recover the classical integrability setting for GBSDEs [18, 45, 48].

Following this discussion on the terminal condition ξ and assumption (H6), a similar rationale applies to the integrability conditions on the drivers f and g given in (H7), which are naturally imposed in the $\mathbb{L}^p$-setting for GBSDEs with stochastic monotonicity and stochastic Lipschitz continuity.

Finally, even in the context of generalized BSDEs where A is bounded or deterministic, the equivalence between the two norms may not always hold. Indeed, it is possible for the standard integrability condition $\mathbb{E}\left[|\xi|^p\right] < +\infty$ to be satisfied while (H6) fails, which illustrates the necessity of our assumption, as supported by the literature on GBSDEs with continuous or jump frameworks [45, 48].

To illustrate such a case, we consider a filtered probability space $\mathbb{G}$ generated by a Brownian motion W, completed and made right-continuous. Let $(X_t)_{t\in[0,T]}$ be a reflected Brownian motion (see, e.g., [42]) satisfying the Skorokhod SDE on $[0,1]$, i.e.,

$$X_t = x + W_t + \mathcal{L}_t^0 - \mathcal{L}_t^1, \quad t \in [0,T],$$

where:

- $x \in [0,1]$ is the initial condition,
- $\mathcal{L}^0 = (\mathcal{L}_t^0)_{t\leq T}$ and $\mathcal{L}^1 = (\mathcal{L}_t^1)_{t\leq T}$ are the local times of X at 0 and 1, respectively. That is, $\mathcal{L}^0$ and $\mathcal{L}^1$ are $\mathbb{G}$-adapted, continuous, nondecreasing processes that increase only when $X_t = 0$ and $X_t = 1$, respectively. Equivalently, we have $\mathcal{L}_t^0 = \int_0^t \mathbf{1}_{\{X_s=0\}}\, d\mathcal{L}_s^0$ and $\mathcal{L}_t^1 = \int_0^t \mathbf{1}_{\{X_s=1\}}\, d\mathcal{L}_s^1$ for all $t \in [0,T]$.

Let X be a version of the sticky reflected Brownian motion (see, e.g., [23, 27]) with local times defined by

$$\mathcal{L}_t^0 = \int_0^t \frac{1}{(T-s)^\ell} \mathbb{1}_{\{X_s=0\}} ds, \quad \mathcal{L}_t^1 = \int_0^t \frac{1}{(T-s)^\ell} \mathbb{1}_{\{X_s=1\}} ds, \quad \ell > 0.$$

Define $\kappa_t := \mathcal{L}_t^0 + \mathcal{L}_t^1$, which is continuous, $\mathbb{G}$-adapted, and nondecreasing, that represents the total time the process X has reflected at the boundary. For $\ell < 1$, the integral $\int_0^t (T-s)^{-\ell} ds$ converges for all $t \in [0,T]$, so κ_T is finite, bounded and assumption (H6) holds. However, for $\ell \geq 1$, κ_t is finite for $t < T$ since $\kappa_t \leq \frac{2t}{(T-t)^\ell} < +\infty$ but diverges at T, i.e., $\kappa_T = +\infty$. Since $\xi := X_T \in [0,1]$, we have $\mathbb{E}\left[|\xi|^p\right] < +\infty$, but

$$\mathbb{E}\left[e^{\frac{p}{2}\beta A_T + \frac{p}{2}\mu\kappa_T}|\xi|^p\right] \geq \mathbb{E}\left[e^{\frac{p}{2}\mu\kappa_T}|\xi|^p\right] = +\infty.$$

Therefore, the a priori estimates in the paper, which rely on assumption (H6), may not hold and the existence result would not be valid. This highlights the necessity of assumptions (H6) and (H7). Accordingly, the conditions we impose extend rather than restrict the existing results for $\mathbb{L}^p$-solutions of GBSDEs.

Remark 1. Throughout this paper, $\mathfrak{c}$ denotes a positive constant that may change from one line to another. Additionally, the notation $\mathfrak{c}_\gamma$ is used to emphasize the dependence of the constant $\mathfrak{c}$ on a specific set of parameters γ.

To derive the optimal constants in the a priori estimates of the solutions, we use the following remark.

Remark 2. Let (Y, Z, M) be a solution of the GBSDE (5) associated with (ξ, f, g, κ). Let $\varepsilon \geq 0$, $\varrho \in \mathbb{R}$ and $(\lambda_t^{(\alpha),\varepsilon,\varrho})_{t\leq T}$ be a progressively measurable nondecreasing continuous process defined by $\lambda_t^{(\alpha),\varepsilon,\varrho} = \exp\left\{\int_0^t \alpha_s ds + \varepsilon A_t + \varrho \kappa_t\right\}$. Then we define

$$\hat{Y}_t := \lambda_t^{(\alpha),\varepsilon,\varrho} Y_t, \quad \hat{Z}_t := \lambda_t^{(\alpha),\varepsilon,\varrho} Z_t, \quad d\hat{M}_t := \lambda_t^{(\alpha),\varepsilon,\varrho} dM_t.$$

By applying the integration-by-parts formula, we obtain

$$\hat{Y}_t = \hat{\xi} + \int_t^T \hat{f}\left(s, \hat{Y}_s, \hat{Z}_s\right) ds + \int_t^T \hat{g}\left(s, \hat{Y}_s\right) d\kappa_s - \int_t^T \hat{Z}_s dW_s - \int_t^T d\hat{M}_s,$$

with $\hat{\xi} = \lambda_T^{(\alpha),\varepsilon,\varrho}\,\xi$ and drivers

$$\hat{f}(t, y, z) = \lambda_t^{(\alpha),\varepsilon,\varrho} f\left(t, \lambda_t^{(-\alpha),-\varepsilon,-\varrho} y, \lambda_t^{(-\alpha),-\varepsilon,-\varrho} z\right) - (\alpha_s + \varepsilon a_s^2) y,$$
$$\hat{g}(t, y) = \lambda_t^{(\alpha),\varepsilon,\varrho} g\left(t, \lambda_t^{(-\alpha),-\varepsilon,-\varrho} y\right) - \varrho y.$$

Thus, if (Y, Z, M) is a solution of the GBSDE (5) associated with (ξ, f, g, κ), then the process $(\hat{Y}, \hat{Z}, \hat{M})$ satisfies a similar GBSDE associated with $(\hat{\xi}, \hat{f}, \hat{g}, \kappa)$. The driver $\hat{f}$ satisfies the stochastic Lipschitz condition described in (H3) with the same stochastic process $(\eta_t)_{t\leq T}$. Moreover, the coefficient $\hat{f}$ satisfies an analogous monotonicity condition (H2) with a modified real-valued stochastic process $(\hat{\alpha}_t^\varepsilon)$ given by $\hat{\alpha}_t^\varepsilon = \alpha_t - (\alpha_t + \varepsilon a_t^2) = -\varepsilon a_t^2 \leq 0$. Consequently, for any $t \in [0, T]$ and each $\varepsilon \geq 0$, we have $\hat{\alpha}_t^\varepsilon + \varepsilon a_t^2 = 0$. On the other hand, we can see that $\theta_t < 0$ is not a severe restriction. Indeed, if g satisfies (H3), it follows that the coefficient $\hat{g}$ also satisfies a similar monotonicity condition with the real-valued stochastic process $(\widehat{\delta_t})$ given by $\widehat{\delta_t} = \theta_t - \varrho$. By choosing ϱ large enough so that $\theta_t < \varrho$, we can always reduce the case where $\widehat{g}$ satisfies (H3) with θ negative.

Finally, let $\varepsilon \geq 0$ and $(Y^\varepsilon, Z^\varepsilon, M^\varepsilon)$ be a solution of the GBSDE (5) associated with $(\xi, f_\varepsilon, g, \kappa)$, where f_ε satisfies (H2) with (α_t) replaced by $(-\varepsilon a_t^2)$ and also verifies (H4). Set $\lambda_t^{(\alpha),\varepsilon} = \exp\left\{-\int_0^t \alpha_s ds - \varepsilon A_t\right\}$ and define

$$\hat{Y}_t^\varepsilon := \lambda_t^{(\alpha),\varepsilon} Y_t^\varepsilon, \quad \hat{Z}_t^\varepsilon := \lambda_t^{(\alpha),\varepsilon} Z_t^\varepsilon, \quad d\hat{M}_t^\varepsilon := \lambda_t^{(\alpha),\varepsilon} dM_t^\varepsilon.$$

Using the integration-by-parts formula, we obtain

$$\hat{Y}_t^\varepsilon = \hat{\xi}^\varepsilon + \int_t^T \hat{f}_\varepsilon\left(s, \hat{Y}_s^\varepsilon, \hat{Z}_s^\varepsilon\right) ds + \int_t^T \hat{g}_\varepsilon\left(s, \hat{Y}_s^\varepsilon\right) d\kappa_s - \int_t^T \hat{Z}_s^\varepsilon dW_s - \int_t^T d\hat{M}_s^\varepsilon$$

with $\hat{\xi}^\varepsilon = \lambda_T^{(\alpha),\varepsilon}\,\xi$ and drivers

$$\hat{f}_\varepsilon(t, y, z) = \lambda_t^{(\alpha),\varepsilon} f_\varepsilon\left(t, \lambda_t^{(-\alpha),-\varepsilon} y, \lambda_t^{(-\alpha),-\varepsilon} z\right) + (\alpha_t + \varepsilon a_t^2)\, y,$$
$$\hat{g}_\varepsilon(t, y) = \lambda_t^{(\alpha),\varepsilon} g\left(t, \lambda_t^{(-\alpha),-\varepsilon} y\right).$$

Then, $\widehat{g}_\varepsilon$ satisfies (H3) with the negative process (θ_t), and $\widehat{f_\varepsilon}$ satisfies (H2) and (H4) with respect to the processes (α_t) and (η_t), respectively.

To facilitate the calculations, we shall assume, for the remainder of this paper, that condition (H2) is satisfied with a process $(\alpha_t)_{t\leq T}$ such that $\alpha_t + \varepsilon a_t^2 = 0$ for each given $\varepsilon \geq 0$ and for all $t \in [0,T]$, and that (H3) holds with $\theta_t < 0$ for all $t \in [0,T]$. If this is not the case, the same change of variables as described in Remark 2 can be applied to reduce the situation to this case, while condition (H4) remains valid with the same process $(\eta_t)_{t\leq T}$. Moreover, in order to avoid any ambiguity regarding integrability, we shall also assume that conditions (H6)–(H7) hold for the transformed data introduced in Remark 2. For simplicity of notation, all these requirements, for both the original and, when relevant, the transformed data, will be collectively referred to as $\mathbf{(H\text{-}M)_p}$.

Remark 3. In particular, when A and κ are bounded, the exponential weight is finite, and one recovers the equivalence of the conditions on ξ and $\hat{\xi}$. In the general case, this equivalence may break down, as illustrated in the example with the sticky reflected Brownian motion. This shows that, contrary to the classical setting with deterministic monotonicity and Lipschitz conditions, such an equivalence can no longer be expected under stochastic monotonicity and stochastic Lipschitz coefficients. Therefore, assumptions (H6) and (H7) must also be imposed for the transformed data.

Since the function $x \mapsto |x|^p$ is not sufficiently smooth for $p < 2$, Itô's formula cannot be applied directly. Therefore, we need a generalization of Tanaka's formula for the multidimensional case. To this end, we introduce the notation $\check{x} = \frac{x}{|x|}\mathbb{1}_{x\neq 0}$ for $x \in \mathbb{R}^d$. The following lemma extends the Meyer–Itô formula as referenced in [6]. Although this result likely appears in earlier works, its proof is provided in [36, Lemma 7] (see also [6, Lemma 2.2] for the Brownian case). In the context of the BSDE in a general filtration considered in [36], the additional generator term associated with $(\kappa_t)_{t\leq T}$ in our formulation (5) is a continuous stochastic process, and our filtration supports a Brownian motion without requiring an independent jump measure. Consequently, the proof remains unchanged. The same observation applies if $\int_0^\cdot g(s, Y_s)\, d\kappa_s$ is replaced by an $\mathbb{R}$-valued, continuous, adapted process with locally integrable variation on $[0,T]$, denoted by $(G_t)_{t\leq T}$ with $G_0 = 0$. Therefore, we omit the proof for brevity.

Lemma 1. *Let $(F_t)_{t\leq T}$ and $(Z_t)_{t\leq T}$ be two progressively measurable processes with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times k}$ such that $\mathbb{P}$-a.s.*

$$\int_0^T \left\{ \left(|F_s| + \|Z_s\|^2\right) ds + d\,[M]_s \right\} < +\infty.$$

We consider the $\mathbb{R}^d$-valued semimartingale $(X_t)_{t\leq T}$ defined by

$$X_t = X_0 + \int_0^t F_s ds + \int_0^t dG_s + \int_0^t Z_s dW_s + \int_0^t dM_s. \tag{6}$$

Then, for any $p \geq 1$, the process $(|X_t|^p)_{t\leq T}$ is an $\mathbb{R}$-semimartingale with the decomposition

$$\begin{aligned}&|X_t|^p \\ &= |X_0|^p + \frac{1}{2}\mathbb{1}_{p=1} L_t + p\int_0^t |X_s|^{p-1}\, \check{X}_s F_s ds + p\int_0^t |X_s|^{p-1}\, \check{X}_s dG_s\end{aligned}$$

$$
\begin{aligned}
&+p\int_0^t |X_s|^{p-1}\check{X}_s Z_s dW_s + p\int_0^t |X_{s-}|^{p-1}\check{X}_{s-}dM_s\\
&+\sum_{0<s\le t}\left\{|X_{s-}+\Delta M_s|^p - |X_{s-}|^p - p\,|X_{s-}|^{p-1}\check{X}_{s-}\Delta M_s\right\}\\
&+\frac{p}{2}\int_0^t |X_s|^{p-2}\mathbb{1}_{X_s\neq 0}\left\{(2-p)\left(\|Z_s\|^2-(\check{X}_s)^*Z_sZ_s^*\check{X}_s\right)+(p-1)\|Z_s\|^2\right\}ds\\
&+\frac{p}{2}\int_0^t |X_s|^{p-2}\mathbb{1}_{X_s\neq 0}\left\{(2-p)\left(d\,[M]_s^c-(\check{X}_s)^*d\,[M,M]_s^c\,\check{X}_s\right)+(p-1)d\,[M]_s^c\right\},
\end{aligned}
$$

where $(L_t)_{t\le T}$ is a continuous, nondecreasing process with $L_0=0$, which increases only on the boundary of the random set $\{t\in[0,T]:X_{t-}=X_t=0\}$.

From this point forward, throughout the remainder of the paper, we assume $p\in(1,2)$ and set $c(p)=\frac{p(p-1)}{2}$.

3 A priori estimates and uniqueness

Let $(Y_t^1,Z_t^1,M_t^1)_{t\le T}$ and $(Y_t^2,Z_t^2,M_t^2)_{t\le T}$ be two $\mathbb{L}^p$-solutions of the GBSDE (5) associated with the data (ξ^1,f^1,g^1,κ^1) and (ξ^2,f^2,g^2,κ^2), respectively, satisfying condition $\mathbf{(H\text{-}M)_p}$. Define $\widehat{\mathcal{R}}=\mathcal{R}^1-\mathcal{R}^2$ with $\mathcal{R}\in\{Y,Z,M,\xi,f,g,\kappa\}$, and set $\bar{\kappa}_t:=\|\widehat{\kappa}\|_t+\kappa_t^2$ for $t\in[0,T]$. Then we have the following proposition.

Proposition 1. *For any $\beta,\mu>\frac{2(p-1)}{p}$, there exists a constant $\mathfrak{c}_{\beta,\mu,p,\epsilon}$ such that*

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right]+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p(dA_s+d\|\widehat{\kappa}\|_s+d\kappa_s^2)\\
&\quad+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left\|\widehat{Z}_s\right\|^2 ds\right)^{\frac{p}{2}}\right]+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}d\left[\widehat{M}\right]_s\right)^{\frac{p}{2}}\right]\\
&\quad\le \mathfrak{c}_{\beta,\mu,p,\epsilon}\Bigg(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p\right]+\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p ds\\
&\quad+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y_s^2)\right|^p d\|\widehat{\kappa}\|_s+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y_s^2)\right|^p d\kappa_s^2\Bigg).
\end{aligned}
$$

Proof. As in [21, Proposition 1], it suffices to prove the result in the case where $\|\widehat{\kappa}\|_T+\kappa_T^2$ is a bounded random variable, and then apply Fatou's lemma.

Using assumptions (H2)–(H4) on the drivers f and g, along with the basic inequality $ab\le\frac{a^2}{2\varepsilon}+\frac{\varepsilon b^2}{2}$ for all $\varepsilon>0$ and Remark 2, we derive the following:

$$
\widehat{Y}_s\left(f^1(s,Y_s^1,Z_s^1)-f^2(s,Y_s^2,Z_s^2)\right)\le\frac{c(p)}{2}\left\|\widehat{Z}_s\right\|^2+\left|\widehat{Y}_s\right|\left|\widehat{f}(s,Y_s^2,Z_s^2)\right| \tag{7}
$$

and

$$
\begin{aligned}
&\widehat{Y}_s\left(g^1(s,Y_s^1)-g^1(s,Y_s^2)\right)d\kappa_s^1+\widehat{Y}_s g^1(s,Y_s^2)d\widehat{\kappa}_s+\widehat{Y}_s\left(g^1(s,Y_s^2)-g^2(s,Y_s^2)\right)d\kappa_s^2\\
&\quad\le\left|\widehat{Y}_s\right|\left|g^1(s,Y_s^2)\right|d\|\widehat{\kappa}\|_s+\left|\widehat{Y}_s\right|\left|\widehat{g}(s,Y_s^2)\right|d\kappa_s^2
\end{aligned}\tag{8}
$$

Next, using (7)–(8) along with Lemma 1, and the integration-by-parts formula [51, Corollary 2, p. 68] for the product of semimartingales $\left(e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}|\widehat{Y}_t|^p\right)$, and the fact that $\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s = \left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\widehat{Y}_s$, we obtain

$$
\begin{aligned}
&e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p + \frac{p}{2}\beta\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p dA_s + \frac{p}{2}\mu\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p d\|\widehat{\kappa}\|_s \\
&\quad + \frac{p}{2}\mu\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p d\kappa_s^2 + \frac{c(p)}{2}\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left\|\widehat{Z}_s\right\|^2 ds \\
&\quad + c(p)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}d\left[\widehat{M}\right]_s^c \\
&\leq e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p + p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|ds \\
&\quad + p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\left\{\left|g^1(s,Y_s^2)\right|d\|\widehat{\kappa}\|_s + \left|\widehat{g}(s,Y_s^2)\right|d\kappa_s^2\right\} \\
&\quad - p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s - p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}d\widehat{M}_s \\
&\quad - \sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left\{\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|^p - \left|\widehat{Y}_{s-}\right|^p - p\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}\Delta\widehat{M}_s\right\}. \qquad (9)
\end{aligned}
$$

By applying Hölder's inequality $\int|h|^{p-1}|\ell|\,d\vartheta \leq \left(\int|h|^p\,d\vartheta\right)^{\frac{p-1}{p}}\left(\int|\ell|^p\,d\vartheta\right)^{\frac{1}{p}}$, Young's inequality $a^{\frac{p-1}{p}}b^{\frac{1}{p}} \leq \frac{p-1}{p}\,a + \frac{1}{p}\,b$, and assumption (H5), we derive

$$
\begin{aligned}
&p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|ds \\
&= p\int_t^T \left(e^{\frac{p-1}{2}\beta A_s+\frac{p-1}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}a_s^{\frac{2(p-1)}{p}}\right)\left(e^{\frac{\beta}{2}A_s+\frac{\mu}{2}\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|a_s^{\frac{2(1-p)}{p}}\right)ds \\
&\leq (p-1)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p dA_s + \int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p a_s^{2(1-p)}ds \\
&\leq (p-1)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p dA_s + \frac{1}{\epsilon^{2(p-1)}}\int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p ds.
\end{aligned}
\qquad (10)
$$

Using a similar argument, we obtain

$$
\begin{aligned}
&p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\left\{\left|g^1(s,Y_s^2)\right|d\|\widehat{\kappa}\|_s + \left|g^1(s,Y_s^2)-g^2(s,Y_s^2)\right|d\kappa_s^2\right\} \\
&\quad \leq (p-1)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p\left\{d\|\widehat{\kappa}\|_s + d\kappa_s^2\right\} \\
&\qquad + \int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y_s^2)\right|^p d\|\widehat{\kappa}\|_s + \int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y_s^2)\right|^p d\kappa_s^2 \qquad (11)
\end{aligned}
$$

Next, from [36, Lemma 9], the jump part of the quadratic variation is controlled by a nondecreasing process involving the jumps of Y, as described below:

$$\sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left\{\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|^p-\left|\widehat{Y}_{s-}\right|^p-p\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}\Delta\widehat{M}_s\right\}$$
$$\geq c(p)\sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|\neq 0}. \tag{12}$$

Additionally, from the dynamics of the process $\widehat{Y}$, we know that $\Delta\widehat{Y}_s=\Delta\widehat{M}_s$. Therefore, $\widehat{Y}_s=\widehat{Y}_{s-}+\Delta\widehat{M}_s$ for any $s\in[0,T]$.

Returning to (9) and using (10), (11), and (12), we have

$$\begin{aligned}
&e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p+\left(\frac{p}{2}\beta-(p-1)\right)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p dA_s\\
&\quad+\left(\frac{p}{2}\beta-(p-1)\right)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p(d\|\kappa\|_s+d\kappa^2_s)\\
&\quad+\frac{c(p)}{2}\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left\|\widehat{Z}_s\right\|^2 ds\\
&\quad+c(p)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}d\left[\widehat{M}\right]^c_s\\
&\quad+c(p)\sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}\\
&\leq e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p+\int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y^2_s,Z^2_s)\right|^p ds\\
&\quad+\int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y^2_s)\right|d\|\widehat{\kappa}\|_s+\int_t^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y^2_s)\right|^p d\kappa^2_s\\
&\quad-p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s-p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}d\widehat{M}_s.
\end{aligned} \tag{13}$$

Let us set

$$\Lambda:=\int_0^{\cdot} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s$$

and

$$\Xi:=\int_0^{\cdot} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}d\widehat{M}_s.$$

It follows from the Burkholder–Davis–Gundy inequality (BDG; see, e.g, Theorem 48 in [51, p. 193]) that the local martingales Λ and Ξ are uniformly integrable martingales with zero expectation.

Indeed, for the continuous local martingale part Λ, we have, by Young's inequality,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Lambda_t\right|\right]\leq \mathfrak{c}\mathbb{E}\left[\left(\int_0^T e^{p\beta A_s+p\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{2(p-1)}\|\widehat{Z}_s\|^2 ds\right)^{\frac{1}{2}}\right]$$

$$
\begin{aligned}
&\leq \mathfrak{c}\mathbb{E}\left[\left(\sup_{t\in[0,T]} e^{\frac{p-1}{2}\beta A_t+\frac{p-1}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^{p-1}\right)\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\|\widehat{Z}_s\|^2 ds\right)^{\frac{1}{2}}\right]\\
&\leq \frac{(p-1)\mathfrak{c}}{p}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^{p}\right]+\frac{\mathfrak{c}}{p}\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\|\widehat{Z}_s\|^2 ds\right)^{\frac{p}{2}}\right]\\
&< +\infty. \qquad (14)
\end{aligned}
$$

A similar argument holds for the RCLL local martingale part Ξ, where we have

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\Xi_t\right|\right] &\leq \mathfrak{c}\mathbb{E}\left[\left(\int_0^T e^{p\beta A_s+p\mu\bar{\kappa}_s}\left|\widehat{Y}_{s-}\right|^{2(p-1)} d\left[\widehat{M}\right]_s\right)^{\frac{1}{2}}\right]\\
&\leq \frac{(p-1)\mathfrak{c}}{p}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^{p}\right]+\frac{\mathfrak{c}}{p}\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s} d\left[\widehat{M}\right]_s\right)^{\frac{p}{2}}\right]\\
&< +\infty. \qquad (15)
\end{aligned}
$$

Then, taking the expectation on both sides and setting $t = 0$, and $\beta,\mu > 0$ such that $\beta,\mu > \frac{2(p-1)}{p}$, we deduce the existence of a constant $\mathfrak{c}_{\beta,\mu,p,\epsilon}$ such that

$$
\begin{aligned}
&\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p dA_s+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^p (d\|\widehat{\kappa}\|_s+d\kappa_s^2)\\
&\quad+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left\|\widehat{Z}_s\right\|^2 ds\\
&\quad+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0} d\left[\widehat{M}\right]_s^c\\
&\quad+\mathbb{E}\left[\sum_{0<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}\right]\\
&\leq \mathfrak{c}_{\beta,\mu,p,\epsilon}\left(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p\right]+\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p ds\right.\\
&\quad\left.+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y_s^2)\right|^p d\|\widehat{\kappa}\|_s+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y_s^2)\right|^p d\kappa_s^2\right). \qquad (16)
\end{aligned}
$$

Let

$$
\begin{aligned}
\mathcal{X} &= e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p+\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p ds\\
&\quad+\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y_s^2)\right| d\|\widehat{\kappa}\|_s+\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y_s^2)\right|^p d\kappa_s^2.
\end{aligned}
$$

Then, using (13) with $\beta,\mu > \frac{2(p-1)}{p}$, we have, a.s., for each $t\in[0,T]$

$$
e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p+\frac{c(p)}{2}\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left\|\widehat{Z}_s\right\|^2 ds
$$

$$
\begin{aligned}
&+ c(p)\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}d\left[\widehat{M}\right]_s^c \\
&+ c(p)\sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|\neq 0} \\
&\quad\leq \mathcal{X} - p\left(\Lambda_T-\Lambda_t\right) - p\left(\Xi_T-\Xi_t\right).
\end{aligned}
\tag{17}
$$

Using again the BDG inequality, we derive

$$
\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right] \leq \mathbb{E}\left[\mathcal{X}\right] + \mathfrak{c}_p\left(\mathbb{E}\left[\Lambda\right]_T^{1/2} + \mathbb{E}\left[\Xi\right]_T^{1/2}\right). \tag{18}
$$

The term $[\Lambda]_T^{1/2}$ can be controlled as in [6]:

$$
\begin{aligned}
&\mathfrak{c}_p\mathbb{E}\left[\Lambda\right]_T^{1/2} \\
&\quad\leq \mathfrak{c}_p\mathbb{E}\left[\left(\sup_{t\in[0,T]} e^{\frac{p}{4}\beta A_t+\frac{p}{4}\mu\bar{\kappa}_t}\left|Y_t\right|^{\frac{p}{2}}\right)\left(\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\|\widehat{Z}_s\|^2 ds\right)^{\frac{1}{2}}\right] \\
&\quad\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right] + \mathfrak{c}_p^2\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left|\widehat{Z}_s\right|^2 ds.
\end{aligned}
\tag{19}
$$

For the term $[\Xi]_T^{1/2}$, which is more complicated to handle, we follow [36] to obtain a bound in terms of the estimation (16):

$$
\begin{aligned}
&\mathfrak{c}_p\mathbb{E}\left[\Xi\right]_T^{1/2} \\
&= \mathfrak{c}_p\mathbb{E}\left[\left(\int_0^T e^{p\beta A_s+p\mu\bar{\kappa}_s}\left|\widehat{Y}_{s-}\right|^{2(p-1)}d\left[\widehat{M}\right]_s\right)^{\frac{1}{2}}\right] \\
&\leq \mathfrak{c}_p\mathbb{E}\left[\left(\int_0^T e^{p\beta A_s+p\mu\bar{\kappa}_s}\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{p-1}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s\right)^{\frac{1}{2}}\right] \\
&\leq \mathfrak{c}_p\mathbb{E}\left[\left(\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left(\left|\widehat{Y}_{t-}\right|^2\vee\left|\widehat{Y}_t\right|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{2}}\right. \\
&\qquad\left.\times\left(\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s\right)^{\frac{1}{2}}\right] \\
&\leq \frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right] + \mathfrak{c}_p^2\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s.
\end{aligned}
$$

Using the pathwise decomposition of the bracket process $\left[\widehat{M}\right]$ (see, e.g., [51, p. 70]), we have

$$
d\left[\widehat{M}\right]_t = d\left[\widehat{M}\right]_t^c + \left|\Delta\widehat{M}_t\right|^2, \quad t\in[0,T], \tag{20}
$$

Therefore,

$$\begin{aligned}
&\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s\\
&\quad=\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s^c\\
&\qquad+\sum_{0<s\leq T}e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\left|\Delta\widehat{M}_s\right|^2\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}.
\end{aligned}$$

Injecting this into the above estimation and using the basic inequality $\sqrt{ab}\leq\frac{1}{4}a+b$ for any $a,b\geq 0$, we obtain

$$\begin{aligned}
\mathfrak{c}_p\mathbb{E}\left[\Xi\right]_T^{1/2}&\leq\frac{1}{4}\mathbb{E}\left[\sup_{t\in[0,T]}e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right]\\
&\quad+\mathfrak{c}_p\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s^c\\
&\quad+\mathfrak{c}_p^2\mathbb{E}\sum_{0<s\leq T}e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\left|\widehat{Y}_{s-}\right|^2\vee\left|\widehat{Y}_s\right|^2\right)^{\frac{p-2}{2}}\left|\Delta\widehat{M}_s\right|^2\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}.
\end{aligned}\tag{21}$$

Plugging (19) and (21) into (18), along with (16), we derive

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right]\\
&\leq\mathfrak{c}_{\beta,\mu,p,\epsilon}\left(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\bar{\kappa}_T}\left|\widehat{\xi}\right|^p\right]+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{f}(s,Y_s^2,Z_s^2)\right|^p ds\right.\\
&\quad\left.+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|g^1(s,Y_s^2)\right|^p d\|\widehat{\kappa}\|_s+\mathbb{E}\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left|\widehat{g}(s,Y_s^2)\right|^p d\kappa_s^2\right).
\end{aligned}\tag{22}$$

Showing the estimation for the remaining term

$$\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left\|\widehat{Z}_s\right\|^2ds\right)^{\frac{p}{2}}\right]+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}d\left[\widehat{M}\right]_s\right)^{\frac{p}{2}}\right]$$

is the last step in proving the assertion. To this end, since $p\in(1,2)$, we apply the equality $\mathbb{1}_{\widehat{Y}_s=0}\left\{\left\|\widehat{Z}_s\right\|^2ds+d\left[\widehat{M}\right]_s\right\}=0$ on $[0,T]$ (we refer to [36, Lemma 8] for a detailed proof). Using this result, along with Young's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}\left\|\widehat{Z}_s\right\|^2ds\right)^{\frac{p}{2}}\right]\\
&=\mathbb{E}\left[\left(\int_0^T e^{\frac{2-p}{2}\beta A_s+\frac{2-p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{2-p}\left(e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left\|\widehat{Z}_s\right\|^2\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{\{\widehat{Y}_s\neq 0\}}\right)ds\right)^{\frac{p}{2}}\right]\\
&\leq\frac{2-p}{2}\mathbb{E}\left[\sup_{t\in[0,T]}e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right]
\end{aligned}$$

$$+\frac{p}{2}\mathbb{E}\left[\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\left\|\widehat{Z}_s\right\|^2\mathbb{1}_{\{\widehat{Y}_s\neq 0\}}ds\right].$$

Similarly, we can show that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\bar{\kappa}_s}d\left[\widehat{M}\right]_s^c\right)^{\frac{p}{2}}\right]\\
&\quad\leq \frac{2-p}{2}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right]\\
&\qquad+\frac{p}{2}\mathbb{E}\left[\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{\left|\widehat{Y}_s\right|\neq 0}d\left[\widehat{M}\right]_s^c\right].
\end{aligned}$$

By virtue of (20), it remains to show a similar estimation for the quadratic jump part of the state process $\widehat{Y}$ given by $\left|\Delta\widehat{M}\right|^2$. To this end, we will use an approximating procedure via a smooth function that is widely considered in the literature (see, e.g., [6, Lemma 2.2], [36, Lemma 7], or [14, Lemma 2.2]).

Let $\varepsilon>0$, and consider the function $\nu_\varepsilon:\mathbb{R}\longrightarrow\mathbb{R}_+$ defined by $\nu_\varepsilon(y)=\sqrt{|y|^2+\varepsilon^2}$. Then, for any $q>0$, we have

$$\begin{aligned}
e^{\frac{2-p}{2}\beta A_s+\frac{2-p}{2}\mu\bar{\kappa}_s}\left(\nu_\varepsilon(y)\right)^q &=\left(\left(e^{\frac{2-p}{q}\beta A_s+\frac{2-p}{q}\mu\bar{\kappa}_s}\left(\left|y\right|^2+\varepsilon^2\right)\right)^{\frac{1}{2}}\right)^q\\
&\leq\left(\nu_{\widehat{\varepsilon}_s^{p,q}}\left(e^{\frac{2-p}{2q}\beta A_s+\frac{2-p}{2q}\mu\bar{\kappa}_s}y\right)\right)^q,
\end{aligned}$$

where $\widehat{\varepsilon}_{p,q}=\varepsilon\mathcal{C}^*_{p,q}$ with $\mathcal{C}^*_{p,q}:=\operatorname{ess\,sup}_{\omega\in\Omega}e^{\frac{p}{2}\beta A_T(\omega)+\frac{p}{2}\mu\bar{\kappa}_T(\omega)}$. Without loss of generality, instead of replacing A with $A\wedge k$ and then passing to the limit using the monotone convergence theorem, we may assume that A is bounded. Therefore, we have $\mathcal{C}^*_{p,q}<+\infty$, and consequently, $\lim_{\varepsilon\downarrow 0}\widehat{\varepsilon}_{p,q}=0$. Furthermore,

$$\lim_{\varepsilon\downarrow 0}\nu_{\widehat{\varepsilon}_{p,q}}\left(e^{\frac{2-p}{2q}\beta A_s+\frac{2-p}{2q}\mu\bar{\kappa}_s}y\right)=e^{\frac{2-p}{2q}\beta A_s+\frac{2-p}{2q}\mu\bar{\kappa}_s}|y|\,\mathbb{1}_{y\neq 0}\quad\text{a.s., } q>0.$$

To simplify notation, we denote $\mathcal{X}_*=\sup_{t\in[0,T]}\left|\mathcal{X}_t\right|$ for any RCLL process $\mathcal{X}=(\mathcal{X}_t)_{t\leq T}$. Then, using Hölder's and Young's inequalities

$$\mathbb{E}\left[A^{p(2-p)/2}B^{p/2}\right]\leq\left(\mathbb{E}\left[A^p\right]\right)^{(2-p)/2}\left(\mathbb{E}\left[B\right]\right)^{p/2}\leq\frac{2-p}{2}\mathbb{E}\left[A^p\right]+\frac{p}{2}\mathbb{E}\left[B\right]$$

for some random variables $A,B\geq 0$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{0<s\leq T}e^{\beta A_s+\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\right)^{\frac{p}{2}}\right]\\
&=\mathbb{E}\left[\left(\sum_{0<s\leq T}e^{\beta A_s+\mu\bar{\kappa}_s}\left(\nu_\varepsilon\left(\left|Y_{s-}\right|\vee\left|Y_s\right|\right)\right)^{2-p}\left(\nu_\varepsilon\left(\left|Y_{s-}\right|\vee\left|Y_s\right|\right)\right)^{p-2}\left|\Delta\widehat{M}_s\right|^2\right)^{\frac{p}{2}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\mathbb{E}\left[\left(\nu_{\widehat{\varepsilon}_s^{p,2-p}}\left(\left(e^{\frac{\beta}{2}A+\frac{\mu}{2}\bar{\kappa}}\widehat{Y}\right)_*\right)\right)^p\right]\right)^{\frac{2-p}{2}} \\
&\qquad \times \left(\mathbb{E}\left[\sum_{0<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\nu_{\varepsilon}\left(\left|Y_{s-}\right|\vee\left|Y_s\right|\right)\right)^{p-2}\left|\Delta\widehat{M}_s\right|^2\right]\right)^{\frac{p}{2}} \\
&\leq \frac{2-p}{2}\mathbb{E}\left[\left(\nu_{\widehat{\varepsilon}_s^{p,2-p}}\left(\left(e^{\frac{\beta}{2}A+\frac{\mu}{2}\bar{\kappa}}\widehat{Y}\right)_*\right)\right)^p\right] \\
&\qquad + \frac{p}{2}\mathbb{E}\left[\sum_{0<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\nu_{\varepsilon}\left(\left|Y_{s-}\right|\vee\left|Y_s\right|\right)\right)^{p-2}\left|\Delta\widehat{M}_s\right|^2\right]. \qquad (23)
\end{aligned}
$$

We know that

$$
\lim_{\varepsilon\downarrow 0}\left(\nu_{\widehat{\varepsilon}_s^{p,2-p}}\left(\left(e^{\frac{\beta}{2}A+\frac{\mu}{2}\bar{\kappa}}\widehat{Y}\right)_*\right)\right)^p = \sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p \quad \text{a.s.}
$$

and that

$$
\lim_{\varepsilon\downarrow 0}\left(\nu_{\varepsilon}\left(\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\right)\right)^{p-2} \nearrow \left(\left|Y_{s-}\right|\vee\left|Y_s\right|\right)^{p-2}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0} \quad \text{a.s.,}
$$

since $p<2$.

Letting $\varepsilon \longrightarrow 0$ and then applying the Lebesgue dominated convergence theorem for the left term in the last inequality of the estimation (23) and the monotone convergence theorem for the right term in the last inequality of the estimation (23) (see also the proof of Lemma 9 in [36]), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{0<s\leq T} e^{\beta A_s+\mu\bar{\kappa}_s}\left|\Delta\widehat{M}_s\right|^2\right)^{\frac{p}{2}}\right] \\
&\quad \leq \frac{2-p}{2}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\bar{\kappa}_t}\left|\widehat{Y}_t\right|^p\right] \\
&\qquad + \frac{p}{2}\mathbb{E}\left[\sum_{0<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\bar{\kappa}_s}\left(\left|Y_{s-}\right|^2\vee\left|Y_s\right|^2\right)^{\frac{p-2}{2}}\mathbb{1}_{\left|\widehat{Y}_{s-}\right|\vee\left|\widehat{Y}_s\right|\neq 0}\left|\Delta\widehat{M}_s\right|^2\right].
\end{aligned}
$$

Finally, the proof is completed by applying the inequalities (16) and (22). □

From Proposition 1, we obtain the uniqueness of the solution.

Corollary 1. *Let (ξ, f, g, κ) be any set of data satisfying assumption* **(H-M)$_{\mathbf{p}}$**. *Then, there exists at most one triplet of processes $(Y_t, Z_t, M_t)_{t\leq T}$ corresponding to the $\mathbb{L}^p$-solution of the GBSDE (5) associated with (ξ, f, g, κ).*

4 Existence of $\mathbb{L}^p$-solutions

First, note that the uniqueness result has been established in Corollary 1. Using Proposition 1, we obtain an important result that provides an $\mathbb{L}^p$-estimate for the solutions of the GBSDE (5) associated with the given data (ξ, f, g, κ).

Corollary 2. *Let (ξ, f, g, κ) be any set of data satisfying assumption* **(H-M)$_\mathbf{p}$**. *For any $\beta, \mu > \frac{2(p-1)}{p}$, there exists a constant $\mathfrak{c}_{\beta,\mu,p,\epsilon}$ such that, whenever (Y, Z) is an $\mathbb{L}^p$-solution of the GBSDE (5), we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\kappa_t}\left|Y_t\right|^p\right] + \mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|Y_s\right|^p dA_s$$
$$+\,\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|Y_s\right|^p d\kappa_s + \mathbb{E}\left[\left(\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left\|Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right]$$
$$+\,\mathbb{E}\left[\left(\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s} d\left[M\right]_s\right)^{\frac{p}{2}}\right]$$
$$\leq \mathfrak{c}_{\beta,\mu,p,\epsilon}\left(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\xi\right|^p\right]+\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\varphi_s\right|^p ds+\mathbb{E}\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\psi_s\right|^p d\kappa_s\right).$$

To establish the existence and uniqueness of $\mathbb{L}^p$-solutions for $p \in (1,2)$ of the GBSDE (5), we first need to establish the existence and uniqueness of $\mathbb{L}^2$-solutions. While this result may have appeared in previous works (see, e.g., [20, 21]), we could not find an equivalent result for the GBSDE (5) under our general condition **(H-M)$_\mathbf{2}$**, which assumes stochastic monotonicity, Lipschitz continuity, and linear growth in a general filtration with larger integrability conditions. The proof of this result can be obtained using the Yosida approximation method, following an approach similar to that adopted in [21, Theorems 1–2]. We also refer to Proposition 4.1 and Theorem 4.1 in [19]. The latter reference deals with a more general class of GBSDEs with reflecting obstacles. By letting the lower and upper obstacles tend to $-\infty$ and $+\infty$, respectively, one recovers the classical GBSDE studied in our case.

Theorem 1. *Suppose that* **(H-M)$_\mathbf{2}$** *holds. Then, the GBSDE (5) admits a unique $\mathbb{L}^2$-solution.*

Now, we can state the main result of this section, which is described as follows.

Theorem 2. *Suppose that* **(H-M)$_\mathbf{p}$** *holds. Then, the GBSDE (5) admits a unique $\mathbb{L}^p$-solution.*

Proof. We assume that

$$e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\xi\right|^p + \sup_{t\in[0,T]} e^{\beta A_t+\mu\kappa_t}\left|\varphi_t\right|^p + \sup_{t\in[0,T]} e^{\beta A_t+\mu\kappa_t}\left|\psi_t\right|^p \leq \mathsf{C}. \tag{24}$$

Note that since φ and ψ are $[1,+\infty)$-valued stochastic processes, we deduce that $e^{\beta A_t+\mu\kappa_t} \leq \mathsf{C}$ for any $t \in [0,T]$. Moreover, from (H7), it follows that

$$\mathbb{E}\left[\int_0^T e^{\beta A_s+\mu\kappa_s}\left(ds + d\kappa_s\right)\right] < +\infty.$$

Additionally, using the definition of the process $\left(e^{\beta A_t+\mu\kappa_t}\right)_{t\leq T}$, we obviously have $e^{\beta A_t+\mu\kappa_t} \geq 1$ for any $t \in [0,T]$. Consequently,

$$|\xi|^2 + \sup_{t\in[0,T]} |\varphi_t|^2 + \sup_{t\in[0,T]} |\psi_t|^2 \leq \mathsf{C}^{\frac{2}{p}}.$$

Therefore, using this and condition (H5), we conclude that

$$\mathbb{E}\left[e^{\beta A_T+\mu\kappa_T}\left|\xi\right|^2+\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\frac{\varphi_s}{a_s}\right|^2 ds+\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\psi_s\right|^2 d\kappa_s\right]$$
$$\leq \mathsf{C}^{\frac{2+p}{p}}+\left(\frac{1}{\epsilon^2}+1\right)\mathsf{C}^{\frac{2}{p}}\mathbb{E}\left[\int_0^T e^{\beta A_s+\mu\kappa_s}(ds+d\kappa_s)\right]<+\infty.$$

Then, using (24) and assumptions (H1)–(H5) and (H7), we place ourselves within the framework of Theorem 1. Consequently, there exists a unique $\mathbb{L}^2$-solution (Y^n, Z^n, M^n) and, hence, an $\mathbb{L}^p$-solution for any $p \in (1,2)$ for the GBSDE (5). Note also that, by applying Corollary 2, we observe that the triplet (Y, Z, M) satisfies the same estimation stated for any $\beta, \mu > \frac{2(p-1)}{p}$.

Using (24), we construct a sequence of GBSDEs associated with some data $(\xi_n, f_n, g_n, \kappa)$ such that (24) is satisfied and that approximates the GBSDE (5). To this end, and to simplify notation, we set $f_0(t) = f(t,0,0)$ and $g_0(t) = g(t,0)$. For each $n \geq 1$, we define a set of data (ξ_n, f_n, g_n) as follows:

$$\begin{cases} \xi_n = \begin{cases} \dfrac{\left|\xi\right| \wedge \sqrt[p]{n}e^{-\frac{\beta}{2}A_T-\frac{\mu}{2}\kappa_T}}{\left|\xi\right|}\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \\ f_n(t,y,z) = \begin{cases} f(t,y,z)-f_0(t)+\dfrac{\left|f_0(t)\right| \wedge \sqrt[p]{n}e^{-\frac{\beta}{p}A_t-\frac{\mu}{p}\kappa_t}}{\left|f_0(t)\right|}f_0(t) & \text{if } f_0(t) \neq 0, \\ 0 & \text{if } f_0(t) = 0, \end{cases} \\ g_n(t,y) = \begin{cases} g(t,y)-g_0(t)+\dfrac{\left|g_0(t)\right| \wedge \sqrt[p]{n}e^{-\frac{\beta}{p}A_t-\frac{\mu}{p}\kappa_t}}{\left|g_0(t)\right|}g_0(t) & \text{if } g_0(t) \neq 0, \\ 0 & \text{if } g_0(t) = 0. \end{cases} \end{cases} \tag{25}$$

For each $n \geq 1$, the data (ξ_n, f_n, g_n) satisfies condition (24). Indeed, it is straightforward to observe the inequality

$$e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\xi_n\right|^p+\sup_{t\in[0,T]} e^{\beta A_t+\mu\kappa_t}\left|f_n(t,0,0)\right|^p+\sup_{t\in[0,T]} e^{\beta A_t+\mu\kappa_t}\left|g_n(t,0)\right|^p \leq n.$$

Therefore, from the previous step, for each $n \geq 1$, there exists a unique triplet (Y^n, Z^n, M^n) that solves the GBSDE

$$Y_t^n = \xi_n+\int_t^T f_n(s,Y_s^n,Z_s^n)ds+\int_t^T g_n(s,Y_s^n)d\kappa_s-\int_t^T Z_s^n dW_s-\int_t^T dM_s^n \tag{26}$$

for any $t \in [0,T]$.

Let $n \geq m \geq 1$. Set $\widehat{R} := \mathcal{R}^n - \mathcal{R}^m$ for $\mathcal{R} \in \{\xi, Y, Z, M\}$. Using again Lemma 1 with the integration-by-parts formula, we get

$$e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\kappa_t}\left|\widehat{Y}_t\right|^p+\frac{p}{2}\beta\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^p dA_s+\frac{p}{2}\mu\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^p d\kappa_s$$

$$
\begin{aligned}
&+c(p)\int_0^t e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^{p-2}\mathbb{1}_{Y_s\neq 0}\left(\left\|\widehat{Z}_s\right\|^2 ds + d\left[\widehat{M}\right]_s^c\right)\\
&\leq e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\widehat{\xi}\right|^p + p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\left(f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m)\right)ds\\
&+p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\left(g_n(s,Y_s^n)-g_m(s,Y_s^m)\right)d\kappa_s\\
&-p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^{p-1}\check{\widehat{Y}}_s\widehat{Z}_s dW_s - p\int_t^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}d\widehat{M}_s\\
&-\sum_{t<s\leq T} e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left\{\left|\widehat{Y}_{s-}+\Delta\widehat{M}_s\right|^p-\left|\widehat{Y}_{s-}\right|^p-p\left|\widehat{Y}_{s-}\right|^{p-1}\check{\widehat{Y}}_{s-}\Delta\widehat{M}_s\right\}. \qquad (27)
\end{aligned}
$$

From (25) and assumptions (H2)–(H4) on f and g, we have

$$
\begin{aligned}
&\widehat{Y}_t\left(f_n(s,Y_s^n,Z_s^n)-f_m(s,Y_s^m,Z_s^m)\right)\\
&\quad\leq \frac{c(p)}{2}\|\widehat{Z}_s\|^2+\left|\widehat{Y}_t\right|\left|f_n(s,0,0)-f_m(s,0,0)\right|
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{Y}_t\left(g_n(s,Y_s^n)-g_m(s,Y_s^m)\right) &= \widehat{Y}_t\left(g(s,Y_s^n)-g(s,Y_s^m)\right)+\widehat{Y}_t\left(g_n(s,0)-g_m(s,0)\right)\\
&\leq \left|\widehat{Y}_s\right|\left|g_n(s,0)-g_m(s,0)\right|
\end{aligned}
$$

Then, we obtain an analogous estimation to (7) for the driver f_n and a simpler one than (8) for the coefficient g. Following this, using (27) and re-performing the calculations from Proposition 1, we deduce that, for any $\beta,\mu>\frac{2(p-1)}{p}$, there exists a constant $\mathfrak{c}_{\beta,\mu,p,\epsilon}$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\kappa_t}\left|\widehat{Y}_t\right|^p\right]+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^p dA_s\\
&\quad+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|\widehat{Y}_s\right|^p d\kappa_s+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s}\left\|\widehat{Z}_s\right\|^2 ds\right)^{\frac{p}{2}}\right]\\
&\quad+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s}d\left[\widehat{M}\right]_s\right)^{\frac{p}{2}}\right]\\
&\leq \mathfrak{c}_{\beta,\mu,p,\epsilon}\left(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\widehat{\xi}\right|^p\right]+\mathbb{E}\int_0^T e^{\beta A_t+\mu\kappa_s}\left|f_n(s,0,0)-f_m(s,0,0)\right|^p ds\right.\\
&\qquad\left.+\mathbb{E}\int_0^T e^{\beta A_s+\mu\kappa_s}\left|g_n(s,0)-g_m(s,0)\right|^p d\kappa_s\right). \qquad (28)
\end{aligned}
$$

By using the basic inequality

$$
\left(\sum_{i=1}^n |X_i|\right)^p \leq n^p\sum_{i=1}^n |X_i|^p \quad \forall (n,p)\in\mathbb{N}^*\times(0,+\infty),
$$

along with (25) and assumption (H5), we obtain

$$\begin{cases} \left|\widehat{\xi}\right|^{p} \leq 2^{p+1}\left|\xi\right|^{p}, & \\ \left|f_{n}(s,0,0)-f_{m}(s,0,0)\right|^{p} \leq 2^{p+1}\left|\varphi_{s}\right|^{p}, & \mathbb{P}\otimes dt\text{-a.e.}, \\ \left|g_{n}(s,0)-g_{m}(s,0)\right|^{p} \leq 2^{p+1}\left|\psi_{s}\right|^{p}, & \mathbb{P}\otimes d\kappa_{t}\text{-a.e.} \end{cases} \tag{29}$$

Since $\lim_{n\to+\infty} f_n(t,0,0) = f_0(t)$ $\mathbb{P}\otimes dt$-a.e. and $\lim_{n\to+\infty} g_n(t,0) = g_0(t)$ $\mathbb{P}\otimes d\kappa_t$-a.e., it follows from (H6) and (H7) that we can apply the Lebesgue dominated convergence theorem. Hence, we deduce that the right-hand side of (28) tends to zero as $n, m \to +\infty$. Therefore, the left-hand side of (28) also tends to zero. Consequently, we derive the convergence

$$\lim_{n,m\to+\infty}\left(\left\|Y^{n}-Y^{m}\right\|_{\mathfrak{B}^{p}_{\beta,\mu}}^{p}+\left\|Z^{n}-Z^{m}\right\|_{\mathcal{H}^{p}_{\beta,\mu}}^{p}+\left\|M^{n}-M^{m}\right\|_{\mathcal{M}^{p}_{\beta,\mu}}^{p}\right)=0. \tag{30}$$

Hence, $\{(Y^n, Z^n, M^n)\}_{n\geq 1}$ is a Cauchy sequence in the Banach space $\mathcal{E}^p_{\beta,\mu}$ for any $\beta, \mu > \frac{2(p-1)}{p}$. It then converges to a process $(Y, Z, M) \in \mathcal{E}^p_{\beta,\mu}$. Moreover, using (29) and Corollary 2, we deduce that, for any $\beta, \mu > \frac{2(p-1)}{p}$, there exists a constant $\mathfrak{c}_{\beta,\mu,p,\epsilon}$ (independent of n) such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} e^{\frac{p}{2}\beta A_t+\frac{p}{2}\mu\kappa_t}\left|Y_t^n\right|^p\right]+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|Y_s^n\right|^p dA_s \\
&\quad+\mathbb{E}\int_0^T e^{\frac{p}{2}\beta A_s+\frac{p}{2}\mu\kappa_s}\left|Y_s^n\right|^p d\kappa_s+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s}\left\|Z_s^n\right\|^2 ds\right)^{\frac{p}{2}}\right] \\
&\quad+\mathbb{E}\left[\left(\int_0^T e^{\beta A_s+\mu\kappa_s} d\left[M^n\right]_s\right)^{\frac{p}{2}}\right] \\
&\leq \mathfrak{c}_{\beta,\mu,p,\epsilon}\left(\mathbb{E}\left[e^{\frac{p}{2}\beta A_T+\frac{p}{2}\mu\kappa_T}\left|\xi\right|^p\right]\right. \\
&\qquad\left.+\mathbb{E}\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\varphi_s\right|^p ds+\mathbb{E}\int_0^T e^{\beta A_s+\mu\kappa_s}\left|\psi_s\right|^p d\kappa_s\right).
\end{aligned} \tag{31}$$

It remains to confirm that the limiting process solves the generalized BSDE (5). To this end, since $p > 1$, we apply the BDG inequality along with (7) to obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T Z^n dW_s-\int_t^T Z_s dW_s\right|^p\right] \leq \mathfrak{c}\mathbb{E}\left[\left(\int_0^T \left\|Z_s^n-Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] \xrightarrow[n\to+\infty]{} 0$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T dM_s^n-\int_t^T dM_s\right|^p\right] \leq \mathfrak{c}\mathbb{E}\left[\left(\int_0^T d\left[M^n-M\right]_s\right)^{\frac{p}{2}}\right] \xrightarrow[n\to+\infty]{} 0.$$

From assumptions (H1) and (H4), we have

$$\left|f(t,Y_t^n,Z_t^n)-f(t,Y_t,Z_t)\right|$$

$$\leq \eta_t \|Z_t^n - Z_t\| + \left|f(t, Y_t^n, Z_t) - f(t, Y_t, Z_t)\right| \xrightarrow[n\to+\infty]{} 0. \tag{32}$$

Moreover, from assumptions (H5) and (H7), and Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \left|\frac{f(t, Y_t^n, Z_t) - f(t, Y_t, Z_t)}{a_s}\right|^p ds \\
&\leq 2^{2p}\left(2\mathbb{E}\int_0^T \|Z_s\|^p\, ds + \frac{2}{\epsilon^p}\mathbb{E}\int_0^T |\varphi_s|^p\, ds + \mathbb{E}\int_0^T \left(\left|Y_s^n\right|^p + |Y_s|^p\right) ds\right) \\
&\leq 2^{2p}\left(2T^{\frac{2-p}{2}}\mathbb{E}\left[\left(\int_0^T \|Z_s\|^2\, ds\right)^{\frac{p}{2}}\right] + \frac{2}{\epsilon^p}\mathbb{E}\int_0^T |\varphi_s|^p\, ds\right. \\
&\qquad \left. + \frac{2}{\epsilon^2}\mathbb{E}\int_0^T \left(\left|Y_s^n\right|^p + |Y_s|^p\right) dA_s\right).
\end{aligned}$$

Using (25), (30), (31), (32), and the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T \left|\frac{f_n(t, Y_t^n, Z_t^n) - f(t, Y_t, Z_t)}{a_s}\right|^p ds = 0.$$

Similarly, we can show that

$$\lim_{n\to+\infty} \mathbb{E}\int_0^T \left|g_n(s, Y_s^n) - g(s, Y_s)\right|^p d\kappa_s = 0.$$

Finally, by passing to the limit term by term in (26), we deduce that the limiting process (Y, Z, M) is the $\mathbb{L}^p$-solution of the GBSDE (5).

This completes the proof. □

Remark 4. An interesting direction and perspective for future research would be to use the results presented in this paper, along with the arguments in [36, Section 6], [43, Theorem 4.1], and [48, Theorem 53.2], to extend the current framework to the case where the deterministic terminal time T is replaced by a stopping time τ in the general filtration $\mathbb{F}$, which may be unbounded.

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