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Laws of the iterated logarithm for iterated perturbed random walks

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Abstract Let $(\xi_k, \eta_k)_{k\geq 1}$ be independent identically distributed random vectors with arbitrarily dependent positive components and $T_k := \xi_1 + \cdots + \xi_{k-1} + \eta_k$ for $k \in \mathbb{N}$. The random sequence $(T_k)_{k\geq 1}$ is called a (globally) perturbed random walk. Consider a general branching process generated by $(T_k)_{k\geq 1}$ and let $Y_j(t)$ denote the number of the jth generation individuals with birth times $\leq t$. Assuming that $\operatorname{Var}\xi_1 \in (0,\infty)$ and allowing the distribution of η_1 to be arbitrary, a law of the iterated logarithm (LIL) is proved for $Y_j(t)$. In particular, an LIL for the counting process of $(T_k)_{k\geq 1}$ is obtained. The latter result was previously established in the article by Iksanov, Jedidi and Bouzeffour (2017) under the additional assumption that $\mathbb{E}\eta_1^a < \infty$ for some $a > 0$. In this paper, it is shown that the aforementioned additional assumption is not needed.

Keywords General branching process, iterated perturbed random walk, law of the iterated logarithm
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1 Introduction and main results

Let $(\xi_k, \eta_k)_{k\geq 1}$ be independent copies of a random vector (ξ, η) with positive arbitrarily dependent components. Put

$$S_0 := 0, \quad S_k := \xi_1 + \cdots + \xi_k, \quad k \in \mathbb{N} := \{1, 2, \ldots\}$$

and then

$$T_k := S_{k-1} + \eta_k, \quad k \in \mathbb{N}.$$

The random sequences $S := (S_k)_{k\geq 0}$ and $T := (T_k)_{k\geq 1}$ are known in the literature as the *standard random walk* and a *(globally) perturbed random walk*. A survey of various results for the so defined perturbed random walks can be found in the book [9].

Put

$$Y(t) := \sum_{k\geq 1} \mathbb{1}_{\{T_k \leq t\}}, \quad t \geq 0.$$

A law of the iterated logarithm (LIL) for $Y(t)$, properly normalized and centered, was proved as $t \to \infty$ *along integers* in Proposition 2.3 of [11] under the assumptions that $\mathbb{E}\eta^a < \infty$ for some $a > 0$ and $\sigma^2 := \operatorname{Var}\xi \in (0, \infty)$. We improve the aforementioned result by showing that the assumption $\mathbb{E}\eta^a < \infty$ for some $a > 0$ can be dispensed with and also that the LIL holds as $t \to \infty$ *along reals*, thereby obtaining an ultimate version of the LIL for $Y(t)$. For a family (x_t) of real numbers denote by $C((x_t))$ the set of its limit points.

Theorem 1. *Assume that $\sigma^2 = \operatorname{Var}\xi \in (0, \infty)$. Then*

$$C\Bigg(\Bigg(\frac{Y(t) - \mu^{-1}\int_0^t \mathbb{P}\{\eta \leq y\}\mathrm{d}y}{(2\sigma^2\mu^{-3}t\log\log t)^{1/2}} \; : \; t > \mathrm{e}\Bigg)\Bigg) = [-1, 1] \quad a.s.,$$

where $\mu := \mathbb{E}\xi < \infty$.

Next, we consider a general branching process generated by the random sequence $(T_k)_{k\geq 1}$. Thus, the random variables $T_1, T_2, \ldots$ are interpreted as the birth times of the first generation individuals. The first generation produces the second generation. The shifts of birth times of the second generation individuals with respect to their mothers' birth times are distributed according to copies of T, and for different mothers these copies are independent. The second generation produces the third one, and so on.

Let $Y_j(t)$ be the number of the jth generation individuals with birth times $\leq t$. Following [3], we call the sequence of processes $((Y_j(t))_{t\geq 0})_{j\geq 2}$ an iterated perturbed random walk. Note that, for $t \geq 0$, $Y_1(t) = Y(t)$ and the following decomposition holds:

$$Y_j(t) = \sum_{r\geq 1} Y_{j-1}^{(r)}(t - T_r)\, \mathbb{1}_{\{T_r \leq t\}}, \quad j \geq 2, \tag{1}$$

where $Y_{j-1}^{(r)}(t)$ is the number of the jth generation individuals who are descendants of the first generation individual with birth time T_r. Put $V(t) := V_1(t) = \mathbb{E}Y(t)$ for $t \geq 0$. Taking expectations in (1) we infer, for $j \geq 2$ and $t \geq 0$,

$$V_j(t) = (V_{j-1} * V)(t) = \int_{[0,\,t]} V_{j-1}(t - y)\mathrm{d}V(y). \tag{2}$$

The iterated perturbed random walks are interesting objects on their own, see [14, 16]. Also, these are the main auxiliary tool in investigations of nested infinite occupancy schemes in random environment. Details can be found in the papers [4–6, 15]. Attention was also paid to iterated standard random walks, which are a rather particular instance of the iterated perturbed random walks which corresponds to $\eta = \xi$. An LIL for the iterated standard random walks was recently proved in [12]. Continuing this line of investigation we formulate and prove an LIL for $Y_j(t)$, properly normalized and centered, as $t \to \infty$.

Theorem 2. *Assume that* $\sigma^2 = \mathrm{Var}\,\xi \in (0, \infty)$. *Then, for* $j \geq 2$,

$$C\Big(\Big(\frac{Y_j(t) - V_j(t)}{(2((2j-1)(j-1)!)^{-1}\sigma^2\mu^{-2j-1}t^{2j-1}\log\log t)^{1/2}} \, : \, t > \mathrm{e}\Big)\Big) = [-1, 1] \quad a.s., \tag{3}$$

where $\mu = \mathbb{E}\xi < \infty$.

Although the beginning of our proof of Theorem 2 is similar to that of Theorem 1.1 in [12], the subsequent technical details are essentially different. The main difficulty is that the distribution of η is arbitrary. Imposing a moment assumption on the distribution of η would greatly simplify an argument.

The remainder of the paper is structured as follows. After proving Theorem 1 in Section 2, we give a number of auxiliary results in Section 3 and then prove Theorem 2 in Section 4.

2 Proof of Theorem 1

We shall denote by n an integer argument and by t a real argument. For $t \in \mathbb{R}$, put $F(t) := \mathbb{P}\{\eta \leq t\}$ and

$$\nu(t) := \sum_{k \geq 0} \mathbb{1}_{\{S_k \leq t\}}, \tag{4}$$

and observe that $F(t) = 0$ and $\nu(t) = 0$ for $t < 0$. For $t > \mathrm{e}$, write

$$Y(t) - \mu^{-1}\int_0^t F(y)\mathrm{d}y = Y(t) - \int_{[0,\,t]} F(t-y)\mathrm{d}\nu(y) + \int_{[0,\,t]} F(t-y)\mathrm{d}(\nu(y) - \mu^{-1}y) =: X(t) + Z(t)$$

and put $a(t) := (2\sigma^2\mu^{-3}t\log\log t)^{1/2}$. It is shown in the proof of Proposition 2.3 in [11] that

$$C\big(\big(Z(n)/a(n) \, : \, n \geq 3\big)\big) = [-1, 1] \quad \text{a.s.} \tag{5}$$

This result holds irrespective of whether $\mathbb{E}\eta^a < \infty$ for some $a > 0$ or $\mathbb{E}\eta^a = \infty$ for all $a > 0$. We intend to show that (5) entails

$$C\big(\big(Z(t)/a(t) \, : \, t > \mathrm{e}\big)\big) = [-1, 1] \quad \text{a.s.} \tag{6}$$

Given $t \geq 4$ there exists $n \in \mathbb{N}$ such that $t \in (n-1, n]$. Hence, by monotonicity,

$$\frac{Z(t)}{a(t)} \leq \frac{Z(n) + \mu^{-1}\int_{n-1}^n F(y)\mathrm{d}y}{a(n-1)} \leq \frac{Z(n) + \mu^{-1}}{a(n-1)} \quad \text{a.s.}$$

Analogously,

$$\frac{Z(t)}{a(t)} \geq \frac{Z(n-1) - \mu^{-1}}{a(n)} \quad \text{a.s.}$$

We conclude that (6) does indeed hold.

It is known (see the proof of Theorem 3.2 in [1]) that

$$\lim_{n\to\infty} n^{-1/2}\Big(Y(n)-\int_{[0,\,n]}F(n-y)\mathrm{d}\nu(y)\Big)=0\quad \text{a.s.}$$

whenever $\mathbb{E}\eta^a<\infty$ for some $a>0$. We note that the latter limit relation may fail to hold if $\mathbb{E}\eta^a=\infty$ for all $a>0$. For instance, it follows from Remark 4.4 in [13] that the upper limit in the last displayed formula is equal to $+\infty$ a.s. whenever $\mathbb{P}\{\xi=c\}=1$ for some $c>0$ and $\lim_{t\to\infty}(\log\log t)(1-F(t))=1$.

The proof of Theorem 3.2 in [1] operates with power moments and relies heavily upon the assumption $\mathbb{E}\eta^a<\infty$ for some $a>0$. Without such an assumption another argument is needed, which operates with exponential rather than power moments. In the remainder of the proof we present such an argument, which enables us to prove that

$$\lim_{t\to\infty}(X(t)/b(t))=0\quad \text{a.s.,}\tag{7}$$

thereby completing the proof of the theorem; here, $b(t):=(t\log\log t)^{1/2}$ for $t>\mathrm{e}$.

Fix any $u\neq 0$ and $t>0$. Put $W_0:=1$ and, for $j\in\mathbb{N}$,

$$\begin{aligned}W_j:=\exp\Big(u\sum_{k=0}^{j-1}(\mathbb{1}_{\{\eta_{k+1}+S_k\le t\}}-F(t-S_k)\,\mathbb{1}_{\{S_k\le t\}})\\ -(u^2\mathrm{e}^{|u|}/2)\sum_{k=0}^{j-1}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\le t\}}\Big),\end{aligned}$$

and denote by $\mathcal{G}_0$ the trivial σ-algebra and, for $j\in\mathbb{N}$, by $\mathcal{G}_j$ the σ-algebra generated by $(\xi_k,\eta_k)_{1\le k\le j}$. Observe that the variable W_j is $\mathcal{G}_j$-measurable for $j\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. Now we prove that $(W_j,\mathcal{G}_j)_{j\ge 0}$ is a positive supermartingale. Indeed, writing $\mathbb{E}_j(\cdot)$ for $\mathbb{E}(\cdot|\mathcal{G}_j)$ and using the inequality $\mathrm{e}^x\le 1+x+x^2\mathrm{e}^{|x|}/2$ for $x\in\mathbb{R}$ in combination with

$$\mathbb{E}_{j-1}\big(\mathbb{1}_{\{\eta_j+S_{j-1}\le t\}}-F(t-S_{j-1})\,\mathbb{1}_{\{S_{j-1}\le t\}}\big)=0\quad \text{a.s.}$$

we infer

$$\begin{aligned}&\mathbb{E}_{j-1}\exp\big(u(\mathbb{1}_{\{\eta_j+S_{j-1}\le t\}}-F(t-S_{j-1})\,\mathbb{1}_{\{S_{j-1}\le t\}})\big)\\ &\quad\le 1+(u^2/2)\mathbb{E}_{j-1}(\mathbb{1}_{\{T_j\le t\}}-F(t-S_{j-1}))^2\\ &\qquad\times\exp(|u(\mathbb{1}_{\{T_j\le t\}}-F(t-S_{j-1})\,\mathbb{1}_{\{S_{j-1}\le t\}})|)\,\mathbb{1}_{\{S_{j-1}\le t\}}.\end{aligned}$$

In view of $|\,\mathbb{1}_{\{T_j\le t\}}-F(t-S_{j-1})\,\mathbb{1}_{\{S_{j-1}\le t\}}|\le 1$ a.s., the right-hand side does not exceed

$$\begin{aligned}&1+(u^2\mathrm{e}^{|u|}/2)F(t-S_{j-1})(1-F(t-S_{j-1}))\,\mathbb{1}_{\{S_{j-1}\le t\}}\\ &\quad\le 1+(u^2\mathrm{e}^{|u|}/2)(1-F(t-S_{j-1}))\,\mathbb{1}_{\{S_{j-1}\le t\}}\\ &\quad\le \exp((u^2\mathrm{e}^{|u|}/2)(1-F(t-S_{j-1}))\,\mathbb{1}_{\{S_{j-1}\le t\}}).\end{aligned}$$

For the latter inequality we have used $1+x\le\mathrm{e}^x$ for $x\ge 0$. Thus, we have proved that, for $j\in\mathbb{N}$, $\mathbb{E}_{j-1}(W_j/W_{j-1})\le 1$ a.s. and thereupon $\mathbb{E}_{j-1}W_j\le W_{j-1}$ a.s., that is,

$(W_j, \mathcal{G}_j)_{j\geq 0}$ is indeed a positive supermartingale. As a consequence, the a.s. limit

$$\lim_{j\to\infty} W_j =: W_\infty = \exp\Big(uX(t) - (u^2 e^{|u|}/2)\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}}\Big)$$

satisfies $\mathbb{E}W_\infty \leq \mathbb{E}W_0 = 1$. In other words, with $u\in\mathbb{R}$ and $t>0$ fixed,

$$\mathbb{E}\exp\Big(uX(t) - (u^2 e^{|u|}/2)\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}}\Big) \leq 1. \tag{8}$$

We shall also need another auxiliary result.

$$\lim_{t\to\infty} t^{-1}\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}} = 0 \quad \text{a.s.} \tag{9}$$

Proof. To prove (9), write, for fixed $a>0$ and $t>a$,

$$\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}} = \sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t-a\}}$$
$$+\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{t-a<S_k\leq t\}} \leq (1-F(a))\nu(t) + (\nu(t)-\nu(t-a)).$$

By the strong law of large numbers for renewal processes, $\lim_{t\to\infty} t^{-1}\nu(t) = \mu^{-1}$ a.s. and $\lim_{t\to\infty} t^{-1}(\nu(t)-\nu(t-a)) = \mu^{-1}-\mu^{-1} = 0$ a.s. Hence, for each fixed $a>0$,

$$\limsup_{t\to\infty} t^{-1}\sum_{k\geq 0}(1-F(t-S_k))\,\mathbb{1}_{\{S_k\leq t\}} \leq \mu^{-1}(1-F(a)) \quad \text{a.s.}$$

Letting $a\to\infty$ we arrive at (9). □

Fix any $\varepsilon>0$ and put $t_n := \exp(n^{3/4})$ for $n\in\mathbb{N}$. We intend to prove that

$$\lim_{n\to\infty}(X(t_n)/b(t_n)) = 0 \quad \text{a.s.} \tag{10}$$

To this end, for $n\geq 3$, define the event

$$A_n := \{X(t_n) > \varepsilon b(t_n)\}.$$

In view of (9), for large n,

$$\sum_{k\geq 0}(1-F(t_n-S_k))\,\mathbb{1}_{\{S_k\leq t_n\}} \leq (\varepsilon^2/8)t_n.$$

Using this we obtain, for any $u>0$ and large n,

$$A_n = \{uX(t_n) - (u^2 e^{|u|}/2)\sum_{k\geq 0}(1-F(t_n-S_k))\,\mathbb{1}_{\{S_k\leq t_n\}}$$
$$> \varepsilon u b(t_n) - (u^2 e^{|u|}/2)\sum_{k\geq 0}(1-F(t_n-S_k))\,\mathbb{1}_{\{S_k\leq t_n\}}\}$$
$$\subseteq \{uX(t_n) - (u^2 e^{|u|}/2)\sum_{k\geq 0}(1-F(t_n-S_k))\,\mathbb{1}_{\{S_k\leq t_n\}}$$

$$> \varepsilon u b(t_n) - (\varepsilon^2/8)(u^2 e^{|u|}/2)t_n\} =: B_n.$$

Invoking Markov's inequality in combination with (8) we infer

$$\begin{aligned}\mathbb{P}\{B_n\} &\le \exp\Big(- \varepsilon u b(t_n) + (\varepsilon^2/8)(u^2 e^{|u|}/2)t_n\Big) \\ &\quad \times \mathbb{E}\exp\Big(uX(t_n) - (u^2 e^{|u|}/2)\sum_{k\ge 0}(1-F(t_n-S_k))\,\mathbb{1}_{\{S_k\le t_n\}}\Big) \\ &\le \exp\Big(- \varepsilon u b(t_n) + (\varepsilon^2/8)(u^2 e^{|u|}/2)t_n\Big).\end{aligned}$$

Let $\rho > 0$ satisfy $\exp(8\varepsilon^{-1}\rho) = 3/2$. For large $x > 0$, $x^{-1}\log\log x \le \rho$. Put

$$u = 8\varepsilon^{-1}(t_n^{-1}\log\log t_n)^{1/2}.$$

Then

$$-\varepsilon u b(t_n) + (\varepsilon^2/8)(u^2 e^{|u|}/2)t_n \le -8\log\log t_n + 4e^{8\varepsilon^{-1}\rho}\log\log t_n = -2\log\log t_n.$$

Hence, by the Borel–Cantelli lemma, $\limsup_{n\to\infty}(X(t_n)/b(t_n)) \le 0$ a.s. The converse inequality for the lower limit follows analogously. We start with $A_n^* := \{-X(t_n) > \varepsilon b(t_n)\}$ and show, by the same reasoning as above, that $A_n^* \subseteq B_n^*$, where B_n^* only differs from B_n by the term $-uX(t_n)$ in place of $uX(t_n)$.

It remains to show that (10) can be lifted to (7). To this end, it suffices to prove that

$$\lim_{n\to\infty}\frac{\sup_{u\in[t_n,\,t_{n+1}]}|X(u)-X(t_n)|}{t_n^{1/2}} = 0 \quad \text{a.s.} \tag{11}$$

Indeed, (11) in combination with (10) entails

$$\lim_{n\to\infty}\frac{\sup_{u\in[t_n,\,t_{n+1}]}|X(u)|}{b(t_n)} = 0 \quad \text{a.s.}$$

This ensures (7) because, for large enough n,

$$\frac{|X(t)|}{b(t)} \le \frac{\sup_{u\in[t_n,\,t_{n+1}]}|X(u)|}{b(t_n)} \quad \text{a.s.}$$

whenever $t \in [t_n,\, t_{n+1}]$.

We denote by $I = I_n$ a sequence of positive integers to be chosen later. For $j \in \mathbb{N}_0$ and $n \in \mathbb{N}$, put

$$F_j(n) := \{v_{j,m}(n) := t_n + 2^{-j}m(t_{n+1}-t_n) : 0 \le m \le 2^j\}.$$

In what follows, we write $v_{j,m}$ for $v_{j,m}(n)$. Observe that $F_j(n) \subseteq F_{j+1}(n)$. For any $u \in [t_n, t_{n+1}]$, put

$$u_j := \max\{v \in F_j(n) : v \le u\} = t_n + 2^{-j}(t_{n+1}-t_n)\left\lfloor \frac{2^j(u-t_n)}{t_{n+1}-t_n}\right\rfloor.$$

An important observation is that either $u_{j-1} = u_j$ or $u_{j-1} = u_j - 2^{-j}(t_{n+1} - t_n)$. Necessarily, $u_j = v_{j,m}$ for some $0 \le m \le 2^j$, so that either $u_{j-1} = v_{j,m}$ or $u_{j-1} = v_{j,m-1}$. Write

$$\begin{aligned}\sup_{u\in[t_n,\,t_{n+1}]} &|X(u) - X(t_n)| \\ &= \max_{0\le j\le 2^I-1} \sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} |(X(v_{I,\,j}) - X(t_n)) + (X(v_{I,\,j}+z) - X(v_{I,\,j}))| \\ &\le \max_{0\le j\le 2^I-1} |X(v_{I,\,j}) - X(t_n)| \\ &\quad + \max_{0\le j\le 2^I-1} \sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} |X(v_{I,\,j}+z) - X(v_{I,\,j})| \quad \text{a.s.}\end{aligned}$$

For $u \in F_I(n)$,

$$\begin{aligned}|X(u) - X(t_n)| &= \Big|\sum_{j=1}^{I}(X(u_j) - X(u_{j-1})) + X(u_0) - X(t_n)\Big| \\ &\le \sum_{j=0}^{I} \max_{1\le m\le 2^j} |X(v_{j,\,m}) - X(v_{j,\,m-1})|.\end{aligned}$$

With this at hand, we obtain

$$\begin{aligned}\sup_{u\in[t_n,\,t_{n+1}]} |X(u) - X(t_n)| &\le \sum_{j=0}^{I} \max_{1\le m\le 2^j} |X(v_{j,\,m}) - X(v_{j,\,m-1})| \\ &+ \max_{0\le j\le 2^I-1} \sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} |X(v_{I,\,j}+z) - X(v_{I,\,j})| \quad \text{a.s.}\end{aligned} \tag{12}$$

We first show that, for all $\varepsilon > 0$,

$$\sum_{n\ge1} \mathbb{P}\Big\{\sum_{j=0}^{I} \max_{1\le m\le 2^j} |X(v_{j,\,m}) - X(v_{j,\,m-1})| > \varepsilon t_n^{1/2}\Big\} < \infty. \tag{13}$$

Let $\ell \in \mathbb{N}$. As a preparation, we derive an appropriate upper bound for $\mathbb{E}(X(u) - X(v))^{2\ell}$ for $u, v > 0$, $u > v$. Observe that $X(u) - X(v)$ is equal to the a.s. limit $\lim_{j\to\infty} R(j,u,v)$, where $(R(j,u,v),\mathcal{G}_j)_{j\ge0}$ is a martingale defined by

$$R(0,u,v) := 0, \quad R(j,u,v) := \sum_{k=0}^{j-1}(\mathbb{1}_{\{v<\eta_{k+1}+S_k\le u\}} - F(u-S_k) + F(v-S_k)), \quad j \in \mathbb{N},$$

and, as before, $\mathcal{G}_0$ denotes the trivial σ-algebra and, for $j \in \mathbb{N}$, $\mathcal{G}_j$ denotes the σ-algebra generated by $(\xi_k, \eta_k)_{1\le k\le j}$. Recall that $F(t) = 0$ for $t < 0$. By the Burkholder–Davis–Gundy inequality, see, for instance, Theorem 11.3.2 in [7],

$$\begin{aligned}\mathbb{E}(X(u) - X(v))^{2\ell} \le C\Big(&\mathbb{E}\Big(\sum_{k\ge0}\mathbb{E}\big((R(k+1,u,v) - R(k,u,v))^2|\mathcal{G}_k\big)\Big)^{\ell} \\ &+ \sum_{k\ge0}\mathbb{E}(R(k+1,u,v) - R(k,u,v))^{2\ell}\Big)\end{aligned}$$

$$= C\Big(\mathbb{E}\Big(\sum_{k\geq 0}(F(u-S_k)-F(v-S_k))(1-F(u-S_k)+F(v-S_k))\Big)^{\ell}$$

$$+\sum_{k\geq 0}\mathbb{E}(\mathbb{1}_{\{v<\eta_{k+1}+S_k\leq u\}}-F(u-S_k)+F(v-S_k))^{2\ell}\Big) =: C(A(u,v)+B(u,v))$$

for a positive constant C. Let $f:[0,\infty)\to[0,\infty)$ be a locally bounded function. It is shown in the proof of Lemma A.3 in [1] that $\mathbb{E}(\nu(1))^{\ell}<\infty$ and that

$$\mathbb{E}\Big(\int_{[0,t]}f(t-y)\mathrm{d}\nu(y)\Big)^{\ell}\leq \mathbb{E}(\nu(1))^{\ell}\Big(\sum_{n=0}^{\lfloor t\rfloor}\sup_{y\in[n,n+1)}f(y)\Big)^{\ell}. \tag{14}$$

Further,

$$\begin{aligned}
A(u,v) &= \mathbb{E}\Big(\int_{(v,u]}F(u-y)(1-F(u-y))\mathrm{d}\nu(y)\\
&\quad+\int_{[0,v]}(F(u-y)-F(v-y))(1-F(u-y)+F(v-y))\mathrm{d}\nu(y)\Big)^{\ell}\\
&\leq 2^{\ell-1}\Big(\mathbb{E}\Big(\int_{(v,u]}F(u-y)(1-F(u-y))\mathrm{d}\nu(y)\Big)^{\ell}\\
&\quad+\mathbb{E}\Big(\int_{[0,v]}(F(u-y)-F(v-y))(1-F(u-y)+F(v-y))\mathrm{d}\nu(y)\Big)^{\ell}\\
&\leq 2^{\ell-1}\Big(\mathbb{E}\Big(\int_{[0,u]}\mathbb{1}_{[0,u-v)}(u-y)\mathrm{d}\nu(y)\Big)^{\ell}\\
&\quad+\mathbb{E}\Big(\int_{[0,v]}(F(u-y)-F(v-y))\mathrm{d}\nu(y)\Big)^{\ell}\Big)\\
&=: 2^{\ell-1}(A_1(u,v)+A_2(u,v)).
\end{aligned}$$

Using (14) with $t=u$ and $f(y)=\mathbb{1}_{[0,u-v)}(y)$ and then with $t=v$ and $f(y)=F(u-v+y)-F(y)$ we infer

$$A_1(u,v)\leq \mathbb{E}(\nu(1))^{\ell}\Big(\sum_{n=0}^{\lfloor u\rfloor}\sup_{y\in[n,n+1)}\mathbb{1}_{[0,u-v)}(y)\Big)^{\ell}=\mathbb{E}(\nu(1))^{\ell}(\lceil u-v\rceil)^{\ell},$$

where $x\mapsto\lceil x\rceil$ is the ceiling function, and

$$\begin{aligned}
A_2(u,v) &\leq \mathbb{E}(\nu(1))^{\ell}\Big(\sum_{n=0}^{\lfloor v\rfloor}\sup_{y\in[n,n+1)}(F(u-v+y)-F(y))\Big)^{\ell}\\
&\leq \mathbb{E}(\nu(1))^{\ell}\Big(\sum_{n=0}^{\lfloor v\rfloor}(F(\lceil u-v\rceil+n+1)-F(n))\Big)^{\ell}\\
&= \mathbb{E}(\nu(1))^{\ell}\Big(\sum_{n=0}^{\lceil u-v\rceil}(F(\lfloor v\rfloor+1+n)-F(n))\Big)^{\ell}\leq \mathbb{E}(\nu(1))^{\ell}(\lceil u-v\rceil+1)^{\ell}.
\end{aligned}$$

Finally,

$$B(u,v) \le \sum_{k\ge 0} \mathbb{E}(\mathbb{1}_{\{v<\eta_{k+1}+S_k\le u\}} - F(u-S_k) + F(v-S_k))^2$$
$$\le 2\mathbb{E}\nu(1)(\lceil u-v\rceil + 1) \le 2\mathbb{E}\nu(1)(\lceil u-v\rceil+1)^\ell$$

and thereupon

$$\mathbb{E}(X(u)-X(v))^{2\ell} \le C_1(\lceil u-v\rceil+1)^\ell. \tag{15}$$

Note that $v_{j,m} - v_{j,m-1} = 2^{-j}(t_{n+1}-t_n)$. Put $I = I_n := \lfloor \log_2(2^{-1}(t_{n+1}-t_n))\rfloor$. We claim that there exists a constant $C_2 > 0$ such that $C_1(\lceil 2^{-j}(t_{n+1}-t_n)\rceil + 1)^\ell \le C_2 2^{-j\ell}(t_{n+1}-t_n)^\ell$ whenever $j \in \mathbb{N}$, $j \le I$. Indeed,

$$(\lceil 2^{-j}(t_{n+1}-t_n)\rceil+1)^\ell \le (2^{-j}(t_{n+1}-t_n)+2)^\ell \le 2^{\ell-1}(2^{-j\ell}(t_{n+1}-t_n)^\ell + 2^\ell)$$
$$\le 2^\ell 2^{-j\ell}(t_{n+1}-t_n)^\ell$$

having utilized $2^{-j}(t_{n+1}-t_n) \ge 2$ for $j \le I$. Invoking (15) we then obtain, for nonnegative integer $j \le I$,

$$\mathbb{E}(X(v_{j,m}) - X(v_{j,m-1}))^{2\ell} \le C_1(\lceil 2^{-j}(t_{n+1}-t_n)\rceil+1)^\ell \le C_2 2^{-j\ell}(t_{n+1}-t_n)^\ell \tag{16}$$

and thereupon

$$\mathbb{E}\Big(\max_{1\le m\le 2^j}(X(v_{j,m})-X(v_{j,m-1}))^{2\ell}\Big) \le \sum_{m=1}^{2^j}\mathbb{E}(X(v_{j,m})-X(v_{j,m-1}))^{2\ell}$$
$$\le C_2 2^{-j(\ell-1)}(t_{n+1}-t_n)^\ell.$$

By the triangle inequality for the $L_{2\ell}$-norm,

$$\mathbb{E}\Big(\sum_{j=0}^{I}\max_{1\le m\le 2^j}|X(v_{j,m})-X(v_{j,m-1})|\Big)^{2\ell}$$
$$\le \Big(\sum_{j=0}^{I}\Big(\mathbb{E}\Big(\max_{1\le m\le 2^j}(X(v_{j,m})-X(v_{j,m-1}))^{2\ell}\Big)\Big)^{1/(2\ell)}\Big)^{2\ell}$$
$$\le C_2(t_{n+1}-t_n)^\ell\Big(\sum_{j\ge 0}2^{-j(\ell-1)/(2\ell)}\Big)^{2\ell} =: C_3(t_{n+1}-t_n)^\ell.$$

By Markov's inequality,

$$\mathbb{P}\Big\{\sum_{j=0}^{I}\max_{1\le m\le 2^j}|X(v_{j,m})-X(v_{j,m-1})| > \varepsilon t_n^{1/2}\Big\} \le C_3\varepsilon^{-2\ell}t_n^{-\ell}(t_{n+1}-t_n)^\ell.$$

Since $t_n^{-1}(t_{n+1}-t_n) \sim (3/4)n^{-1/4}$ as $n\to\infty$, (13) follows upon setting $\ell = 6$, say. Invoking the Borel–Cantelli lemma we infer

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{I}\max_{1\le m\le 2^j}|X(v_{j,m})-X(v_{j,m-1})|}{t_n^{1/2}} = 0 \quad \text{a.s.}$$

Now we proceed with the analysis of the second summand in (12). Put $M(t) := \int_{[0,\,t]} F(t-y)\mathrm{d}\nu(y)$ for $t \geq 0$. Using the equality $X(t) = N(t) - M(t)$ and a.s. monotonicity of N and M we infer

$$\begin{aligned}
&\sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} |X(v_{I,\,j}+z) - X(v_{I,\,j})| \\
&\quad\leq \sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} (N(v_{I,\,j}+z) - N(v_{I,\,j})) \\
&\qquad+ \sup_{z\in[0,\,v_{I,\,j+1}-v_{I,\,j}]} (M(v_{I,\,j}+z) - M(v_{I,\,j})) \\
&\quad= (N(v_{I,\,j+1}) - N(v_{I,\,j})) + (M(v_{I,\,j+1}) - M(v_{I,\,j})).
\end{aligned}$$

Observe that

$$\begin{aligned}
\max_{0\leq j\leq 2^I-1} (N(v_{I,\,j+1}) - N(v_{I,\,j})) \leq &\max_{0\leq j\leq 2^I-1} |X(v_{I,\,j+1}) - X(v_{I,\,j})| \\
&+ \max_{0\leq j\leq 2^I-1} (M(v_{I,\,j+1}) - M(v_{I,\,j})).
\end{aligned}$$

Hence, according to the Borel–Cantelli lemma, it is enough to prove that, for all $\varepsilon > 0$,

$$\sum_{n\geq 1} \mathbb{P}\{\max_{0\leq j\leq 2^I-1} (M(v_{I,\,j+1}) - M(v_{I,\,j})) > \varepsilon t_n^{1/2}\} < \infty \tag{17}$$

and

$$\sum_{n\geq 1} \mathbb{P}\{\max_{0\leq j\leq 2^I-1} |X(v_{I,\,j+1}) - X(v_{I,\,j})| > \varepsilon t_n^{1/2}\Big\} < \infty. \tag{18}$$

Arguing as above we conclude that, for $u, v > 0$, $u > v$,

$$\begin{aligned}
\mathbb{E}(M(u) - M(v))^\ell &= \mathbb{E}\Big(\int_{(v,\,u]} F(u-y)\mathrm{d}\nu(y) + \int_{[0,\,v]} (F(u-y) - F(v-y))\mathrm{d}\nu(y)\Big)^\ell \\
&\leq 2^{\ell-1}\mathbb{E}(\nu(1))^\ell(\lceil u-v\rceil + 1)^\ell.
\end{aligned}$$

As a consequence, for nonnegative integer $j \leq I$ and a constant $C_4 > 0$,

$$\mathbb{E}(M(v_{I,\,j+1}) - M(v_{I,\,j}))^\ell \leq C_4 2^{-I\ell}(t_{n+1} - t_n)^\ell.$$

By Markov's inequality and our choice of I,

$$\begin{aligned}
\mathbb{P}\{\max_{0\leq j\leq 2^I-1} (M(v_{I,\,j+1}) - M(v_{I,\,j})) > \varepsilon t_n^{1/2}\} &\leq C_4\varepsilon^{-\ell} 2^{-I(\ell-1)} t_n^{-\ell/2}(t_{n+1} - t_n)^\ell \\
&\leq C_4\varepsilon^{-\ell} 2^{2(\ell-1)} t_n^{-\ell/2}(t_{n+1} - t_n).
\end{aligned}$$

Hence, (17) follows upon choosing $\ell > 2$. To prove (18), we invoke (16) which enables us to conclude that

$$\begin{aligned}
\mathbb{P}\{\max_{0\leq j\leq 2^I-1} |X(v_{I,\,j+1}) - X(v_{I,\,j})| > \varepsilon t_n^{1/2}\} &\leq C_2\varepsilon^{-2\ell} 2^{-I(\ell-1)} t_n^{-\ell}(t_{n+1} - t_n)^\ell \\
&\leq C_2\varepsilon^{-2\ell} 2^{2(\ell-1)} t_n^{-\ell}(t_{n+1} - t_n).
\end{aligned}$$

Choosing $\ell > 1$ we arrive at (18).

The proof of Theorem 1 is complete.

3 Auxiliary results

To prove Theorem 2, we need some auxiliary results on the iterated perturbed random walks. Lemma 1 is a known result, see Assertion 1 in [16].

Lemma 1. *Assume that* $\mu = \mathbb{E}\xi < \infty$. *Then, for fixed* $j \in \mathbb{N}$,

$$\lim_{t\to\infty} \frac{V_j(t)}{t^j} = \frac{1}{j!\mu^j}. \tag{19}$$

Put

$$U(t) := \mathbb{E}\nu(t) = \sum_{k\geq 0} \mathbb{P}\{S_k \leq t\} \quad \text{for } t \geq 0, \tag{20}$$

so that U is the renewal function.

Lemma 2. *For every* $x, h > 0$ *and* $k \in \mathbb{N}$,

$$V_k(x+h) - V_k(x) \leq U(h)(V(x+h))^{k-1}. \tag{21}$$

Proof. We use mathematical induction. For $k = 1$, write

$$\begin{aligned} V(x+h) - V(x) &= \int_{[0,\,x+h]} U(x+h-y)\mathrm{d}F(y) - \int_{[0,\,x]} U(x-y)\mathrm{d}F(y) \\ &= \int_{(x,\,x+h]} U(x+h-y)\mathrm{d}F(y) + \int_{[0,\,x]} (U(x+h-y) - U(x-y))\mathrm{d}F(y) \\ &\leq U(h)(F(x+h) - F(x)) + U(h)F(x) \leq U(h). \end{aligned} \tag{22}$$

The penultimate inequality is justified by subadditivity of the renewal function U, see Theorem 1.7 on p. 10 in [17], and its monotonicity.

Assume that inequality (21) holds for $k \leq l-1$. Note that (2) implies that $V_{l-1}(h) \leq (V(h))^{l-1} \leq U(h)(V(x+h))^{l-2}$ for $l \geq 2$ and $h \geq 0$. Using this and the induction assumption, we have

$$\begin{aligned} V_l(x+h) - V_l(x) &= \int_{[0,\,x]} (V_{l-1}(x+h-y) - V_{l-1}(x-y))\mathrm{d}V(y) \\ &\quad + \int_{(x,x+h]} V_{l-1}(x+h-y)\mathrm{d}V(y) \\ &\leq U(h) \int_{[0,\,x]} (V(x+h-y))^{l-2}\mathrm{d}V(y) + V_{l-1}(h)(V(x+h) - V(x)) \\ &\leq U(h)(V(x+h))^{l-2} \cdot V(x) + U(h)(V(x+h))^{l-2}(V(x+h) - V(x)) \\ &= U(h)(V(x+h))^{l-1}. \end{aligned}$$

□

Lemma 3. *Assume that* $\operatorname{Var}\xi \in (0,\infty)$. *Then, for* $k \in \mathbb{N}$,

$$a_k(t) := \operatorname{Var} Y_k(t) = O(t^{2k-1}), \quad t \to \infty. \tag{23}$$

Proof. We use mathematical induction. For $k = 1$, write

$$\begin{aligned}Y_1(t) - V_1(t) &= \sum_{j\ge1}(\mathbb{1}_{\{\eta_j+S_{j-1}\le t\}} - F(t-S_{j-1})) + (\sum_{j\ge0} F(t-S_j) - V_1(t))\\ &=: I_1(t) + I_2(t).\end{aligned}$$

Note that $\mathrm{Var}\, Y_1(t) = \mathbb{E}(Y_1(t) - V_1(t))^2 \le 2(\mathbb{E}(I_1(t))^2 + \mathbb{E}(I_2(t))^2)$. Let U be as in (20). We have

$$\mathbb{E}(I_1(t))^2 = \int_{[0,t]} F(t-y)(1-F(t-y))\mathrm{d}U(y) \le \int_{[0,t]} (1-F(t-y))\mathrm{d}U(y).$$

If $\mathbb{E}\eta = \infty$, then Lemma 6.2.9 in [9] with $r_1 = 0$ and $r_2 = 1$ yields

$$\int_{[0,t]} (1-F(t-y))\mathrm{d}U(y) \sim \frac{1}{\mu}\int_0^t (1-F(y))\mathrm{d}y = o(t), \quad t \to \infty,$$

where $\mu = \mathbb{E}\xi < \infty$. If $\mathbb{E}\eta < \infty$, then $\int_{[0,t]}(1-F(t-y))\mathrm{d}U(y) = O(1)$ as $t \to \infty$ by the key renewal theorem. Thus, in any case, $\mathbb{E}(I_1(t))^2 = o(t)$ as $t \to \infty$.

In the proof of Lemma 4.2 in [8] it is shown that

$$\mathbb{E} \sup_{s\in[0,t]} (\nu(s) - U(s))^2 = O(t), \quad t \to \infty, \tag{24}$$

where $\nu(s)$ is the same as in (4). Therefore, almost surely

$$\begin{aligned}|I_2(t)| &= \Big|\int_{[0,t]} F(t-y)\mathrm{d}(\nu(y) - U(y))\Big| = \Big|\int_{[0,t]} (\nu(t-y) - U(t-y)\mathrm{d}F(y)\Big|\\ &\le \int_{[0,t]} |\nu(t-y) - U(t-y)|\mathrm{d}F(y) \le \sup_{s\in[0,t]} |\nu(s) - U(s)| \cdot F(t)\\ &\le \sup_{s\in[0,t]} |\nu(s) - U(s)|.\end{aligned} \tag{25}$$

Consequently, according to (24), $\mathbb{E}(I_2(t))^2 \le \mathbb{E}\sup_{s\in[0,t]}(\nu(s) - U(s))^2 = O(t)$ as $t \to \infty$. We have proved that $a_1(t) = O(t)$ as $t \to \infty$.

Assume that relation (23) holds for $k \le l-1$. We shall use the representation

$$Y_l(t) - V_l(t) = \sum_{r\ge1} \big(Y_{l-1}^{(r)}(t-T_r) - V_{l-1}(t-T_r)\big) + \big(\sum_{r\ge1} V_l(t-T_r) - V_l(t)\big) =: J_l(t) + K_l(t),$$

which particularly entails

$$a_l(t) = \mathbb{E}(Y_l(t) - V_l(t))^2 = \mathbb{E}(J_l(t))^2 + \mathbb{E}(K_l(t))^2.$$

Note that, according to the induction assumption, there exist $A > 0$ and $t_0 > 0$ such that $a_{l-1}(t) \le At^{2l-3}$ for all $t \ge t_0$. Therefore, using (19) and (22),

$$\mathbb{E}(J_l(t))^2 = \int_{[0,t]} a_{l-1}(t-y)\mathrm{d}V(y)$$

$$\begin{aligned}&= \int_{[0,\,t-t_0]} a_{l-1}(t-y)\mathrm{d}V(y) + \int_{(t-t_0,\,t]} a_{l-1}(t-y)\mathrm{d}V(y)\\&\le A\int_{[0,\,t]}(t-y)^{2l-3}\mathrm{d}V(y) + \sup_{s\in[0,t_0]} a_{l-1}(s)(V(t)-V(t-t_0))\\&\le At^{2l-3}V(t)+O(1)=O(t^{2l-2}),\quad t\to\infty.\end{aligned} \tag{26}$$

Further,

$$\begin{aligned}K_l(t) &= \sum_{r\ge 0}\big(V_{l-1}(t-T_r)-(V_{l-1}*F)(t-S_{r-1})\big)\\&\quad+\Big(\sum_{r\ge 0}(V_{l-1}*F)(t-S_{r-1})-V_l(t))\Big)\\&=: K_{l1}(t)+K_{l2}(t).\end{aligned}$$

Using $V_l = V_{l-1}*F*U$ and the same reasoning as in (25) we obtain

$$|K_{l2}(t)| = \Big|\int_{[0,\,t]}(V_{l-1}*F)(t-y)\mathrm{d}(\nu(y)-U(y))\Big| \le \sup_{s\in[0,\,t]}|\nu(s)-U(s)|\cdot V_{l-1}(t) \ \text{ a.s.}$$

Therefore, in view of (19) and (24),

$$\mathbb{E}(K_{l2}(t))^2 \le \mathbb{E}\sup_{s\in[0,\,t]}(\nu(s)-U(s))^2(V_{l-1}(t))^2 = O(t^{2l-1}),\quad t\to\infty.$$

Finally,

$$\begin{aligned}\mathbb{E}(K_{l1}(t))^2 &= \sum_{r\ge 1}\mathbb{E}\big(V_{l-1}(t-T_r)-(V_{l-1}*F)(t-S_{r-1})\big)^2\\&\le \sum_{r\ge 1}\Big[\mathbb{E}\big(V_{l-1}(t-T_r)\big)^2+\mathbb{E}\big((V_{l-1}*F)(t-S_{r-1})\big)^2\Big]\\&= \int_{[0,\,t]}\big(V_{l-1}(t-y)\big)^2\mathrm{d}V(y)+\int_{[0,\,t]}\big((V_{l-1}*F)(t-y)\big)^2\mathrm{d}U(y)\\&\le \big(V_{l-1}(t)\big)^2\cdot V(t)+\big((V_{l-1}*F)(t)\big)^2\cdot U(t) = O(t^{2l-1}),\quad t\to\infty.\end{aligned}$$

For the last equality we have used $(V_{l-1}*F)(t)\le V_{l-1}(t)$ for $t\ge 0$. The proof of the Lemma 3 is complete. □

We shall also need two results on the standard random walks. The next lemma is a consequence of formula (33) in [1], with $\eta=\xi$.

Lemma 4. *For all positive b and c,*

$$\lim_{t\to\infty}\frac{\nu(t+b)-\nu(t)}{t^c}=0 \quad a.s.$$

Lemma 5. *Let $K_1,\ K_2\ :\ [0,\infty)\to[0,\infty)$ be nondecreasing functions and $K_1(t)\ge K_2(t)$ for $t\ge 0$. Assume that*

$$\limsup_{t\to\infty}\frac{K_1(t)+K_2(t)}{\int_0^t(K_1(y)-K_2(y))\mathrm{d}y}\le\lambda\in(0,\infty). \tag{27}$$

Then, for all $c > 0$*,*

$$\lim_{t\to\infty} \frac{\int_{[0,t]}(K_1(t-y)-K_2(t-y))\mathrm{d}\nu(y)}{t^c \int_0^t (K_1(y)-K_2(y))\mathrm{d}y} = 0 \quad a.s. \tag{28}$$

Proof. We use the decomposition

$$\int_{[0,t]}(K_1(t-y)-K_2(t-y))\mathrm{d}\nu(y) = \int_{[0,\lfloor t\rfloor]} \dots + \int_{[\lfloor t\rfloor, t]} \dots =: I_1(t)+I_2(t).$$

For $I_2(t)$ we have

$$I_2(t) \le \int_{[\lfloor t\rfloor,t]} K_1(t-y)\mathrm{d}\nu(y) \le K_1(t-\lfloor t\rfloor)(\nu(t)-\nu(\lfloor t\rfloor)) \le K_1(1)(\nu(t)-\nu(t-1)).$$

Hence, by Lemma 4, for all $c > 0$, $\lim_{t\to\infty} t^{-c} I_2(t) = 0$ a.s. It remains to consider $I_1(t)$:

$$\begin{aligned}
I_1(t) &= K_1(t)-K_2(t) + \sum_{j=0}^{\lfloor t\rfloor-1} \int_{(j,j+1]} (K_1(t-y)-K_2(t-y))\mathrm{d}\nu(y) \\
&\le K_1(t)-K_2(t) + \sum_{j=0}^{\lfloor t\rfloor-1} (K_1(t-j)-K_2(t-j-1))(\nu(j+1)-\nu(j)) \\
&\le K_1(t) + \sup_{s\in[0,\lfloor t\rfloor]} (\nu(s+1)-\nu(s)) \sum_{j=0}^{\lfloor t\rfloor-1} (K_1(t-j)-K_2(t-j-1)) \\
&\le K_1(t) + \sup_{s\in[0,\lfloor t\rfloor]} (\nu(s+1)-\nu(s)) \sum_{j=0}^{\lfloor t\rfloor-1} (K_1(\lfloor t\rfloor+1-j)-K_2(\lfloor t\rfloor-1-j)) \\
&= \sup_{s\in[0,\lfloor t\rfloor]} (\nu(s+1)-\nu(s)) \left(\int_2^{\lfloor t\rfloor} (K_1(y)-K_2(y))\mathrm{d}y + O(K_1(t)+K_2(t)) \right).
\end{aligned}$$

Another application of Lemma 4 yields

$$\lim_{t\to\infty} \frac{I_1(t)}{t^c \int_0^t (K_1(y)-K_2(y))\mathrm{d}y} = 0 \quad \text{a.s.}$$

The proof of Lemma 5 is complete. □

4 Proof of Theorem 2

We use a decomposition

$$Y_j(t)-V_j(t) = \sum_{k\ge 1}\left(Y_{j-1}^{(k)}(t-T_k)-V_{j-1}(t-T_k)\right) + \sum_{k\ge 1} V_{j-1}(t-T_k)-V_j(t), \quad j\ge 2,\ t\ge 0. \tag{29}$$

The first term of the decomposition is treated in Proposition 1.

Proposition 1. *Assume that* $\operatorname{Var}\xi \in (0, \infty)$. *Then, for* $j \geq 2$,

$$\lim_{t\to\infty} \frac{\sum_{k\geq 1} \big(Y_{j-1}^{(k)}(t-T_k) - V_{j-1}(t-T_k)\big)}{(t^{2j-1}\log\log t)^{1/2}} = 0 \quad a.s.$$

We first prove Theorem 2 with the help of Proposition 1. Afterwards, a proof of Proposition 1 will be given.

Proof of Theorem 2. By Proposition 1, the contribution of the first term in (29) normalized by $(t^{2j-1}\log\log t)^{1/2}$ vanishes as $t \to \infty$.

For the second term in (29), write

$$\begin{aligned}\sum_{k\geq 1} V_{j-1}(t-T_k) - V_j(t) &= \int_{[0,\,t]} Y(t-x)\mathrm{d}V_{j-1}(x) - V_j(t)\\ &= \int_{[0,\,t]} \big(Y(t-x) - (F*\nu)(t-x)\big)\mathrm{d}V_{j-1}(x)\\ &\quad + \Big(\int_{[0,\,t]} (F*\nu)(t-x)\mathrm{d}V_{j-1}(x) - V_j(t)\Big) =: A_1(t) + A_2(t).\end{aligned}$$

According to (7), $\lim_{t\to\infty} \frac{Y(t)-(F*\nu)(t)}{(t\log\log t)^{1/2}} = 0$ a.s., whence

$$\lim_{t\to\infty}\sup_{z\in[0,\,t]} \frac{|Y(z)-(F*\nu)(z)|}{(t\log\log t)^{1/2}} = 0 \quad \text{a.s.}$$

With this at hand,

$$\frac{|A_1(t)|}{t^{j-1/2}(\log\log t)^{1/2}} \leq \sup_{z\in[0,\,t]} \frac{|Y(z)-(F*\nu)(z)|}{(t\log\log t)^{1/2}} \cdot \frac{V_{j-1}(t)}{t^{j-1}} \longrightarrow 0 \quad \text{a.s.,} \quad t\to\infty,$$

using $\dfrac{V_{j-1}(t)}{t^{j-1}} \longrightarrow \dfrac{1}{(j-1)!\mu^{j-1}}, \quad t\to\infty.$

Further,

$$\begin{aligned}A_2(t) = (F*\nu*V_{j-1})(t) - V_j(t) &= \int_{[0,\,t]} (F*V_{j-1})(t-x)\mathrm{d}(\nu(x)-U(x))\\ &= \int_{[0,\,t]} (\nu(t-x)-U(t-x))\mathrm{d}(F*V_{j-1})(x).\end{aligned}$$

Recall that the distribution of η is arbitrary. Now we show that in the subsequent proof F can be replaced with an absolutely continuous distribution function that has a directly Riemann integrable (dRi) density.

Put $G(x) := 1 - \mathrm{e}^{-x}$ for $x \geq 0$. The function $H := F * G$ is absolutely continuous with the density $h(x) = \int_{[0,\,x]} \mathrm{e}^{-(x-y)}\mathrm{d}F(y)$ for $x \geq 0$. Since $x \mapsto \mathrm{e}^{-x}$ is dRi on $[0,\infty)$, so is h as a Lebesgue–Stieltjes convolution of a dRi function and a distribution function, see Lemma 6.2.1 (c) in [9]. Note that $H(x) \leq F(x)$ for $x \geq 0$. To show that we can work with H instead of F, it suffices to check that

$$\lim_{t\to\infty} \frac{(F*V_{j-1}*\nu)(t) - (H*V_{j-1}*\nu)(t)}{t^{j-1/2}} = 0 \quad \text{a.s.} \qquad (30)$$

and

$$\lim_{t\to\infty} \frac{(F * V_{j-1} * U)(t) - (H * V_{j-1} * U)(t)}{t^{j-1/2}} = 0. \tag{31}$$

For (31), write

$$\begin{aligned}(F * V_{j-1} * U)(t) - (H * V_{j-1} * U)(t) &= \int_{[0,t]} (1 - G(t-x))\mathrm{d}V_j(x) \\ = \int_{[0,t]} \mathrm{e}^{-(t-x)}\mathrm{d}V_j(x) \sim \Big(\int_0^\infty \mathrm{e}^{-y}\mathrm{d}y\Big) V_{j-1}(t) \\ = V_{j-1}(t) \sim \frac{t^{j-1}}{(j-1)!\mu^{j-1}}, \quad t\to\infty,\end{aligned}$$

where the asymptotic equalities are justified by (19) and Theorem 2 in [16]. This proves (31).

To prove (30), we use Lemma 5 with $K_1(t) = (F * V_{j-1})(t)$ and $K_2(t) = (H * V_{j-1})(t)$ for $t \geq 0$. Note that $K_2(t) = \mathbb{E}K_1(t-\theta)\,\mathbb{1}_{\{\theta\leq t\}}$, where θ is a random variable with the distribution function G, and that

$$\begin{aligned}0 \leq \int_0^t (K_1(y) - K_2(y))\mathrm{d}y &= \int_0^t (K_1(y) - \mathbb{E}K_1(y-\theta)\,\mathbb{1}_{\{\theta\leq y\}})\mathrm{d}y \\ &= \int_0^t K_1(y)\mathrm{d}y \cdot \mathrm{e}^{-t} + \mathbb{E}\int_{t-\theta}^t K_1(y)\mathrm{d}y\,\mathbb{1}_{\{\theta\leq t\}}.\end{aligned}$$

Using the Laplace transforms and (19), we have

$$K_1(t) \sim K_2(t) \sim V_{j-1}(t) \sim \frac{t^{j-1}}{(j-1)!\mu^{j-1}}, \quad t\to\infty.$$

Therefore,

$$\lim_{t\to\infty} \frac{\int_0^t K_1(y)\mathrm{d}y \cdot \mathrm{e}^{-t}}{K_1(t)} = 0$$

and, in view of monotonicity,

$$\frac{\mathbb{E}\theta K_1(t-\theta)\,\mathbb{1}_{\{\theta\leq t\}}}{K_1(t)} \leq \frac{\mathbb{E}\int_{t-\theta}^t K_1(y)\mathrm{d}y\,\mathbb{1}_{\{\theta\leq t\}}}{K_1(t)} \leq \mathbb{E}\theta = 1.$$

By Lebesgue's dominated convergence theorem

$$\int_0^t (K_1(t) - K_2(t))\mathrm{d}y \sim K_1(t) \sim \frac{t^{j-1}}{(j-1)!\mu^{j-1}}, \quad t\to\infty.$$

Thus, condition (27) holds with $\lambda = 2$. Consequently, (30) holds by (28) with $c = 1/2$.

As a consequence of (30) and (31), we can and do investigate $\hat{A}_2(t) = \int_{[0,t]}(\nu(t-x) - U(t-x))\mathrm{d}(H * V_{j-1})(x)$ in place of $A_2(t)$. By Lemma 3.1 in [12], there exists a standard Brownian motion $(W(t))_{t\geq 0}$ such that

$$\lim_{t\to\infty} \frac{\sup_{z\in[0,t]} |\nu(z) - U(z) - \sigma\mu^{-3/2}W(z)|}{(t\log\log t)^{1/2}} = 0 \quad \text{a.s.} \tag{32}$$

With this specific $(W(t))_{t\geq 0}$, write

$$\hat{A}_2(t) = \int_{[0,\,t]} \Big(\nu(t-x) - U(t-x) - \sigma\mu^{-3/2}W(t-x)\Big)\mathrm{d}(H * V_{j-1})(x) \\ + \sigma\mu^{-3/2}\int_{[0,\,t]} W(t-x)\mathrm{d}(H * V_{j-1})(x) =: B_1(t) + \sigma\mu^{-3/2}B_2(t).$$

Then, using (32) and (19), we have

$$\begin{aligned}|B_1(t)| &\leq \sup_{z\in[0,\,t]} |\nu(z) - U(z) - \sigma\mu^{-3/2}W(z)| \cdot (H * V_{j-1})(t) \\ &\leq \sup_{z\in[0,\,t]} |\nu(z) - U(z) - \sigma\mu^{-3/2}W(z)| \cdot V_{j-1}(t) \\ &= o((t^{2j-1}\log\log t)^{1/2}), \quad t\to\infty.\end{aligned}$$

We are left with showing that

$$C\bigg(\bigg(\frac{(j-1)!\mu^{j-1}B_2(t)}{(2(2j-1)^{-1}t^{2j-1}\log\log t)^{1/2}} \ :\ t > \mathrm{e}\bigg)\bigg) = [-1, 1] \quad \text{a.s.}$$

Since H is absolutely continuous with a dRi density, the function $H * V_{j-1}$ is almost everywhere differentiable with

$$(H * V_{j-1})'(x) = \int_{[0,\,x]} h(x-y)\mathrm{d}V_{j-1}(y) \quad \text{for almost every } x \geq 0.$$

Consequently,

$$\int_{[0,\,t]} W(t-x)\mathrm{d}(H * V_{j-1})(x) = \int_{[0,\,t]} W(t-x)(H * V_{j-1})'(x)\mathrm{d}x.$$

By Theorem 2 in [16], for $j \geq 2$,

$$\int_{[0,\,x]} h(x-y)\mathrm{d}V_{j-1}(y) \sim \int_0^\infty h(y)\mathrm{d}y \cdot \frac{x^{j-2}}{(j-2)!\mu^{j-1}} = \frac{x^{j-2}}{(j-2)!\mu^{j-1}}, \quad x\to\infty. \tag{33}$$

In particular, $(H * V_{j-1})'$ varies regularly at infinity with index $j-2$, and Proposition 2.4 in [10] yields

$$C\bigg(\bigg(\frac{(j-1)!\mu^{j-1}}{t^{j-1}} \frac{\int_{[0,\,t]} W(t-x)(H * V_{j-1})'(x)\mathrm{d}x}{(2(2j-1)^{-1}t\log\log t)^{1/2}} \ :\ t > \mathrm{e}\bigg)\bigg) = [-1, 1] \quad \text{a.s.}$$

Here, we have used that (33) entails

$$(H * V_{j-1})(x) \sim \frac{x^{j-1}}{(j-1)!\mu^{j-1}}, \quad x\to\infty,$$

see Proposition 1.5.8 in [2]. The proof of Theorem 2 is complete. □

Finally, we prove Proposition 1.

Proof of Proposition 1. Put $Z_j(t) = \sum_{k\geq 1}\big(Y^{(k)}_{j-1}(t-T_k) - V_{j-1}(t-T_k)\big)$ for $t \geq 0$. Relation (23) implies that there exist $t_0 > 0$ and $A > 0$ such that $a_{j-1}(t) \leq At^{2j-3}$ for all $t \geq t_0$. Using the same reasoning as in (26), we have

$$\mathbb{E}(Z_j(t))^2 = \int_{[0,\,t]} a_{j-1}(t-x)\mathrm{d}V(x) = O(t^{2j-2}), \quad t \to \infty. \tag{34}$$

By Markov's inequality and (34), for all $\varepsilon > 0$,

$$\sum_{n\geq 1} \mathbb{P}\left\{\frac{|Z_j(n^{3/2})|}{n^{(3/2)(j-1/2)}} > \varepsilon\right\} \leq \sum_{n\geq 1} \frac{\mathbb{E}(Z_j(n^{3/2}))^2}{\varepsilon^2 n^{3(j-1/2)}} < \infty.$$

Hence, by the Borel–Cantelli lemma,

$$\lim_{n\to\infty} \frac{Z_j(n^{3/2})}{n^{(3/2)(j-1/2)}} = 0 \quad \text{a.s.} \tag{35}$$

It remains to pass from an integer argument to a continuous argument. For any $t \geq 0$ there exists $n \in \mathbb{N}_0$ such that $t \in [n^{3/2}, (n+1)^{3/2})$. By monotonicity,

$$\frac{Z_j(t)}{t^{j-1/2}} \leq \frac{Z_j((n+1)^{3/2})}{n^{(3/2)(j-1/2)}} + \frac{\int_{[0,\,(n+1)^{3/2}]} V_{j-1}((n+1)^{3/2}-x)\mathrm{d}Y(x) - \int_{[0,\,n^{3/2}]} V_{j-1}(n^{3/2}-x)\mathrm{d}Y(x)}{n^{(3/2)(j-1/2)}}.$$

Relation (35) implies that the first summand on the right-hand side converges to 0 a.s. as $n \to \infty$. The second summand is equal to

$$\int_{(n^{3/2},(n+1)^{3/2}]} V_{j-1}((n+1)^{3/2}-x)\mathrm{d}Y(x) + \int_{[0,n^{3/2}]} (V_{j-1}((n+1)^{3/2}-x) - V_{j-1}(n^{3/2}-x))\mathrm{d}Y(x) =: X_{j,1}(n) + X_{j,2}(n).$$

By monotonicity, for $j \geq 2$, as $n \to \infty$, a.s.

$$\begin{aligned} X_{j,1}(n) &\leq V_{j-1}((n+1)^{3/2} - n^{3/2})(Y((n+1)^{3/2}) - Y(n^{3/2})) \\ &= O(n^{j/2+1}) = o(n^{(3/2)(j-1/2)}). \end{aligned}$$

Here, the penultimate equality is justified by the inequality $Y(t) \leq \nu(t)$ for $t \geq 0$, the strong law of large numbers for renewal processes $\lim_{n\to\infty} n^{-1}\nu(n) = \mu^{-1}$ a.s. and $V_{j-1}((n+1)^{3/2} - n^{3/2}) = O(n^{(j-1)/2})$ as $n \to \infty$, which holds true by (19).

Using (21), we infer

$$\begin{aligned} X_{j,2}(n) &\leq U((n+1)^{3/2} - n^{3/2})(V((n+1)^{3/2}))^{j-2}Y(n^{3/2}) \\ &= O(n^{(3/2)(j-2/3)}) = o(n^{(3/2)(j-1/2)}) \end{aligned}$$

a.s. as $n \to \infty$. The penultimate equality is secured by the elementary renewal theorem, the strong law of large numbers for renewal processes and the inequality $Y(t) \leq \nu(t)$ for $t \geq 0$.

We have shown that

$$\limsup_{t\to\infty} t^{-(j-1/2)} Z_j(t) \le 0 \quad \text{a.s.}$$

An analogous argument proves the converse inequality for the lower limit. The proof of Proposition 1 is complete. □

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