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A change of measures technique for compound mixed renewal processes with applications in Risk Theory

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Abstract Given a compound mixed renewal process S under a probability measure P, we provide a characterization of all progressively equivalent martingale probability measures Q on the domain of P, that convert S into a compound mixed Poisson process. This result extends earlier works of Delbaen and Haezendonck, Lyberopoulos and Macheras, and the authors, and enables us to find a wide class of price processes satisfying the condition of no free lunch with vanishing risk. Implications to the ruin problem and to the computation of premium calculation principles in an arbitrage-free insurance market are also discussed.

Keywords Compound mixed renewal processes, change of measures, martingales, premium calculation principles, ruin probability
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1 Introduction

Given a price process $U_{\mathbb{T}} := \{U_t\}_{t\in\mathbb{T}}$, where $\mathbb{T} := [0, T]$, $T > 0$, on a probability space (Ω, Σ, P), a basic method in Mathematical Finance is to replace the initial probability measure P by an equivalent one Q, which converts $U_{\mathbb{T}}$ into a martingale with respect to Q. The new probability measure, often called *risk-neutral* or *equivalent martingale measure* (written EMM for short), see Section 3 for the definition, is then used for pricing and hedging contingent claims (e.g., options, futures, etc.). Note that such a method for pricing contingent claims is originated from the field of Actuarial Science

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(see Delbaen and Schachermayer [9], pages 149–150 for more details). However, in contrast to the situation of the classical Black–Scholes option pricing formula, where the EMM is unique, in Actuarial Mathematics which is certainly not the case, as the insurance market is not, in general, complete (see, e.g., Sondermann [31], Section 4). Thus, if $U_{\mathbb{T}}$ represents the liabilities of an insurance company, then there exist infinitely many equivalent martingale measures for $U_{\mathbb{T}}$, so that pricing is directly linked with an attitude towards risk, see [8], pages 269–270, for more details. The latter led Delbaen and Haezendonck [8] to a positive answer to the problem of characterizing all those EMMs Q which preserve the structure of a given compound Poisson process under P, see [8], Proposition 2.2. The work of Delbaen and Haezendonck played a key role in understanding the interplay between financial and actuarial pricing of insurance (see Embrechts [10] for an overview), and has influenced the studies of many researchers (see [22], page 44, and the references therein for more details). Nevertheless, such a characterization of EMMs for $U_{\mathbb{T}}$ does not always provide a viable pricing system in actuarial practice, since it is not appropriate for describing inhomogeneous risk portfolios. For this reason, the work of Delbaen and Haezendonck [8] was generalized by Embrechts and Meister [11] and Lyberopoulos and Macheras [19, 20] to mixed Poisson risk models. However, since the (mixed) Poisson risk model presents some serious deficiencies as far as practical models are considered (see [22], page 44, and [33], page 226, and the references therein), it seems reasonable to investigate the existence of EMMs for the price process $U_{\mathbb{T}}$ in the more general mixed renewal risk model.

In [33], Corollary 4.8, a characterization of all progressively equivalent probability measures that convert a compound mixed renewal process (CMRP for short) into a compound mixed Poisson process (CMPP for short) was proven. In Section 3, relying on the above result, we provide a characterization of all progressively EMMs Q for a canonical price process that convert a CMRP under P into a CMPP under Q, see Theorem 1. This theorem generalizes corresponding results of [19], Proposition 5.1(ii), and [22], Proposition 4.2. A first consequence of Theorem 1 is Theorem 2, where we find out a wide class of *canonical* price processes, inducing a corresponding class of EMMs, satisfying the condition of no free lunch with vanishing risk (written (NFLVR) for short), connecting in this way our results with this basic notion of Mathematical Finance.

Another implication of Theorem 1 concerning the ruin problem is discussed in Section 4, where for a given reserve process $R^u(\Theta)$ induced by the initial reserve u, the stochastic premium intensity $c(\Theta)$ and the aggregate claims process S (see Definition 2), we characterize all those progressively equivalent to P ℓ-martingale measures Q that convert a CMRP under P into a CMPP under Q in such a way that ruin for $R^u(\Theta)$ occurs Q-a.s., see Theorem 3. In Section 5, we discuss some implications of our results to the pricing of actuarial risks (premium calculation principles) in an arbitrage-free insurance market. In Section 6 we present some concrete examples demonstrating how to construct mixed premium calculation principles (see Section 5 for the definition) in an insurance market possessing the property of (NFLVR) and how to obtain explicit formulas for their corresponding ruin probabilities. Finally, in the Appendix a list of symbols is presented.

2 Preliminaries

$\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the natural, the rational and the real numbers, respectively, while $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. If $d \in \mathbb{N}$ then $\mathbb{R}^d$ denotes the Euclidean space of dimension d. For a map $f : A \longrightarrow E$ and for a nonempty set $B \subseteq A$ write $f\upharpoonright B$ for the restriction of f to B and $\mathbb{1}_B$ for the indicator function of the set B.

Throughout this paper, unless stated otherwise, (Ω, Σ, P) *is a fixed but arbitrary probability space and* $\Theta : \Omega \longrightarrow D \subseteq \mathbb{R}^d$ $(d \in \mathbb{N})$ *is a* d*-dimensional random vector.* By $\mathcal{L}^\ell(P)$ we denote the family of all Σ-measurable real-valued functions f on Ω such that $\int |f|^\ell \, dP < \infty$ $(\ell \in \{1, 2\})$. For any Hausdorff topology $\mathfrak{T}$ over Ω, by $\mathfrak{B}(\Omega)$ is denoted the Borel σ-algebra on Ω, i.e., the σ-algebra generated by $\mathfrak{T}$, while $\mathfrak{B} := \mathfrak{B}(\mathbb{R})$ and $\mathfrak{B}_d := \mathfrak{B}(\mathbb{R}^d)$, where $d \in \mathbb{N}$, stand for the Borel σ-algebras of subsets of $\mathbb{R}$ and $\mathbb{R}^d$, respectively, generated by the Euclidean topology $\mathfrak{E}$ over $\mathbb{R}$ and by the product topology $\mathfrak{E}^d$ over $\mathbb{R}^d$, respectively. For any $J \subseteq \mathbb{R}^d$ denote by $\mathfrak{B}(J)$ the σ-algebra on J generated by the topology $\mathfrak{E}^d \cap J := \{J \cap G : G \in \mathfrak{E}^d\}$. In particular, write $\mathfrak{B}(0, \infty) := \mathfrak{B}((0, \infty))$, for simplicity. Our measure-theoretic terminology is standard and generally follows [4]. For the definitions of real-valued random variables and random variables we refer to [4], page 308. We apply the notation $P_X := P_X(\theta) := \mathbf{K}(\theta)$ to mean that X is distributed according to the law $\mathbf{K}(\theta)$, where $\theta \in D \subseteq \mathbb{R}^d$ is the parameter of the distribution. Notation $\mathbf{Ga}(b, a)$, where $a, b \in (0, \infty)$, stands for the law of gamma distribution (cf., e.g., [30], page 180). In particular, $\mathbf{Ga}(b, 1) = \mathbf{Exp}(b)$ stands for the law of exponential distribution. For the unexplained terminology of Probability and Risk Theory we refer to [30].

Given a random variable X, a **conditional distribution of** X **over** Θ is a $\sigma(\Theta)$-$\mathfrak{B}$-Markov kernel (see [3], Definition 36.1 for the definition) denoted by $P_{X|\Theta} := P_{X|\sigma(\Theta)}$ and satisfying for each $B \in \mathfrak{B}$ the equality $P_{X|\Theta}(\bullet, B) = P(X^{-1}(B) \mid \sigma(\Theta))(\bullet)$ $P\upharpoonright\sigma(\Theta)$-almost surely (written a.s. for short). Clearly, for every $\mathfrak{B}(\mathbb{R}^d)$-$\mathfrak{B}$-Markov kernel k, the map $K(\Theta)$ from $\Omega \times \mathfrak{B}$ into $[0, 1]$ defined by

$$K(\Theta)(\omega, B) := \big(k(\bullet, B) \circ \Theta\big)(\omega) \quad \text{for any} \quad (\omega, B) \in \Omega \times \mathfrak{B}$$

is a $\sigma(\Theta)$-$\mathfrak{B}$-Markov kernel. Then for $\theta = \Theta(\omega)$ with $\omega \in \Omega$ the probability measures $k(\theta, \bullet)$ are distributions on $\mathfrak{B}$ and so we may write $\mathbf{K}(\theta)(\bullet)$ instead of $k(\theta, \bullet)$. Consequently, in this case $K(\Theta)$ will be denoted by $\mathbf{K}(\Theta)$.

For any real-valued random variables X, Y on Ω we say that $P_{X|\Theta}$ and $P_{Y|\Theta}$ are $P\upharpoonright\sigma(\Theta)$-equivalent and we write $P_{X|\Theta} = P_{Y|\Theta}$ $P\upharpoonright\sigma(\Theta)$-a.s., if there exists a P-null set $M \in \sigma(\Theta)$ such that for any $\omega \notin M$ and $B \in \mathfrak{B}$ the equality $P_{X|\Theta}(\omega, B) = P_{Y|\Theta}(\omega, B)$ holds true.

A sequence $\{V_n\}_{n\in\mathbb{N}}$ of real-valued random variables on Ω is said to be:

- P-**conditionally (stochastically) independent over** $\sigma(\Theta)$ if, for each $n \in \mathbb{N}$ with $n \geq 2$, we have

$$P\left(\bigcap_{j=1}^{n} \{V_{i_j} \leq v_{i_j}\} \mid \sigma(\Theta)\right) = \prod_{j=1}^{n} P\big(\{V_{i_j} \leq v_{i_j}\} \mid \sigma(\Theta)\big) \quad P\upharpoonright\sigma(\Theta)\text{-a.s.,}$$

  whenever $i_1, \ldots, i_n$ are distinct members of $I \subseteq \mathbb{N}$ and $(v_{i_1}, \ldots, v_{i_n}) \in \mathbb{R}^n$;

- **P-conditionally identically distributed over** $\sigma(\Theta)$ if

$$P\big(F \cap V_k^{-1}[B]\big) = P\big(F \cap V_m^{-1}[B]\big)$$

whenever $k, m \in \mathbb{N}$, $F \in \sigma(\Theta)$ and $B \in \mathfrak{B}$.

We say that the process $\{V_n\}_{n\in\mathbb{N}}$ is **P-conditionally (stochastically) independent or identically distributed given** Θ, if it is conditionally independent or identically distributed over the σ-algebra $\sigma(\Theta)$.

Throughout what follows we write "conditionally" in the place of "conditionally given Θ" whenever conditioning refers to Θ.

For the definitions of a *counting* (or *claim number*) *process* $N := \{N_t\}_{t\in\mathbb{R}_+}$, an *arrival process* $T := \{T_n\}_{n\in\mathbb{N}_0}$ induced by N, an *interarrival process* $W := \{W\}_{n\in\mathbb{N}}$ induced by T, a *claim size process* $X := \{X_n\}_{n\in\mathbb{N}}$ and an *aggregate claims process* $S := \{S_t\}_{t\in\mathbb{R}_+}$ induced by N and X, we refer to [30]. We assume that $P(\{X_n > 0\}) = 1$ for all $n \in \mathbb{N}$. Recall that a pair (N, X) is called a **risk process**, if N is a counting process, X is P-i.i.d. and the processes N and X are P-mutually independent (see [30], page 127).

Recall that a counting process N is said to be a **P-mixed renewal process with mixing parameter Θ and interarrival time conditional distribution $\mathbf{K}(\Theta)$** (written P-MRP($\mathbf{K}(\Theta)$) for short), if the interarrival process W is P-conditionally independent and for all $n \in \mathbb{N}$ the condition $P_{W_n|\Theta} = \mathbf{K}(\Theta)$ $P\restriction\sigma(\Theta)$-a.s. is valid (see [21], Definition 3.1). In particular, if the distribution of Θ is degenerate at some point $\theta_0 \in D$, then the counting process N becomes a *P-renewal process with interarrival time distribution* $\mathbf{K}(\theta_0)$ (written P-RP($\mathbf{K}(\theta_0)$) for short). If N is a P-MRP($\mathbf{K}(\Theta)$) then according to [33], Corollary 3.5(ii), it has zero probability of explosion, i.e., $P(\{\sup_{n\in\mathbb{N}_0} T_n < \infty\}) = 0$.

Accordingly, an aggregate claims process S induced by a P-risk process (N, X) such that N is a P-MRP($\mathbf{K}(\Theta)$) is called a **compound mixed renewal process with parameters $\mathbf{K}(\Theta)$ and P_{X_1}** (P-CMRP($\mathbf{K}(\Theta), P_{X_1}$) for short). If Θ is a real-valued random variable and $\mathbf{K}(\Theta) = \mathbf{Exp}(\Theta)$, we say that S is a **compound mixed Poisson process with parameters Θ and P_{X_1}** and write P-CMPP(Θ, P_{X_1}). In particular, if the distribution of Θ is degenerate at $\theta_0 \in D$, then S is called a **compound renewal process with parameters $\mathbf{K}(\theta_0)$ and P_{X_1}** (P-CRP($\mathbf{K}(\theta_0), P_{X_1}$) for short). In the special case $\theta_0 > 0$ and $\mathbf{K}(\theta_0) = \mathbf{Exp}(\theta_0)$ we say that S is a **compound Poisson process with parameters θ_0 and P_{X_1}** and write P-CPP(θ_0, P_{X_1}).

Throughout what follows we denote again by $\mathbf{K}(\Theta)$ and $\mathbf{K}(\theta)$ the conditional distribution function and the distribution function induced by the conditional probability distribution $\mathbf{K}(\Theta)$ and the probability distribution $\mathbf{K}(\theta)$, respectively.

Since conditioning is involved in the definition of (compound) mixed renewal processes, it is expected that regular conditional probabilities will play a fundamental role in their analysis. To this purpose, recall that if (Z, H, R) is a probability space, then a family $\{P_z\}_{z\in Z}$ of probability measures on Σ is called a **regular conditional probability** (rcp for short) of P over R if for any fixed $E \in \Sigma$ the map $Z \ni z \mapsto P_z(E)$ is H-measurable, and $\int P_z(E)\,R(dz) = P(E)$ for every $E \in \Sigma$. If $f : \Omega \to Z$ is an inverse-measure-preserving function (i.e., $P(f^{-1}(B)) = R(B)$ for each $B \in H$), an rcp $\{P_z\}_{z\in Z}$ of P over R is called **consistent** with f if, for each $B \in H$, the equality

$P_z(f^{-1}(B)) = 1$ holds for R-almost every $z \in B$ (cf., e.g., [33], Definition 3.3). We say that an rcp $\{P_z\}_{z\in Z}$ of P over R consistent with f is **essentially unique**, if for any other rcp $\{\widetilde{P}_z\}_{z\in Z}$ of P over R consistent with f there exists a R-null set $M \in H$ such that for any $z \notin M$ the equality $P_z = \widetilde{P}_z$ holds true.

From now on $(Z, H, R) := (D, \mathfrak{B}(D), P_\Theta)$ *and the family* $\{P_\theta\}_{\theta\in D}$ *is an rcp of* P *over* P_Θ *consistent with* Θ.

Let $\mathbb{T} \subseteq \mathbb{R}_+$. For a process $Y_\mathbb{T} := \{Y_t\}_{t\in\mathbb{T}}$ denote by $\mathcal{F}^Y_\mathbb{T} := \{\mathcal{F}^Y_t\}_{t\in\mathbb{T}}$ the canonical filtration of $Y_\mathbb{T}$. For $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{T} = \mathbb{N}$ we simply write $\mathcal{F}^Y$ instead of $\mathcal{F}^Y_{\mathbb{R}_+}$ or $\mathcal{F}^Y_{\mathbb{N}}$, respectively. Also, we write $\mathcal{F} := \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$, where $\mathcal{F}_t := \sigma(\mathcal{F}^S_t \cup \sigma(\Theta))$, for the canonical filtration of S and Θ, $\mathcal{F}^S_\infty := \sigma(\bigcup_{t\in\mathbb{R}_+} \mathcal{F}^S_t)$ and $\mathcal{F}_\infty := \sigma(\mathcal{F}^S_\infty \cup \sigma(\Theta))$.

Let Q be a probability measures on (Ω, Σ). We say that P and Q are **progressively equivalent**, if P and Q are equivalent (in the sense of absolute continuity) on $\mathcal{F}_t$ (in symbols $Q\restriction\mathcal{F}_t \sim P\restriction\mathcal{F}_t$) for any $t \in \mathbb{R}_+$, see [33], Definition 3.1. If P and Q are equivalent on Σ we write $P \sim Q$.

The following conditions will be useful for our investigations:

(a1) The processes W and X are P-conditionally mutually independent.

(a2) The random vector Θ and the process X are P-(unconditionally) independent.

Next, whenever condition (a1) or (a2) holds true we shall write that the quadruplet (P, W, X, Θ) *or (if no confusion arises) the probability measure* P *satisfies (a1) or (a2), respectively.*

Notations 1. Denote by $\mathfrak{M}^k(D)$, $k \in \mathbb{N}$, the class of all $\mathfrak{B}(D)$-$\mathfrak{B}_k$-measurable functions on D. In the special case $k = 1$, write $\mathfrak{M}(D) := \mathfrak{M}^1(D)$ and $\mathfrak{M}_+(D)$ for the class of all positive elements of $\mathfrak{M}(D)$. Fix an arbitrary $\ell \in \{1, 2\}$ and $\rho \in \mathfrak{M}^k(D)$.

(a) The class of all real-valued $\mathfrak{B}((0,\infty)\times D)$-measurable functions β on $(0,\infty)\times D$, defined by $\beta(x, \theta) := \gamma(x) + \alpha(\theta)$ for any $(x, \theta) \in (0,\infty)\times D$, where $\alpha \in \mathfrak{M}(D)$, and γ is a real-valued $\mathfrak{B}(0,\infty)$-measurable function satisfying conditions $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$ and $\mathbb{E}_P[X_1^\ell \cdot e^{\gamma(X_1)}] < \infty$ (resp. $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$), will be denoted by $\mathcal{F}^\ell_{P,\Theta} := \mathcal{F}^\ell_{P,\Theta,X_1}$ (resp. $\mathcal{F}_{P,\Theta} := \mathcal{F}_{P,\Theta,X_1}$).

(b) The class of all $\xi \in \mathfrak{M}(D)$ such that $P_\Theta(\{\xi > 0\}) = 1$ and $\mathbb{E}_P[\xi(\Theta)] = 1$ is denoted by $\mathcal{R}_+(D) := \mathcal{R}_+(D, \mathfrak{B}(D), P_\Theta)$.

(c) The class of all probability measures Q on Σ, which satisfy conditions (a1) and (a2), are progressively equivalent to P, and such that S is a Q-CMRP$(\mathbf{\Lambda}(\rho(\Theta)), Q_{X_1})$, will be denoted by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta)),P,X_1}$. The class of all elements Q of $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ with $\mathbb{E}_Q[X_1^\ell] < \infty$ will be denoted by $\mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))} := \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta)),P,X_1}$. In the special case $d = k$ and $\rho := id_D$ we write $\mathcal{M}_{S,\mathbf{\Lambda}(\Theta)} := \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ and $\mathcal{M}^\ell_{S,\mathbf{\Lambda}(\Theta)} := \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))}$ for simplicity.

(d) Let $\theta \in D$. Denote by $\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ the class of all probability measures Q_θ on Σ, such that $Q_\theta\restriction\mathcal{F}_t \sim P_\theta\restriction\mathcal{F}_t$ for any $t \in \mathbb{R}_+$ and S is a Q_θ-CRP$(\mathbf{\Lambda}(\rho(\theta)), (Q_\theta)_{X_1})$. The class of all $Q_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ with $\mathbb{E}_{Q_\theta}[X_1^\ell] < \infty$ is denoted by $\mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\theta))}$.

Remarks 1. **(a)** Clearly inclusions $\mathcal{F}^2_{P,\Theta} \subseteq \mathcal{F}^1_{P,\Theta} \subseteq \mathcal{F}_{P,\Theta}$ and $\mathcal{M}^2_{S,\mathbf{\Lambda}(\rho(\Theta))} \subseteq \mathcal{M}^1_{S,\mathbf{\Lambda}(\rho(\Theta))} \subseteq \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ hold true, but simple examples show that $\mathcal{F}^2_{P,\Theta} \neq \mathcal{F}^1_{P,\Theta} \neq \mathcal{F}_{P,\Theta}$ and $\mathcal{M}^2_{S,\mathbf{\Lambda}(\rho(\Theta))} \neq \mathcal{M}^1_{S,\mathbf{\Lambda}(\rho(\Theta))} \neq \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ in general.

(b) For $\ell \in \{1,2\}$ the following statements are equivalent:

(i) $P \in \mathcal{M}^{\ell}_{S,\mathbf{K}(\Theta)}$ with $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$;

(ii) there exists a P_Θ-null set $W_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}^{\ell}_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ and $\mathbb{E}_{P_\theta}[W_1] < \infty$ for any $\theta \notin W_P$.

In fact, since X_1 and Θ are (unconditionally) independent by (a2), we have $X_1 \in \mathcal{L}^{\ell}(P)$ if and only if $X_1 \in \mathcal{L}^{\ell}(P_\theta)$ for all $\theta \in D$, while by [33], Proposition 3.4, we have $P \in \mathcal{M}_{S,\mathbf{K}(\Theta)}$ if and only if there exists a P_Θ-null set $L_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ for all $\theta \notin L_P$. Furthermore, by [18], Lemma 3.5(i), we get that $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$ if and only if there exists a P_Θ-null set $D_P \in \mathfrak{B}(D)$ such that $\mathbb{E}_{P_\theta}[W_1] < \infty$ for all $\theta \notin D_P$. Putting $W_P := L_P \cup D_P \in \mathfrak{B}(D)$ we get the desired equivalence of (i) and (ii).

Recall that, for given $\mathbb{T} \subseteq \mathbb{R}_+$ a **martingale in $\mathcal{L}^{\ell}(P)$ adapted to the filtration** $\mathcal{Z}_{\mathbb{T}} := \{\mathcal{Z}_t\}_{t\in\mathbb{T}}$, or else a **$\mathcal{Z}_{\mathbb{T}}$-martingale in $\mathcal{L}^{\ell}(P)$**, is a family $Z_{\mathbb{T}} := \{Z_t\}_{t\in\mathbb{T}}$ of random variables in $\mathcal{L}^{\ell}(P)$ such that Z_t is $\mathcal{Z}_t$-measurable for each $t \in \mathbb{T}$, and whenever $u \le t$ in $\mathbb{T}$ and $E \in \mathcal{Z}_u$ then $\int_E Z_u dP = \int_E Z_t dP$. For $\mathcal{Z}_{\mathbb{R}_+} = \mathcal{F}$ we simply write that Z is a martingale in $\mathcal{L}^{\ell}(P)$. A $\mathcal{Z}_{\mathbb{T}}$-martingale $\{Z_t\}_{t\in\mathbb{T}}$ in $\mathcal{L}^{\ell}(P)$ is **P-a.s. positive**, if Z_t is P-a.s. positive for each $t \in \mathbb{T}$.

Given $\Omega := (0,\infty)^{\mathbb{N}} \times (0,\infty)^{\mathbb{N}} \times D$ and $\Sigma := \mathfrak{B}(\Omega) = \mathfrak{B}(0,\infty)_{\mathbb{N}} \otimes \mathfrak{B}(0,\infty)_{\mathbb{N}} \otimes \mathfrak{B}(D)$, let μ be a probability measure on $\mathfrak{B}(D)$, and let $P_n(\theta) := \mathbf{K}(\theta)$ and $R_n := R$ be probability measures on $\mathfrak{B}(0,\infty)$ for any $n \in \mathbb{N}$ and fixed $\theta \in D$. Assume that for any fixed $B \in \mathfrak{B}(0,\infty)$ the function $\theta \mapsto \mathbf{K}(\theta)(B)$ is $\mathfrak{B}(D)$-measurable. It then follows by [33], Proposition 4.1, that there exist:

- a family $\{P_\theta\}_{\theta\in D}$ of probability measures on Σ and a probability measure P on Σ such that $\{P_\theta\}_{\theta\in D}$ is an rcp of P over μ consistent with $\Theta := \pi_D$, where π_D is the canonical projection from Ω onto D, and $P_\Theta = \mu$;
- an interarrival process W such that $(P_\theta)_{W_n} = \mathbf{K}(\theta)$ for all $n \in \mathbb{N}$;
- a claim size process X such that $P_{X_n} = R$ for all $n \in \mathbb{N}$, and
- a counting process N and an aggregate process S induced by the risk process (N, X), such that P is an element of $\mathcal{M}_{S,\mathbf{K}(\Theta)}$.

Throughout what follows, unless stated otherwise, (Ω, Σ, P), N, W, X, S, Θ *and* $\{P_\theta\}_{\theta\in D}$ *are as above,* $\Sigma = \mathcal{F}_\infty$, $\ell \in \{1,2\}$ *and* $S^{(\gamma)}_t := \sum_{j=1}^{N_t} \gamma(X_j)$ *for any* $t \in \mathbb{R}_+$ *and for any real-valued* $\mathfrak{B}(0,\infty)$*-measurable function* γ *satisfying condition* $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$.

For the validity of the equality $\Sigma = \mathcal{F}_\infty$ see [33], Remark 4.2.

The following result of [33], concerning a characterization of all progressively equivalent probability measures that convert a CMRP into a CMPP, serves as a useful basic tool for our results.

Proposition 1 (See [33], Corollary 4.8 and Remark 4.9(c))**.** *For given* $P \in \mathcal{M}^{\ell}_{S,\mathbf{K}(\Theta)}$ *with* $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$ *the following hold true:*

(i) *for every pair* $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ *there exists an essentially unique pair* $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}_+(D)$, *where* e^{γ} *and* ξ *are the Radon–Nikodým derivatives of* Q_{X_1} *with respect to* P_{X_1} *and of* Q_{Θ} *with respect to* P_{Θ}, *respectively, such that*

$$\alpha(\Theta) = \ln \rho(\Theta) + \ln \mathbb{E}_P[W_1 \mid \Theta] \quad P\restriction\sigma(\Theta)\text{-a.s.}, \tag{$*$}$$

and

$$Q(A) = \int_A M^{(\beta)}_t(\Theta)\, dP \quad \text{for all } 0 \le s \le t \text{ and } A \in \mathcal{F}_s, \tag{RPM_ξ}$$

where

$$M^{(\beta)}_t(\Theta) := \xi(\Theta) \cdot \frac{e^{S^{(\gamma)}_t - \rho(\Theta)\cdot J_t}}{1 - \mathbf{K}(\Theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\Theta))}{d\mathbf{K}(\Theta)}(W_j),$$

with $J_t := t - T_{N_t}$, *and the family* $M^{(\beta)}(\Theta) := \{M^{(\beta)}_t(\Theta)\}_{t\in\mathbb{R}_+}$ *is a* P*-a.s positive martingale in* $\mathcal{L}^1(P)$;

(ii) *conversely, for every pair* $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}_+(D)$ *there exists a unique pair* $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ *determined by conditions* $(*)$ *and* (RPM_ξ), *such that* e^{γ} *and* ξ *are the Radon–Nikodým derivatives of* Q_{X_1} *with respect to* P_{X_1} *and of* Q_{Θ} *with respect to* P_{Θ}, *respectively;*

(iii) *in both cases (i) and (ii), there exists an essentially unique rcp* $\{Q_\theta\}_{\theta\in D}$ *of* Q *over* Q_Θ *consistent with* Θ *and a* P_Θ*-null set* $\widetilde{L}_{**} \in \mathfrak{B}(D)$, *containing the* P_Θ*-null set* $W_P \in \mathfrak{B}(D)$ *appearing in Remark 1(b), satisfying for any* $\theta \notin \widetilde{L}_{**}$ *conditions* $Q_\theta \in \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\theta))}$,

$$\rho(\theta) = \frac{e^{\alpha(\theta)}}{\mathbb{E}_{P_\theta}[W_1]}, \tag{$\widetilde{*}$}$$

and

$$Q_\theta(A) = \int_A \widetilde{M}^{(\beta)}_t(\theta)\, dP_\theta \quad \text{for all } 0 \le s \le t \text{ and } A \in \mathcal{F}_s, \tag{RPM_θ}$$

where

$$\widetilde{M}^{(\beta)}_t(\theta) := \frac{e^{S^{(\gamma)}_t - \rho(\theta)\cdot J_t}}{1 - \mathbf{K}(\theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\theta))}{d\mathbf{K}(\theta)}(W_j),$$

and the family $\widetilde{M}^{(\beta)}(\theta) := \{\widetilde{M}^{(\beta)}_t(\theta)\}_{t\in\mathbb{R}_+}$ *is a* P_θ*-a.s. positive martingale in* $\mathcal{L}^1(P_\theta)$, *where* $\widetilde{L}_{**}$ *is the* P_Θ*-null sets appearing in [33], Corollary 4.8.*

3 A characterization of progressively equivalent martingale measures for compound mixed renewal processes

In this section we find out a wide class of *canonical* price processes inducing a corresponding class of progressively EMMs and satisfying the condition of *no free* *lunch with vanishing risk* (written (NFLVR) for short) (see [9], Definition 8.1.2), connecting in this way our results with this basic notion in mathematical finance.

In order to present the results of this section we recall the following notions. For a given real-valued process $Y := \{Y_t\}_{t\in\mathbb{R}_+}$ on (Ω, Σ) a probability measure Q on Σ is called an ℓ-**martingale measure** for Y, if Y is a martingale in $\mathcal{L}^\ell(Q)$. We will say that Y satisfies condition (PEMM) if there exists a 2-martingale measure Q for Y, which is progressively equivalent to P. Moreover, let $T > 0$, $\mathbb{T} := [0, T]$, $\mathcal{F}_{\mathbb{T}} := \{\mathcal{F}_t\}_{t\in\mathbb{T}}$, $Q_T := Q\restriction\mathcal{F}_T$, $P_T := P\restriction\mathcal{F}_T$ and $Y_{\mathbb{T}} := \{Y_t\}_{t\in\mathbb{T}}$. We will say that the process $Y_{\mathbb{T}}$ satisfies condition (EMM) if there exists a 2-martingale measure Q_T for $Y_{\mathbb{T}}$, which is equivalent to P_T.

Notations 2. **(a)** For given $\beta \in \mathcal{F}^\ell_{P,\Theta}$, denote by $\mathcal{R}^{*,\ell}_+(D) := \mathcal{R}^{*,\ell}_{+,\beta}(D)$ the class of all functions $\xi \in \mathcal{R}_+(D)$ such that

$$\xi(\Theta)\cdot\left(\frac{e^{\alpha(\Theta)}}{\mathbb{E}_P[W_1 \mid \Theta]}\right)^\ell \in \mathcal{L}^1(P),$$

under the assumption $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$.

(b) Denote by $\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ the class of all measures $Q \in \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))}$ satisfying condition

$$\left(1/\mathbb{E}_Q[W_1 \mid \Theta]\right)^\ell \in \mathcal{L}^1(Q),$$

under the assumption $Q(\{\mathbb{E}_Q[W_1 \mid \Theta] < \infty\}) = 1$.

(c) For arbitrary $\theta \in D$ denote by $\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\theta))}$ the class of all probability measures $Q_\theta \in \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\theta))}$ such that $(1/\mathbb{E}_{Q_\bullet}[W_1])^\ell \in \mathcal{L}^1(Q_\Theta)$ under the assumption $Q_\Theta(\{\mathbb{E}_{Q_\bullet}[W_1] < \infty\}) = 1$.

Remark 2. **(a)** Inclusions $\mathcal{R}^{*,2}_+(D) \subsetneqq \mathcal{R}^{*,1}_+(D) \subsetneqq \mathcal{R}_+(D)$ hold true.

Clearly $\mathcal{R}^{*,2}_+(D) \subseteq \mathcal{R}^{*,1}_+(D) \subseteq \mathcal{R}_+(D)$. Let $D := (0, \infty)$ and $P \in \mathcal{M}^\ell_{S,\mathbf{Exp}(\Theta)}$ and assume that P_Θ is absolutely continuous with respect to the Lebesgue measure λ on $\mathfrak{B}$ restricted to $\mathfrak{B}(D)$. Let f_Θ be the corresponding probability density functions of Θ with respect to P. Consider the real-valued functions $\beta(x, \theta) := \theta$ for all $(x, \theta) \in D\times D$ and $\xi(\theta) := \frac{a\cdot e^{-a\cdot\theta}}{f_\Theta(\theta)}$ for each $\theta \in D$, where $a > 0$ is a real constant. A straightforward computation yields $\beta \in \mathcal{F}^\ell_{P,\Theta}$ and $\xi \in \mathcal{R}_+(D)$. However, for $a \in (0, 1]$ we have $\xi \notin \mathcal{R}^{*,1}_+(D)$, while for $a \in (1, 2]$ we have $\xi \in \mathcal{R}^{*,1}_+(D)\setminus\mathcal{R}^{*,2}_+(D)$; hence the required inclusions follow.

(b) Inclusion $\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))} \subsetneqq \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))}$ holds true.

Clearly, inclusion $\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))} \subseteq \mathcal{M}^\ell_{S,\mathbf{\Lambda}(\rho(\Theta))}$ holds. Take $D := (0, \infty)$ and assume that $\rho(\Theta) = e^\Theta$ and $Q \in \mathcal{M}^\ell_{S,\mathbf{Exp}(\rho(\Theta))}$ with $Q_\Theta = \mathbf{Exp}(\eta)$, where $\eta < \ell$ is a positive constant. It then follows that $\mathbb{E}_Q[e^{\ell\cdot\Theta}] = \infty$, implying that $Q \notin \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$.

Remark 3. The following statements are equivalent:

(i) $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$;

(ii) $P_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ for all $\theta \notin W_P$, where $W_P \in \mathfrak{B}(D)$ is the P_Θ-null set appearing in Remark 1(b).

In fact, by Remark 1(b) there exists a P_Θ-null set $W_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}^{\ell}_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ and $\mathbb{E}_{P_\theta}[W_1] < \infty$ for all $\theta \notin W_P$. Taking now into account the fact

$$\int \left(\frac{1}{\mathbb{E}_P[W_1 \mid \Theta]}\right)^{\ell} dP = \int \left(\frac{1}{\mathbb{E}_{P_\theta}[W_1]}\right)^{\ell} P_\Theta(d\theta),$$

which is a consequence of [18], Lemma 3.5, we get the claimed equivalence.

In the next theorems we provide a characterization of all progressively equivalent martingale measures Q on Σ converting a CMRP under P into a CMPP under Q, in such a way that they are associated to stochastic processes satisfying condition (NFLVR).

Theorem 1. *If $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$ the following statements hold true:*

(i) for every pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ there exists an essentially unique pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$, where e^γ and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively, satisfying conditions $(*)$ and (RPM_ξ);

(ii) conversely, for every pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$ there exists a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), such that e^γ and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively;

(iii) in both cases (i) and (ii), there exists an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ satisfying for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\widetilde{*})$ and (RPM_θ), such that Q_θ is an ℓ-martingale measure for the process $V(\theta) := V(\theta,\beta) := \{V_t(\theta,\beta)\}_{t\in\mathbb{R}_+} =: \{V_t(\theta)\}_{t\in\mathbb{R}_+}$ with $V_t(\theta) := S_t - t \cdot \frac{\mathbb{E}_{P_\theta}[X_1 \cdot e^{\beta(X_1,\theta)}]}{\mathbb{E}_{P_\theta}[W_1]}$ for any $t \in \mathbb{R}_+$, where $\widetilde{L}_{**}$ is the P_Θ-null set appearing in Proposition 1.

(iv) The pair (ρ, Q) is an element of $\mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ if and only if $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ and Q is an ℓ-martingale measure for the process $V(\Theta) := V(\Theta,\beta) := \{V_t(\Theta,\beta)\}_{t\in\mathbb{R}_+} =: \{V_t(\Theta)\}_{t\in\mathbb{R}_+}$ with $V_t(\Theta) := S_t - t \cdot \frac{\mathbb{E}_P[X_1 \cdot e^{\beta(X_1,\Theta)} \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]}$ for every $t \in \mathbb{R}_+$, where $\beta \in \mathcal{F}^{\ell}_{P,\Theta}$ is the function appearing in Proposition 1(i);

(v) the measure Q appearing in both statements (i) and (ii) is an ℓ-martingale measure for $V(\Theta, \beta)$.

Proof. Ad (i): Since $\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))} \subseteq \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, according to Proposition 1(i) there exists an essentially unique pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}_+(D)$, where e^γ and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively, satisfying conditions $(*)$ and (RPM_ξ). Our assumption $Q \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, along with condition $(*)$, yields $\rho(\Theta) \in \mathcal{L}^{\ell}(Q)$, implying $\mathbb{E}_P[\xi(\Theta) \cdot (e^{\alpha(\Theta)}/\mathbb{E}_P[W_1 \mid \Theta])^{\ell}] = \mathbb{E}_Q[(\rho(\Theta))^{\ell}] < \infty$; hence $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$.

Ad (ii): Since $\mathcal{F}^{\ell}_{P,\Theta} \subseteq \mathcal{F}_{P,\Theta}$ and $\mathcal{R}^{*,\ell}_{+}(D) \subseteq \mathcal{R}_{+}(D)$, it follows by Proposition 1(ii) that there exists a unique pair $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_{ξ}), such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_{Θ} with respect to P_{Θ}, respectively, implying, along with the assumptions of (ii), that $\xi(\Theta) \cdot (e^{\alpha(\Theta)}/\mathbb{E}_P[W_1 \mid \Theta])^{\ell} \in \mathcal{L}^1(P)$; hence $1/\mathbb{E}_Q[W_1 \mid \Theta] = \rho(\Theta) \in \mathcal{L}^{\ell}(Q)$. Thus, $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$.

Ad (iii): In both cases (i) and (ii), by Proposition 1(iii), there exists an essentially unique rcp $\{Q_{\theta}\}_{\theta \in D}$ of Q over Q_{Θ} consistent with Θ and a P_{Θ}-null set $\widetilde{L}_{**}$, such that for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_{\theta} \in \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\circledast)$ and (RPM_{θ}) hold true. But since $Q \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, it follows by Remark 3 that $Q_{\theta} \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ for any $\theta \notin \widetilde{L}_{**}$; hence, we can apply [22], Proposition 4.2, to complete the proof of statement (iii).

Ad (iv): Let $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$. It follows by statement (i) that there exists an essentially unique function $\beta \in \mathcal{F}^{\ell}_{P,\Theta}$ with e^{γ} being a Radon–Nikodým derivative of Q_{X_1} with respect to P_{X_1}, such that condition $(*)$ is valid. The latter yields $e^{\alpha(\Theta)} = \rho(\Theta) \cdot \mathbb{E}_P[W_1 \mid \Theta]$ $P\restriction\sigma(\Theta)$-a.s., implying

$$V_t(\Theta) = V_t(\Theta, \beta) = S_t - t \cdot \rho(\Theta) \cdot \mathbb{E}_P\left[X_1 \cdot e^{\gamma(X_1)}\right] \quad \text{for all } t \in \mathbb{R}_{+}. \tag{1}$$

The rest of the proof of the direct implication runs by the arguments appearing in the proof of [19], Proposition 5.1(ii) with "ρ" in the place of "g".

Conversely, if $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ and Q is an ℓ-martingale measure for the process $V(\Theta) := V(\Theta, \beta)$, where $\beta \in \mathcal{F}^{\ell}_{P,\Theta}$ is the function appearing in Proposition 1(i) such that e^{γ} is a Radon–Nikodým derivative of Q_{X_1} with respect to P_{X_1} and condition $(*)$ is valid, then condition (1) holds true. As Q is an ℓ-martingale measure for the process $V(\Theta)$, we get $V_t(\Theta) \in \mathcal{L}^{\ell}(Q)$ for each $t \in \mathbb{R}_{+}$, implying along with condition (1) that $X_1, \rho(\Theta) \in \mathcal{L}^{\ell}(Q)$, i.e., $Q \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$.

Ad (v): Since in both statements (i) and (ii) the pair (ρ, Q) is an element of $\mathfrak{M}_{+}(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, the conclusion of (v) follows by (iv). □

Theorem 2. *Let $P \in \mathcal{M}^{*,2}_{S,\mathbf{K}(\Theta)}$. For every pair $(\beta, \xi) \in \mathcal{F}^{2}_{P,\Theta} \times \mathcal{R}^{*,2}_{+}(D)$ there exist a unique pair $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_{ξ}), such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_{Θ} with respect to P_{Θ}, respectively, and an essentially unique rcp $\{Q_{\theta}\}_{\theta \in D}$ of Q over Q_{Θ} consistent with Θ satisfying for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_{\theta} \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\circledast)$ and (RPM_{θ}), such that:*

(i) the process $V_{\mathbb{T}}(\Theta) := V_{\mathbb{T}}(\Theta, \beta) := \{V_t(\Theta, \beta)\}_{t \in \mathbb{T}} =: \{V_t(\Theta)\}_{t \in \mathbb{T}}$ satisfies condition (NFLVR);

(ii) for any $\theta \notin \widetilde{L}_{**}$ the process $V_{\mathbb{T}}(\theta) := V_{\mathbb{T}}(\theta, \beta) := \{V_t(\theta, \beta)\}_{t \in \mathbb{T}} =: \{V_t(\theta)\}_{t \in \mathbb{T}}$ satisfies condition (NFLVR),

where $\widetilde{L}_{**} \in \mathfrak{B}(D)$ is the P_{Θ}-null set appearing in Theorem 1 and $\mathbb{T} := [0, T]$ with $T > 0$.

Proof. Ad (i): By Theorem 1(ii), conditions $(*)$ and (RPM_{ξ}) determine a unique pair $(\rho, Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$, such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_{Θ} with respect to P_{Θ}, respectively, implying by Theorem 1(v) that the process $V(\Theta)$ is a martingale in $\mathcal{L}^2(Q)$; hence for any $T > 0$ the process $V_{\mathbb{T}}(\Theta)$ is an $\mathcal{F}_{\mathbb{T}}$-martingale in $\mathcal{L}^2(Q_T)$, implying that it is a $\mathcal{F}_{\mathbb{T}}$-semi-martingale in $\mathcal{L}^2(Q_T)$ (cf., e.g., [34], Chapter 1, Section 1.3, Definition on page 23). The latter implies that $V_{\mathbb{T}}(\Theta)$ is also an $\mathcal{F}_{\mathbb{T}}$-semi-martingale in $\mathcal{L}^2(P_T)$ since $Q_T \sim P_T$ (cf., e.g., [34], Theorem 10.1.8). Because the process $V(\Theta)$ satisfies condition (PEMM), we get that the process $V_{\mathbb{T}}(\Theta)$ must satisfy condition (EMM). Thus, we can apply the Fundamental Theorem of Asset Pricing of Delbaen and Schachermayer for unbounded stochastic processes, see [9], Theorem 14.1.1, in order to conclude that the process $V_{\mathbb{T}}(\Theta)$ satisfies condition (NFLVR).

Assertion (ii) follows easily by Theorem 1(iii) and [22], Theorem 4.1. □

4 An application to the ruin problem

The fact that $\{P_\theta\}_{\theta \in D}$ is an rcp of P over P_Θ consistent with Θ, allows for the extension of some well-known results from the Poisson or renewal risk models, to their mixed counterpart. In fact, whenever in the P_θ-Cramér–Lundberg or the P_θ-Sparre Andersen risk model an explicit formula for the (infinite time) ruin probability exists, then one can just mix over the involved parameter in order to obtain explicit formulas for the corresponding mixed risk models (compare Albrecher et al. [1], Sections 3 and 5). However, such explicit formulas, which are also computationally feasible, can be obtained only in certain special cases, e.g., when the claim sizes follow a gamma distribution (see Constantinescu et al. [5]), a Coxian distribution (see Landriault and Willmot [17]), or a general phase type distribution (cf., e.g., Asmussen and Albrecher [2], Chapter IX, Theorem 4.4). If we are interested in an exact value for the ruin probability in a general mixed renewal risk model, then the only method available seems to be simulation. In the setting of a finite horizon ruin probability, it is straightforward to use the Crude Monte Carlo method to simulate it, see [2], page 462. The situation is more complicated for an infinite horizon ruin probability. The difficulty is that the indicator function of the ruin event in such a case cannot be simulated in finite time: no finite segment of S can tell whether ruin will ultimately occur or not. In order to overcome this obstacle, some methods have been developed (cf., e.g., [2], Chapter XV, Sections 2–5). Among them, the most celebrated one is the change of measures technique, which gives us the opportunity to express the ruin event as a quantity under the new measure such that ruin occurs almost surely.

The most common change of measures techniques applied to the Sparre Andersen risk model arise from the martingales constructed via the so-called Backward or Forward Markovization Techniques for the reserve process (see Dassios and Embrechts [7], Section 2.3, and Dassios [6], Section 3.5, respectively, in connection with, e.g., Schmidli [28], Sections 8.1–8.3), where the martingales (and thus the new measures) are obtained as solutions of partial differential equations (see also [33], Proposition 4.15, for a simplified construction of the martingales/measures arising from the Backward Markovization Technique). These techniques have been widely used to solve various ruin related problems (cf., e.g., Embrechts et al. [12], Ng and Yang [23], Schmidli [26, 27] and Tzaninis [32]), as they allow for the construction of a suitable probability measure $Q^{(r)}$, where $r > 0$, so that ruin occurs $Q^{(r)}$-a.s. However, as the main assumption for their construction is the existence of the moment generating function $M_{P_{X_1}}$ of the claim size distribution (for the definition of the moment generating function of a distribution we refer to [30], page 174), heavy-tailed distributions (cf., e.g., [25], Section 2.5, for the definition and their basic properties) are naturally excluded. When P is clear from the context we write M_{X_1} instead of $M_{P_{X_1}}$. The previous discussion raises the question of constructing a probability measure Q being progressively equivalent to P and so the ruin occurs Q-a.s., but without necessarily needing the assumption that M_{X_1} exists.

In this section, we characterize all progressively equivalent martingale measures Q on Σ that convert a P-CMRP into a Q-CMPP, in such a way that ruin occurs Q-a.s., see Theorem 3. Such a characterization allows us to find an explicit formula for the ruin probability under P. To this purpose, we first need to prove the following auxiliary results.

In order to justify the definition of a *conditional premium density* we need the next lemma, which extends a well-known result in the case of a kind of compound mixed Poisson processes (cf., e.g., [13], Proposition 9.1).

Lemma 1. *If* $P \in \mathcal{M}^1_{S,\mathbf{K}(\Theta)}$ *and* $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$ *then*

$$\lim_{t\to\infty} \frac{S_t}{t} = \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P\restriction\sigma(\Theta)\text{-}a.s.$$

Proof. Since $P \in \mathcal{M}^1_{S,\mathbf{K}(\Theta)}$ and $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$, it follows by Remark 1(b) that there exists a P_Θ-null set $W_P \in \mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}^1_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ and $\mathbb{E}_{P_\theta}[W_1] < \infty$ for any $\theta \notin W_P$. Fix an arbitrary $\theta \notin W_P$. We first show the validity of condition

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P\restriction\sigma(\Theta)\text{-a.s.} \tag{2}$$

In fact, consider the function $v := \mathbb{1}_{\{\lim_{t\to\infty} \frac{N_t(\bullet)}{t} = \frac{1}{\mathbb{E}_{P_\bullet}[W_1]}\}} : \Omega \times D \to [0,1]$ and put $g := v \circ (id_\Omega \times \Theta) = \mathbb{1}_{\{\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_P[W_1|\Theta]}\}}$. Since $v \in \mathcal{L}^1(M)$, where $M := P\circ(id_\Omega\times\Theta)^{-1}$, we may apply [18], Proposition 3.8(i), to get $\mathbb{E}_P[g \mid \Theta] = \mathbb{E}_{P_\bullet}[v^\bullet]\circ\Theta$ $P\restriction\sigma(\Theta)$-a.s., implying

$$\begin{aligned} P\left(\left\{\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_P[W_1 \mid \Theta]}\right\}\right) &= \int \mathbb{E}_P[g \mid \Theta]\, dP = \int \mathbb{E}_{P_\bullet}\left[v^\bullet\right] \circ \Theta\, dP \\ &= \int P_\theta\left(\left\{\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_{P_\theta}[W_1]}\right\}\right) P_\Theta(d\theta). \end{aligned}$$

But since N is a P_θ-RP($\mathbf{K}(\theta)$), we may apply [14], Section 2.5, Theorem 5.1, to get

$$\lim_{t\to\infty} \frac{N_t}{t} = \frac{1}{\mathbb{E}_{P_\theta}[W_1]} \quad P_\theta\text{-a.s.} \tag{3}$$

The latter, along with [18], Lemma 3.5(i), yields condition (2).

Since the process S is a P_θ-CRP$(\mathbf{K}(\theta), (P_\theta)_{X_1})$ with $(P_\theta)_{X_1} = P_{X_1}$, we may apply [14], Section 1.2, Theorem 2.3(iii), in order to get

$$\lim_{t\to\infty} \frac{S_t}{N_t} = \mathbb{E}_{P_\theta}[X_1] \quad P_\theta\text{-a.s.}; \tag{4}$$

implying along with condition (a2) and [18], Lemma 3.5(i), that $\lim_{t\to\infty} \frac{S_t}{N_t} = \mathbb{E}_P[X_1]$ P-a.s. The latter, along with condition (2), completes the proof. □

Let $P \in \mathcal{M}^1_{S,\mathbf{K}(\Theta)}$ such that $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$. For any $\theta \notin W_P$, where $W_P \in \mathfrak{B}(D)$ is the P_Θ-null set appearing in Remark 1(b), conditions (3) and (4) imply

$$\lim_{t\to\infty} \frac{S_t}{t} = \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]} \quad P_\theta\text{-a.s.},$$

that is, the limit $\lim_{t\to\infty} \frac{S_t}{t}$ coincides with the **premium density** $p(P_\theta)$, i.e., the monetary payout per unit time, in a P_θ-Sparre Andersen model (see [22], page 54, for more details). This motivates the following definitions.

Definitions 1. For any $P \in \mathcal{M}^1_{S,\mathbf{K}(\Theta)}$ such that $P(\{\mathbb{E}_P[W_1 \mid \Theta] < \infty\}) = 1$, we say that the random vector

$$p(P, \Theta) := \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P\restriction\sigma(\Theta)\text{-a.s.}$$

is a **conditional premium density** for P. In particular, for any $P \in \mathcal{M}^{*,1}_{S,\mathbf{K}(\Theta)}$ define the corresponding **mixed premium density** by means of $p(P) := \mathbb{E}_P[p(P,\Theta)]$.

Remark 4. If $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$ then the following statements are equivalent:

(i) $p(P,\Theta)$ is a conditional premium density for P;

(ii) there exists a P_Θ-null set $W_{P,1} \in \mathfrak{B}(D)$, containing the P_Θ-null set $W_P \in \mathfrak{B}(D)$ appearing in Remark 3, such that $p(P,\theta)$ is the premium density for P_θ, i.e., $p(P,\theta) = p(P_\theta) := \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]}$ for any $\theta \notin W_{P,1}$.

In fact, first note that according to Remark 3, there exists a P_Θ-null set $W_P \in \mathfrak{B}(D)$, such that $P_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$ and $\mathbb{E}_{P_\theta}[W_1] < \infty$ for any $\theta \notin W_P$; hence $p(P_\theta) = \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]}$ for any $\theta \notin W_P$. Furthermore, as statement (i) is equivalent to

$$\int_{\Theta^{-1}[F]} p(P,\Theta)\, dP = \int_{\Theta^{-1}[F]} \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1 \mid \Theta]}\, dP \quad \text{for every } F \in \mathfrak{B}(D),$$

we can apply [18], Lemma 3.5(i), to get

$$\int_F p(P,\theta)\, P_\Theta(d\theta) = \int_F \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_{P_\theta}[W_1]}\, P_\Theta(d\theta) = \int_F \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]}\, P_\Theta(d\theta)$$

for every $F \in \mathfrak{B}(D)$, where the second equality follows by condition (a2), implying that there exists a P_Θ-null set $W'_P \in \mathfrak{B}(D)$, such that $p(P,\theta) = \frac{\mathbb{E}_{P_\theta}[X_1]}{\mathbb{E}_{P_\theta}[W_1]}$ for any $\theta \notin W'_P$. Putting now $W_{P,1} := W'_P \cup W_P \in \mathfrak{B}(D)$, we get the desired equivalence of (i) and (ii).

Definitions 2. Let S be an aggregate claims process induced by a counting process N and a claims size process X. Fix arbitrary $u, t \in \mathbb{R}_+$ and define the function $r_t^u : \Omega \times D \to \mathbb{R}$ by means of $r_t^u(\omega, \theta) := u + c(\theta) \cdot t - S_t(\omega)$ for any $(\omega, \theta) \in \Omega \times D$, where c is a positive $\mathfrak{B}(D)$-measurable function. For arbitrary but fixed $\theta \in D$ the process $r^u(\theta) := \{r_t^u(\theta)\}_{t \in \mathbb{R}_+}$, defined by $r_t^u(\theta)(\omega) := r_t^u(\omega, \theta)$ for any $\omega \in \Omega$, is called the **reserve process induced by the initial reserve u, the premium intensity $c(\theta)$ and the aggregate claims process S** (cf., e.g., [30], pages 155–156). The function $\psi_\theta : [0, \infty) \to [0, 1]$ defined by $\psi_\theta(u) := P_\theta(\{\inf_{t \in \mathbb{R}_+} r_t^u(\theta) < 0\})$ is called the **probability of ruin for the reserve process $r^u(\theta)$ with respect to P_θ** (cf., e.g., [30], page 158). The **ruin time of the reserve process** $r^u(\theta)$ is defined as $\tau_u(\theta) := \inf\{t \in \mathbb{R}_+ : r_t^u(\theta) < 0\}$ (compare, e.g., [28], page 84). Recall that $\psi_\theta(u) = P_\theta(\{\tau_u(\theta) < \infty\})$, see, e.g., [25], page 148.

Define the real-valued function $R_t^u(\Theta)$ on Ω by means of $R_t^u(\Theta) := r_t^u \circ (id_\Omega \times \Theta)$. The process $R^u(\Theta) := \{R_t^u(\Theta)\}_{t \in \mathbb{R}_+}$ is called the **reserve process induced by the initial reserve u, the stochastic premium intensity $c(\Theta)$ and the aggregate claims process** S, where $c(\Theta)$ is a real-valued random variable on Ω. The function ψ defined by $\psi(u) := P(\{\inf_{t \in \mathbb{R}_+} R_t^u(\Theta) < 0\})$ is called the **probability of ruin for the reserve process $R^u(\Theta)$ with respect to** P. We define the **ruin time of the reserve process** $R^u(\Theta)$ by means of $T_u(\Theta) := \tau_u \circ (id_\Omega \times \Theta)$. Clearly, $\psi(u) = P(\{T_u(\Theta) < \infty\})$.

Throughout what follows in this section, unless stated otherwise, $P \in \mathcal{M}_{S,\mathbf{K}(\Theta)}^{*,\ell}$.

Lemma 2. *The following statements are equivalent:*

(i) $c(\Theta) \le p(P, \Theta)$ $P \restriction \sigma(\Theta)$*-a.s.;*

(ii) $\psi(u) = 1$ *for any* $u \in \mathbb{R}_+$.

Proof. Fix arbitrary $u \in \mathbb{R}_+$.

Ad (i)⇒(ii): If (i) holds, we get by [18], Lemma 3.5(i), the existence of a P_Θ-null set $\widetilde{M}_P \in \mathfrak{B}(D)$ such that $c(\theta) \le p(P, \theta)$ for all $\theta \notin \widetilde{M}_P$. But, due to Remark 4, there exists a P_Θ-null set $W_{P,1} \in \mathfrak{B}(D)$ such that $p(P, \theta) = p(P_\theta)$ and $P_\theta \in \mathcal{M}_{S,\mathbf{K}(\theta)}^{*,\ell}$ for all $\theta \notin W_{P,1}$. Putting $M_P := \widetilde{M}_P \cup W_{P,1} \in \mathfrak{B}(D)$, we get $P_\Theta(M_P) = 0$ and $c(\theta) \le p(P_\theta)$ for all $\theta \notin M_P$; hence for any $\theta \notin M_P$ we may apply [30], Corollary 7.1.4, which remains true also for $u = 0$, to obtain $\psi_\theta(u) = 1$. The latter along with [33], Remark 3.6, yields $\psi(u) = \int_D \psi_\theta(u)\, P_\Theta(d\theta) = 1$.

Ad (ii)⇒(i): Since N has zero probability of explosion we may apply [30], Lemma 7.1.2, which remains true also for $u = 0$, along with [33], Remark 3.6, to get that $\psi(u) = P(\{\inf_{n \in \mathbb{N}_0} U_n^u(\Theta) < 0\})$, where $U_n^u(\Theta) := u + \sum_{j=1}^n (c(\Theta) \cdot W_j - X_j)$ for any $n \in \mathbb{N}_0$. But since $\psi(u) = 1$, there exists a positive integer n_0 such that $U_{n_0}^u(\Theta) < 0$ P-a.s., and so there exists a natural number $n_1 \le n_0$ such that $c(\Theta) \cdot W_{n_1} - X_{n_1} < 0$ P-a.s., implying along with condition (a2)

$$\begin{aligned} 0 &\ge c(\Theta) \cdot \mathbb{E}_P[W_{n_1} \mid \Theta] - \mathbb{E}_P[X_{n_1} \mid \Theta] && P\restriction\sigma(\Theta)\text{-a.s.} \\ &= c(\Theta) \cdot \mathbb{E}_P[W_{n_1} \mid \Theta] - \mathbb{E}_P[X_{n_1}] && P\restriction\sigma(\Theta)\text{-a.s.;} \end{aligned}$$

hence $c(\Theta) \le \frac{\mathbb{E}_P[X_{n_1}]}{\mathbb{E}_P[W_{n_1} \mid \Theta]}$ $P\restriction\sigma(\Theta)$-a.s. But since W is P-identically distributed by [33], Remark 2.1, and X is P-i.i.d., it follows that statement (i) holds. □

Remark 5. The following statements are equivalent:

(i) the pair (P, Θ) fulfils condition

$$c(\Theta) > p(P, \Theta) \quad P\restriction\sigma(\Theta)\text{-a.s.}; \tag{5}$$

(ii) there exists a P_Θ-null set $O_P \in \mathfrak{B}(D)$, containing the P_Θ-null set $W_{P,1}$ appearing in Remark 4, such that for any $\theta \notin O_P$ the measure P_θ is an element of $\mathcal{M}^{*,\ell}_{S,\mathbf{K}(\theta)}$ satisfying condition $c(\theta) > p(P_\theta)$.

In fact, (i) holds if and only if there exist a P_Θ-null set $\widetilde{O}_P \in \mathfrak{B}(D)$ such that $c(\theta) > p(P, \theta)$ for any $\theta \notin \widetilde{O}_P$ by [18], Lemma 3.5(i), and a P_Θ-null set $W_{P,1}$ in $\mathfrak{B}(D)$ such that $P_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\theta)}$ and $p(P, \theta) = p(P_\theta)$ for any $\theta \notin \widetilde{W}_{P,1}$ by Remark 4. Putting $O_P := \widetilde{O}_P \cup W_{P,1}$, we get (i)$\Leftrightarrow$(ii).

The next result has been proven in a more general setting for multivariate counting processes in [16], Proposition 3.39(a). However, as it is essential for the proof of the main result of the present section (see Theorem 3), we write it exactly in the form needed for our purposes. Recall that a filtration $\{\mathcal{G}_t\}_{t\in\mathbb{R}_+}$ for (Ω, Σ) is called **right-continuous** if $\mathcal{G}_{t+} := \bigcap_{s>t} \mathcal{G}_s = \mathcal{G}_t$ for any $t \in \mathbb{R}_+$ (cf., e.g., [25], Subsection 10.2.1, page 404).

Proposition 2. *Let (Ω, Σ, P) be an arbitrary probability space, and let S be an arbitrary aggregate claims process induced by a claim number process N and a claim size process X. The canonical filtration $\mathcal{F}$ generated by S and Θ is right-continuous.*

The proof follows by a slight modification of the arguments appearing in Jacod [16], Proposition 3.39(a).

Remark 6. The canonical filtration $\mathcal{F}^S$ of an arbitrary aggregate claims process S is right-continuous. The proof runs with arguments similar to those of the proof of [16], Proposition 3.39(a). An alternative proof for the right-continuity of $\mathcal{F}^S$ works by arguments similar to those of Protter [24], Theorem 25.

Lemma 3. *Let (Ω, Σ, P) be an arbitrary probability space. The following hold true:*

(i) the ruin time $T_u(\Theta)$ of the reserve process $R^u(\Theta)$ is an $\mathcal{F}$-stopping time;

(ii) for any $\theta \in D$ the ruin time $\tau_u(\theta)$ of the reserve process $r^u(\theta)$ is an $\mathcal{F}$-stopping time.

Proof. Ad (i): Let $t \in \mathbb{R}_+$. Since $R^u(\Theta)$ has right-continuous paths it follows that

$$\{T_u(\Theta) < t\} = \bigcup_{q\in\mathbb{Q}_t} \{R^u_q(\Theta) < 0\} \in \mathcal{F}_t,$$

where $\mathbb{Q}_t := \mathbb{Q} \cap [0, t)$ (compare [25], Theorem 10.1.1), implying along with [25], Lemma 10.1.1, that $T_u(\Theta)$ is a $\{\mathcal{F}_{t+}\}_{t\in\mathbb{R}_+}$-stopping time. But by Proposition 2, $\mathcal{F}$ is right-continuous, and thus we may apply again [25], Lemma 10.1.1, in order to get that $T_u(\Theta)$ is an $\mathcal{F}$-stopping time.

Ad (ii): Fix an arbitrary $\theta \in D$. Using the arguments of the proof of statement (i), we get that $\tau_u(\theta)$ is a $\{\mathcal{F}^S_{t+}\}_{t\in\mathbb{R}_+}$-stopping time. Consequently, since $\mathcal{F}^S$ is right-continuous by Remark 6, applying [25], Lemma 10.1.1, we get that $\tau_u(\theta)$ is an $\mathcal{F}^S$-stopping time; hence it is an $\mathcal{F}$-stopping time since $\mathcal{F}^S_t \subseteq \mathcal{F}_t$ for all $t \in \mathbb{R}_+$. □

According to Theorem 1(ii), for every pair $(\beta,\xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_{+}(D)$ there exists a unique pair $(\rho,Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_{ξ}) such that the process $M^{(\beta)}(\Theta)$, involved in condition (RPM_{ξ}), is a P-a.s. positive martingale in $\mathcal{L}^1(P)$, implying

$$P(A) = \mathbb{E}_Q\left[1/M^{(\beta)}_t(\Theta)\,\mathbb{1}_A\right] \quad \text{for all } t \in \mathbb{R}_+ \text{ and } A \in \mathcal{F}_t.$$

But since the ruin time $T_u(\Theta)$ of the reserve process $R^u(\Theta)$ is an $\mathcal{F}$-stopping time by Lemma 3(i), we may apply the Optional Stopping Theorem (cf., e.g., [28], Proposition B.2) in order to get

$$\psi(u) = \mathbb{E}_Q\left[1/M^{(\beta)}_{T_u(\Theta)}(\Theta)\,\mathbb{1}_{\{T_u(\Theta)<\infty\}}\right] \quad \text{for any } u \in \mathbb{R}_+. \tag{6}$$

However, the latter formula is quite complicated mainly due to the (possible) dependence between $1/M^{(\beta)}_{T_u(\Theta)}(\Theta)$ and $\mathbb{1}_{\{T_u(\Theta)<\infty\}}$. Thus, we have to carefully choose an appropriate pair $(\rho,Q) \in \mathfrak{M}_{+}(D)\times\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, or equivalently an appropriate pair $(\beta,\xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_{+}(D)$, in order to eliminate such a dependence. This motivates us to formulate and prove the following result.

Theorem 3. *Assume that the pair (P,Θ) satisfies condition (5). The following hold true:*

(i) for each pair $(\rho,Q) \in \mathfrak{M}_{+}(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying $c(\Theta) \le p(Q,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., there exists an essentially unique pair $(\beta,\xi) \in \mathcal{F}^{\ell}_{P,\Theta}\times\mathcal{R}^{*,\ell}_{+}(D)$, where e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_{Θ} with respect to P_{Θ}, respectively, satisfying conditions $(*)$, (RPM_{ξ}) and

$$\psi(u) = \int \frac{1}{\xi(\Theta)}\cdot e^{-S^{(\gamma)}_{T_u(\Theta)}}\cdot\prod_{j=1}^{N_{T_u(\Theta)}}\frac{d\mathbf{K}(\Theta)}{d\mathbf{Exp}(\rho(\Theta))}(W_j)\,dQ < 1 \qquad (\mathrm{ruin}(P))$$

for all $u \in \mathbb{R}_+$;

(ii) conversely, for every pair $(\beta,\xi) \in \mathcal{F}^{\ell}_{P,\Theta}\times\mathcal{R}^{*,\ell}_{+}(D)$ with

$$c(\Theta) \le \frac{\mathbb{E}_P[X_1\cdot e^{\beta(X_1,\Theta)} \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P\restriction\sigma(\Theta)\text{-a.s.},$$

there exists a unique pair $(\rho,Q) \in \mathfrak{M}_{+}(D)\times\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_{ξ}), such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_{Θ} with respect to P_{Θ}, respectively, satisfying conditions $Q(\{T_u(\Theta) < \infty\}) = 1$ and (ruin(P));

(iii) in both cases (i) and (ii), there exist an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_{Θ} consistent with Θ and a P_{Θ}-null set $M_{*,Q} \in \mathfrak{B}(D)$, containing the P_{Θ}-null set $\widetilde{L}_{**}$ appearing in Theorem 1, satisfying for any $\theta \notin M_{*,Q}$ conditions $Q_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\widetilde{*})$, (RPM_{θ}), $Q_\theta(\{\tau_u(\theta) < \infty\}) = 1$ and

$$\psi_\theta(u) = \int e^{-S^{(\gamma)}_{\tau_u(\theta)}}\cdot\prod_{j=1}^{N_{\tau_u(\theta)}}\frac{d\mathbf{K}(\theta)}{d\mathbf{Exp}(\rho(\theta))}(W_j)\,dQ_\theta < 1 \qquad (\mathrm{ruin}(P_\theta))$$

for all $u \in \mathbb{R}_+$.

Proof. Fix an arbitrary $u \in \mathbb{R}_+$.
Ad (i): Since $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, it follows by Theorem 1(i) that there exists an essentially unique pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_{+}(D)$, where e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively, satisfying conditions $(*)$ and (RPM_ξ) such that the family $M^{(\beta)}(\Theta)$ is a P-a.s. positive martingale in $\mathcal{L}^1(P)$. But since $c(\Theta) \le p(Q, \Theta)$ $P\restriction\sigma(\Theta)$-a.s., we get by Lemma 2 that $Q(\{T_u(\Theta) < \infty\}) = 1$, implying, together with equality (6) and the fact that $J_{T_u(\Theta)} = 0$, the equality in condition (ruin(P)). The inequality in (ruin(P)) follows by (5) and Lemma 2.
Ad (ii): Since $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_{+}(D)$, we can apply Theorem 1(ii) to obtain a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$, (RPM_ξ) and such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively. But since

$$c(\Theta) \le \frac{\mathbb{E}_P[X_1 \cdot e^{\beta(X_1,\Theta)} \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]} \quad P\restriction\sigma(\Theta)\text{-a.s.} \Leftrightarrow c(\Theta) \le p(Q, \Theta) \quad P\restriction\sigma(\Theta)\text{-a.s.},$$

we get by Lemma 2 that $Q(\{T_u(\Theta) < \infty\}) = 1$. Condition (ruin$(P)$) follows as in (i).
Ad (iii): In both cases (i) and (ii), according to Theorem 1(iii), there exist an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$ satisfying for each $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\widetilde{*})$ and (RPM_θ). But since $c(\Theta) \le p(Q, \Theta)$ $P\restriction\sigma(\Theta)$-a.s., it follows as in Remark 4 that there exists a P_Θ-null set $W_{Q,1} \in \mathfrak{B}(D)$ such that $c(\theta) \le p(Q_\theta)$ for each $\theta \notin W_{Q,1}$. Fix an arbitrary $\theta \notin M_{*,Q} := \widetilde{L}_{**} \cup W_{Q,1} \in \mathfrak{B}(D)$. The inequality $c(\theta) \le p(Q_\theta)$, along with [30], Corollary 7.1.4, implies that $Q_\theta(\{\tau_u(\theta) < \infty\}) = 1$. Taking now into account condition (RPM_θ), Lemma 3(ii) and the fact that ruin occurs Q_θ-a.s., and using the arguments appearing in the proof of assertion (i), we get condition (ruin(P_θ)). □

Remark 7. Assume that the pair (P, Θ) satisfies condition (5).
(a) The class of all functions $\beta \in \mathcal{F}^{\ell}_{P,\Theta}$ satisfying condition $\mathbb{E}_P[W_1 \mid \Theta] \cdot c(\Theta) \le \mathbb{E}_P[X_1 \cdot e^{\beta(X_1,\Theta)} \mid \Theta]$ $P\restriction\sigma(\Theta)$-a.s. is a strict subclass of $\mathcal{F}^{\ell}_{P,\Theta}$.

In fact, let $D := (0, \infty)$ and consider a function $\beta \in \mathcal{F}^{\ell}_{P,\Theta}$ with $\beta \le 0$. We then get

$$\frac{\mathbb{E}_P[X_1 \cdot e^{\beta(X_1,\Theta)} \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]} \le \frac{\mathbb{E}_P[X_1 \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]} = \frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1 \mid \Theta]} < c(\Theta) \quad P\restriction\sigma(\Theta)\text{-a.s.},$$

where the equality follows by condition (a2), and the second inequality is due to condition (5).
(b) The class of all probability measures $Q \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ satisfying condition $c(\Theta) \le p(Q, \Theta)$ $P\restriction\sigma(\Theta)$-a.s. is a strict subclass of $\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$.

In fact, consider the pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_{+}(D)$, where β is the function defined in (a). It then follows by Theorem 1(ii) that conditions $(*)$ and (RPM_ξ) determine a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, such that e^{γ} and ξ are the Radon–Nikodým derivatives of Q_{X_1} with respect to P_{X_1} and of Q_Θ with respect to P_Θ, respectively. However, for this particular Q we have that

$$p(Q,\Theta)=\rho(\Theta)\cdot\mathbb{E}_Q[X_1]=\frac{\mathbb{E}_P[X_1\cdot e^{\beta(X_1,\Theta)}\mid\Theta]}{\mathbb{E}_P[W_1\mid\Theta]} \qquad P\restriction\sigma(\Theta)\text{-a.s.}$$
$$\le\frac{\mathbb{E}_P[X_1\mid\Theta]}{\mathbb{E}_P[W_1\mid\Theta]}=\frac{\mathbb{E}_P[X_1]}{\mathbb{E}_P[W_1\mid\Theta]}<c(\Theta) \qquad P\restriction\sigma(\Theta)\text{-a.s.}$$

where the second equality follows from condition $(*)$ and the fact that e^{γ} is a P_{X_1}-a.s. positive Radon–Nikodým derivative of Q_{X_1} with respect to P_{X_1}, the third equality follows by condition (a2), while the second inequality follows by condition (5).

Remark 8. Let $P\in\mathcal{M}^{*,2}_{S,\mathbf{K}(\Theta)}$. Theorems 2 and 3 reveal an interesting connection between the pricing of insurance in an arbitrage-free market and the ruin probability of the corresponding reserve process.

In fact, for a given pair $(\beta,\xi)\in\mathcal{F}^2_{P,\Theta}\times\mathcal{R}^{*,2}_{+}(D)$, consider the reserve process $R^u(\Theta):=\{R^u_t(\Theta)\}_{t\in\mathbb{R}_+}$ defined by

$$R^u_t(\Theta):=R^u_t\big(\Theta,V_t(\Theta)\big):=u-V_t(\Theta)=u+t\cdot\frac{\mathbb{E}_P[X_1\cdot e^{\beta(X_1,\Theta)}\mid\Theta]}{\mathbb{E}_P[W_1\mid\Theta]}-S_t$$

for any $t,u\in\mathbb{R}_+$. It then follows by Theorem 2 that conditions $(*)$ and (RPM_ξ) determine a unique pair $(\rho,Q)\in\mathfrak{M}_+(D)\times\mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$, such that the process $V_{\mathbb{T}}(\Theta)$, where $\mathbb{T}:=[0,T]$ with $T>0$, satisfies condition (NFLVR), and so the process $R^u_{\mathbb{T}}(\Theta)$ does as well. On the other hand, if the pair (P,Θ) satisfies condition (5), we get

$$p(P,\Theta)<c(\Theta)=\frac{\mathbb{E}_P[X_1\cdot e^{\beta(X_1,\Theta)}\mid\Theta]}{\mathbb{E}_P[W_1\mid\Theta]}=p(Q,\Theta)\ \ P\restriction\sigma(\Theta)\text{-a.s.};$$

hence, we can apply Theorem 3, to deduce that the corresponding ruin probability $\psi(u)$ can be computed by using the representation appearing in condition (ruin(P)).

The following example shows that the classical approach to the ruin problem via a change of measures technique in the case of CPPs (cf., e.g., [28], Section 8.2) appears as a special instance of Theorem 3.

Example 1. Let $D=(0,\infty)$, $\xi\in\mathcal{R}^{*,\ell}_{+}(D)$ and $P\in\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\Theta)}$. It follows by Remark 3 that there exists a P_Θ-null set $W_P\in\mathfrak{B}(D)$, such that $P_\theta\in\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\theta)}$ and $(P_\theta)_{X_1}=P_{X_1}$ for all $\theta\notin W_P$; hence we may write $M_{X_1}=M_{P_{X_1}}=M_{(P_\theta)_{X_1}}$ for any $\theta\notin W_P$. Assume that $M_{X_1}(r)<\infty$ for some $r\in[0,r_{X_1})$, where $r_{X_1}:=\sup\{r\ge 0: M_{X_1}(r)<\infty\}$, and that the pair (P,Θ) satisfies condition (5). It then follows by Remark 5, that there exists a P_Θ-null set $O_P\in\mathfrak{B}(D)$, containing the P_Θ-null set $W_P\in\mathfrak{B}(D)$, such that $P_\theta\in\mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\theta)}$ and

$$c(\theta)>p(P_\theta) \tag{7}$$

for every $\theta\notin O_P$.

For arbitrary but fixed $\theta\notin O_P$ define the function $\kappa_\theta:[0,r_{X_1})\to\mathbb{R}$ by means of $\kappa_\theta(r):=\theta\cdot(M_{X_1}(r)-1)-c(\theta)\cdot r$ for every $r\in[0,r_{X_1})$ (compare [28], page 89). Define the function $\kappa:(D\setminus O_P)\times[0,r_{X_1})\to\mathbb{R}$ by means of $\kappa(\theta,r):=\kappa_\theta(r)$ for all

$(\theta, r) \in (D \setminus O_P) \times [0, r_{X_1})$, and for fixed $r \in [0, r_{X_1})$ denote by $\kappa_\Theta(r)$ the random variable defined by $\kappa_\Theta(r)(\omega) := \kappa_{\Theta(\omega)}(r)$ for any $\omega \in \Omega$.

It can be easily seen that for any fixed $\theta \notin O_P$ the function κ_θ is strictly convex on $[0, r_{X_1})$, or equivalently the function κ'_θ is strictly increasing on $[0, r_{X_1})$. By condition (7), we get $\kappa'_\theta(0) < 0$; hence there exists a number $r_m(\theta) \in (0, r_{X_1})$ such that $\kappa_\theta(r_m(\theta)) = \min_{r \in (0, r_{X_1})} \kappa_\theta(r) < 0$. Put $r^* := \sup_{\theta \in D \setminus O_P} r_m(\theta)$ and assume that $r^* < r_{X_1}$.

Fix an arbitrary $r \in [r^*, r_{X_1})$, and define the function $\beta_r : D \times D \longrightarrow \mathbb{R}$ by means of $\beta_r(x, \theta) := r \cdot x$ if $(x, \theta) \in D \times (D \setminus O_P)$ and $\beta_r(x, \theta) := 0$ if $(x, \theta) \in D \times O_P$. A standard computation justifies that $\beta_r \in \mathcal{F}^{\ell}_{P,\Theta}$, with $\gamma_r(x) = r \cdot x - \ln M_{X_1}(r)$ and $\alpha_r(\theta) = M_{X_1}(r)$ if $(x, \theta) \in D \times (D \setminus O_P)$ and $\gamma_r(x) = \alpha_r(\theta) = 0$ if $(x, \theta) \in D \times O_P$.

Fix an arbitrary $\theta \notin O_P$. As the function κ_θ is strictly convex on $[0, r_{X_1})$ and $\kappa'_\theta(r_m(\theta)) = 0$, we infer that $\kappa'_\theta(r) \geq 0$, implying $\theta \cdot M'_{X_1}(r) - c(\theta) \geq 0$, or equivalently $\theta \cdot \mathbb{E}_{P_\theta}[X_1 \cdot e^{\beta_r(X_1, \theta)}] \geq c(\theta)$, and so by [18], Lemma 3.5,

$$\Theta \cdot \mathbb{E}_P\left[X_1 \cdot e^{\beta_r(X_1, \Theta)} \mid \Theta\right] \geq c(\Theta) \quad P\restriction\sigma(\Theta)\text{-a.s..}$$

Thus, we may apply Theorem 3(ii), in order to obtain a unique pair $(\rho_r, Q^{(r)}) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ) and satisfying the conclusions of the statement (ii) of this theorem. Since $\alpha_r(\Theta) = \ln M_{X_1}(r)$ $P\restriction\sigma(\Theta)$-a.s, conditions $(*)$ and (RPM_ξ) become

$$\rho_r(\Theta) = \Theta \cdot M_{X_1}(r) \quad P\restriction\sigma(\Theta)\text{-a.s.,}$$

and

$$Q^{(r)}(A) = \int_A \xi(\Theta) \cdot e^{r \cdot S_t - t \cdot \Theta \cdot (M_{X_1}(r) - 1)}\, dP = \int_A \xi(\Theta) \cdot e^{-r \cdot (R^u_t(\Theta) - u) - t \cdot \kappa_\Theta(r)}\, dP$$

for every $u \in \mathbb{R}_+$, $0 \leq s \leq t$, and $A \in \mathcal{F}_s$, respectively. Conditions (RPM_ξ) and $(\text{ruin}(P))$ yield

$$\psi(u) = \mathbb{E}_{Q^{(r)}}\left[\frac{e^{r \cdot R^u_{T_u(\Theta)}(\Theta) + \kappa_\Theta(r) \cdot T_u(\Theta)}}{\xi(\Theta)}\right] \cdot e^{-r \cdot u} \quad \text{for all } u \in \mathbb{R}_+. \tag{8}$$

In particular, if the distribution of Θ is degenerate, i.e., if $P_\Theta = \delta_{\theta_0}$ for some $\theta_0 \in D$, where δ_{θ_0} is the Dirac measure on $\mathfrak{B}(D)$ concentrated on θ_0, then S is reduced to a P-CPP(θ_0, P_{X_1}). Since ξ is a P_Θ-a.s. positive Radon–Nikodým derivative of $Q^{(r)}_\Theta := (Q^{(r)})_\Theta$ with respect to P_Θ, we deduce that $\xi(\theta_0) = 1$ and $Q^{(r)}_\Theta = \delta_{\theta_0}$; hence $Q^{(r)} \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho_r(\theta_0))}$. Thus, condition (8) reduces to

$$\psi(u) = \mathbb{E}_{Q^{(r)}}\left[e^{r \cdot R^u_{T_u(\theta_0)}(\theta_0) + \kappa_{\theta_0}(r) \cdot T_u(\theta_0)}\right] \cdot e^{-r \cdot u} \quad \text{for all } u \in \mathbb{R}_+. \tag{9}$$

If in addition an adjustment coefficient $\eta := \eta(\theta_0)$ for the reserve process $R^u_t(\theta_0)$ exists (cf., e.g., [28], page 90, for the definition), then by choosing $r = \eta$, condition (9) becomes

$$\psi(u) = \mathbb{E}_{Q^{(\eta)}}\left[e^{\eta \cdot R^u_{T_u(\theta_0)}(\theta_0)}\right] \cdot e^{-\eta \cdot u} \quad \text{for all } u \in \mathbb{R}_+,$$

compare [28], page 173.

Note that the assumption $r^* < r_{X_1}$ is fulfilled, e.g., in the case $P_{X_1} = \mathbf{Exp}(z)$, where $z > 0$ is a real constant, $P_\Theta = \mathbf{Ga}(a, b)$ with $a, b \in (0, \infty)$ and $c(\theta) = (1+e^{-\theta})\cdot\theta\cdot z^{-1}$, since by standard computations we have $r_m(\theta) = z - z\cdot(1 + e^{-\theta})^{-1/2} \in (0, z)$ and $r^* = z - \frac{z}{\sqrt{2}} \in (0, z)$.

One of the main drawbacks of the method used in Example 1 is the assumption that $M_{X_1}(r)$ exists for some $r > 0$, since it excludes heavy-tailed distributions. In the following example we consider again $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\Theta)}$ and we demonstrate how Theorem 3 can be used to handle such cases.

Example 2. Let $D := (0, \infty)$, $\xi \in \mathcal{R}^{*,\ell}_+(D)$ and $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\Theta)}$ with $P_{X_1} = \mathbf{Par}(a, b)$ for some real constants $a > 1$ and $b > 0$ (cf., e.g., [30], page 180 for the definition of the Pareto distribution). Assume that the pair (P, Θ) satisfies condition (5). It then follows by Remark 5, that there exists a P_Θ-null set $O_P \in \mathfrak{B}(D)$, containing the P_Θ-null set $W_P \in \mathfrak{B}(D)$ appearing in Remark 3, such that for any $\theta \notin O_P$ condition (7) is valid and $P_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\theta)}$ with $(P_\theta)_{X_1} = P_{X_1}$. For any function $z \in \mathfrak{M}_+(D)$ with $z(\Theta) > c(\Theta)$ $P\restriction\sigma(\Theta)$-a.s., define the function $\beta_z : D\times D \to \mathbb{R}$ by means of $\beta_z(x, \theta) := \ln\frac{z(\theta)}{\theta\cdot\mathbb{E}_P[X_1]}$ if $(x, \theta) \in D \times (D \setminus O_P)$ and $\beta_z(x, \theta) := 0$ if $(x, \theta) \in D \times O_P$. It then follows that $\beta_z \in \mathcal{F}^\ell_{P,\Theta}$, with $\gamma_z(x) = 0$ and $\alpha_z(\theta) = \ln\frac{z(\theta)}{\theta\cdot\mathbb{E}_P[X_1]}$ if $(x, \theta) \in D \times (D \setminus O_P)$ and $\gamma_z(x) = \alpha_z(\theta) = 0$ if $(x, \theta) \in D \times O_P$. Since

$$\frac{\mathbb{E}_P[X_1\cdot e^{\beta(X_1,\Theta)} \mid \Theta]}{\mathbb{E}_P[W_1 \mid \Theta]} = z(\Theta)\cdot\frac{\mathbb{E}_P[X_1 \mid \Theta]}{\mathbb{E}_P[X_1]} = z(\Theta) > c(\Theta) \quad P\restriction\sigma(\Theta)\text{-a.s.},$$

where the second equality follows by condition (a2), we may apply Theorem 3(ii) to construct a unique pair $(\rho_z, Q^{(z)}) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ) and satisfying the conclusions of the statement (ii) of this theorem. Since $\alpha_z(\Theta) = \ln\frac{z(\Theta)}{\Theta\cdot\mathbb{E}_P[X_1]}$ $P\restriction\sigma(\Theta)$-a.s, conditions $(*)$ and (RPM_ξ) become

$$\rho_z(\Theta) = \frac{z(\Theta)}{\mathbb{E}_P[X_1]} \quad P\restriction\sigma(\Theta)\text{-a.s.},$$

and

$$\begin{aligned} Q^{(z)}(A) &= \int_A \xi(\Theta)\cdot\left(\frac{\rho_z(\Theta)}{\Theta}\right)^{N_t}\cdot e^{-t\cdot(\rho_z(\Theta)-\Theta)}\,dP \\ &= \int_A \xi(\Theta)\cdot\left(\frac{z(\Theta)}{\Theta\cdot\mathbb{E}_P[X_1]}\right)^{N_t}\cdot e^{-\frac{t}{\mathbb{E}_P[X_1]}\cdot(z(\Theta)-\Theta\cdot\mathbb{E}_P[X_1])}\,dP \end{aligned}$$

for every $0 \le s \le t$ and $A \in \mathcal{F}_s$, respectively. Conditions (RPM_ξ) and $(\mathrm{ruin}(P))$ imply

$$\begin{aligned} \psi(u) &= \int \frac{1}{\xi(\Theta)}\cdot\left(\frac{\Theta\cdot\mathbb{E}_P[X_1]}{z(\Theta)}\right)^{N_{T_u(\Theta)}}\cdot e^{-\frac{T_u(\Theta)}{\mathbb{E}_P[X_1]}\cdot(\Theta\cdot\mathbb{E}_P[X_1]-z(\Theta))}\,dQ^{(z)} \\ &= \int \frac{1}{\xi(\Theta)}\cdot\left(\frac{\Theta\cdot b}{(a-1)\cdot z(\Theta)}\right)^{N_{T_u(\Theta)}}\cdot e^{-\frac{(a-1)\cdot T_u(\Theta)}{b}\cdot(\Theta\cdot\frac{b}{a-1}-z(\Theta))}\,dQ^{(z)} \end{aligned}$$

for any $u \in \mathbb{R}_+$.

Note that the arguments appearing in the above example, remain true for any claim size distribution P_{X_1} with finite expectations.

5 Mixed premium calculation principles and change of measures

In this section, we discuss implications of our results for the computation of premium calculation principles in a model of an insurance market possessing the property (NFLVR). In this context, the financial pricing of insurance (FPI for short) approach proposed by Delbaen and Haezendonck [8] plays a key role.

Let $T > 0$. According to the FPI approach, the liabilities of an insurance company over a fixed period of time $\mathbb{T} := [0, T]$ can be represented as a price process $U_{\mathbb{T}} := \{U_t\}_{t\in\mathbb{T}}$ defined by $U_t := p_t + S_t$ for any $t \in \mathbb{T}$, where S_t represents the total amount of claims paid up to time t and p_t represents the total premium for the remaining risk $S_T - S_t$. Under the assumption that the random behavior of the price process $U_{\mathbb{T}}$ is described by the given probability measure P on Σ, and that the insurance market is liquid enough (see [8], Section 1, for more details) by applying the Harrison–Kreps theory (see Harrison and Kreps [15]) it follows that the existence of a 2-martingale measure Q on Σ for $U_{\mathbb{T}}$ which is equivalent to P implies the elimination of arbitrage opportunities in the insurance market and vice versa (see [8], Section 1). However, as the insurance market is not, in general, complete (see, e.g., [11], page 20, or [31], Section 4) the measure Q is not unique; hence the next step should be the selection of such a measure Q.

Under the assumption that the aggregate process S is a P-CPP, Delbaen and Haezendonck [8] were interested in all those measures Q with linear premiums of the form

$$p_t = (T - t) \cdot p(Q) \quad \text{for any } \ t \in \mathbb{T},$$

where $p(Q)$ is the premium density under Q. But as $U_{\mathbb{T}}$ is a martingale under Q if and only if $U_{\mathbb{T}} - p_0 = \{S_t - p(Q) \cdot t\}_{t\in\mathbb{T}}$ is such, Delbaen and Haezendonck faced the problem of characterizing all those risk-neutral measures Q on Σ under which the compensator of $S_{\mathbb{T}}$ is a linear deterministic function of $t \in \mathbb{T}$. As the linearity of the premiums implies that $S_{\mathbb{T}}$ is a CPP under Q and vice versa (see [8], Section 1), the above problem is equivalent to the following one:

If S is a CPP under P, then characterize all progressively equivalent to P probability measures Q on Σ such that S remains a CPP under Q.

Recall that under the FPI framework, a **premium calculation principle** (written PCP for short) is a probability measure Q on Σ which is progressively equivalent to P, the process S is a Q-CPP and $X_1 \in \mathcal{L}^1(Q)$, see [8], Definition 3.1. If the distribution Q_Θ is not degenerate, then the probability measure $Q \in \mathcal{M}^{*,1}_{S,\mathbf{Exp}(\rho(\Theta))}$ constructed in Theorem 1(ii) fails to be a PCP. Nevertheless, by virtue of Theorem 1(iii) there exists an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ such that for P_Θ-a.a. $\theta \in D$ the probability measures Q_θ are PCPs. Thus, it seems natural to call every probability measure $Q \in \mathcal{M}^{*,1}_{S,\mathbf{Exp}(\rho(\Theta))}$ a **mixed PCP**.

In order for a mixed PCP to provide a realistic and viable pricing framework it should give more weight to unfavorable events in a risk-averse environment, i.e., conditions

$$p(P, \Theta) < p(Q, \Theta) < \infty \quad P\restriction\sigma(\Theta)\text{-a.s..} \tag{10}$$

and

$$p(P) < p(Q) < \infty \tag{11}$$

must hold true.

Remark 9. Let $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ and let $\{Q_\theta\}_{\theta \in D}$ be the rcp of Q over Q_Θ consistent with Θ appearing in Theorem 1(iii). The following statements are equivalent:

(i) $p(P, \Theta) < p(Q, \Theta) < \infty \quad P\restriction\sigma(\Theta)$-a.s.;

(ii) there exists a P_Θ-null set $\widetilde{M}_{P,Q} \in \mathfrak{B}(D)$ containing the P_Θ-null sets $\widetilde{L}_{**}$ and $W_{P,1} \cup W_{Q,1}$ appearing in Theorem 1(iii) and Remark 4, respectively, such that $Q_\theta \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\theta))}$ and $p(P_\theta) < p(Q_\theta) < \infty$ for any $\theta \notin \widetilde{M}_{P,Q}$.

In fact, first note that by [18], Lemma 3.5(i), there exists a P_Θ-null set $M_{P,Q} \in \mathfrak{B}(D)$ such that statement (i) holds if and only if $p(P, \theta) < p(Q, \theta) < \infty$ for all $\theta \notin M_{P,Q}$. Putting $\widetilde{M}_{P,Q} := \widetilde{L}_{**} \cup M_{P,Q} \cup W_{P,1} \cup W_{Q,1} \in \mathfrak{B}(D)$ and applying Theorem 1(iii) and Remark 4 we infer that $p(P_\theta) < p(Q_\theta) < \infty$ for all $\theta \notin \widetilde{M}_{P,Q}$, i.e., (i)⇔(ii).

However, the existence of a mixed PCP does not, in general, guarantee the validity of conditions (10) and (11) as the next two examples demonstrate. In the first example, we construct a mixed PCP that does not satisfy condition (10) and leads to a P-a.s. ruin.

Example 3. Let $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$ and $D := (0, \infty)$. Consider the pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$ with $\beta \le 0$. Applying Theorem 1(ii) and (v) we obtain a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, and such that Q is an ℓ-martingale measure for the process $V(\Theta)$; hence for the reserve process $R^u(\Theta)$ (see Definition 2) with $c(\Theta) = p(Q, \Theta) = \frac{\mathbb{E}_P[X_1 \cdot e^{\beta(X_1,\Theta)} | \Theta]}{\mathbb{E}_P[W_1 | \Theta]}$ $P\restriction\sigma(\Theta)$-a.s. However, for this particular choice of mixed PCP one has that $p(Q, \Theta) < p(P, \Theta)$ $P\restriction\sigma(\Theta)$-a.s. which according to Lemma 2 leads to a P-a.s. ruin.

In the next example, we construct a mixed PCP for which condition (10) holds but condition (11) fails.

Example 4. Let $D := (0, \infty)$ and $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\Theta)}$ with $P_\Theta = \mathbf{Exp}(1)$. Consider the pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$ with $\beta(x, \theta) := \ln 2$ for all $(x, \theta) \in D \times D$ and $\xi(\theta) := 4 \cdot e^{-3\cdot\theta}$ for any $\theta \in D$. By Theorem 1(ii) and (v) we have that conditions $(*)$ and (RPM_ξ) determine a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{Exp}(\rho(\Theta))}$, satisfying the conclusions of the statement (ii) of this theorem, and such that the reserve process $R^u(\Theta)$, with $c(\Theta) = p(Q, \Theta) = 2 \cdot \Theta \cdot \mathbb{E}_P[X_1]$ $P\restriction\sigma(\Theta)$-a.s., is a martingale in $\mathcal{L}^\ell(Q)$. Even though the conditional premium densities satisfy condition (10), the corresponding mixed premium densities satisfy the reverse inequality since $p(Q) = \frac{\mathbb{E}_P[X_1]}{2}$ and $p(P) = \mathbb{E}_P[X_1]$.

Examples 3 and 4 raise the question when a mixed PCP satisfies conditions (10) and (11) or the implication (10)⇒(11). In the next proposition we find sufficient conditions for the validity of the implication (10)⇒(11). To prove it, we need the following lemma, which is a consequence of Schmidt [29], Theorem 2.2, but we write it exactly in the form needed for our purposes.

Lemma 4. *Let (Ω, Σ, P) be an arbitrary probability space. If $Z : \Omega \to \mathbb{R}$ is a random variable, $J \in \mathfrak{B}$ is a Borel set satisfying $P(\{Z \in J\}) = 1$, and $f, g : J \to \mathbb{R}$ are monotonic functions of the same monotonicity which are either positive or for which $f(Z), g(Z), f(Z) \cdot g(Z) \in \mathcal{L}^1(P)$, then*

$$\mathbb{E}_P\big[f(Z) \cdot g(Z)\big] \geq \mathbb{E}_P\big[f(Z)\big] \cdot \mathbb{E}_P\big[g(Z)\big].$$

If the functions f, g have different monotonicity, then the opposite inequality is valid.

Proposition 3. *Let $D := (0, \infty)$, $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$, $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ and let $\xi \in \mathcal{R}^{*,\ell}_+(D)$, $\widetilde{L}_{**}$, $\{Q_\theta\}_{\theta \in D}$ be as in Theorem 1. If for all $\theta \notin \widetilde{L}_{**}$ the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ are monotonic of the same monotonicity, then*

(i) $p(P_\theta) \leq p(Q_\theta)$ for P_Θ-a.a. $\theta \in D$ implies $p(P) \leq p(Q) < \infty$.

(ii) $p(P_\theta) < p(Q_\theta)$ for P_Θ-a.a. $\theta \in D$ implies $p(P) < p(Q) < \infty$.

Proof. First note that given $P \in \mathcal{M}^{*,\ell}_{S,\mathbf{K}(\Theta)}$ and $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ the existence of a function $\xi \in \mathcal{R}^{*,\ell}_+(D)$ follows by Theorem 1(i), according to which there exists an essentially unique pair $(\beta, \xi) \in \mathcal{F}^{\ell}_{P,\Theta} \times \mathcal{R}^{*,\ell}_+(D)$ satisfying among others condition $(*)$.

Ad (i): Since for all $\theta \notin \widetilde{L}_{**}$ the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ are monotonic of the same monotonicity, if $p(P_\theta) \leq p(Q_\theta)$ for P_Θ-a.a. $\theta \in D$ we get

$$\begin{aligned} p(P) = \mathbb{E}_{P_\Theta}\big[p(P_\theta)\big] &\leq \mathbb{E}_{P_\Theta}\big[p(Q_\theta)\big] = \mathbb{E}_{Q_\Theta}\Big[p(Q_\theta) \cdot \big(\xi(\theta)\big)^{-1}\Big] \\ &\leq \mathbb{E}_{Q_\Theta}\big[p(Q_\theta)\big] \cdot \mathbb{E}_{Q_\Theta}\Big[\big(\xi(\theta)\big)^{-1}\Big] = \mathbb{E}_{Q_\Theta}\big[p(Q_\theta)\big] < \infty, \end{aligned}$$

where the second inequality follows by Lemma 4, the last equality is a consequence of the fact that ξ is a Radon–Nikodym derivative of Q_Θ with respect to P_Θ, and the last inequality follows by $p(Q_\bullet) \in \mathcal{L}^1(Q_\Theta)$; hence $p(P) \leq p(Q) < \infty$.

Ad (ii): If $p(P_\theta) < p(Q_\theta)$ for P_Θ-a.a. $\theta \in D$, then $\mathbb{E}_{P_\Theta}[p(P_\theta)] < \mathbb{E}_{P_\Theta}[p(Q_\theta)]$. The latter along with the arguments of the proof of (i) yields condition $p(P) = \mathbb{E}_{P_\Theta}[p(P_\theta)] < \mathbb{E}_{P_\Theta}[p(Q_\theta)] = p(Q) < \infty$. This completes the proof. □

6 Examples

In this section, applying our results, we provide some examples to show how to construct mixed PCPs Q satisfying conditions (10) and (11) and such that for any $T > 0$ the processes $V_{\mathbb{T}}(\Theta)$ and $R^u_{\mathbb{T}}(\Theta)$, for $u \in \mathbb{R}_+$, appearing in Theorem 2 and Remark 8, respectively, have the property (NFLVR). Moreover, we provide explicit formulas for the ruin probability for the reserve process $R^u(\Theta)$ with respect to the original measure P.

Example 5. Take $D := (1, \infty)^2$, and let $\Theta = (\Theta_1, \Theta_2)$ be a D-valued random vector on Ω with $\Theta_1, \Theta_2 \in \mathcal{L}^1(P)$. Moreover, assume that $P \in \mathcal{M}^{*,2}_{S,\mathbf{K}(\Theta)}$ with $P_{X_1} = \mathbf{Ga}(\zeta, 2)$ for a real constant $\zeta > 0$, and

$$\mathbf{K}(\Theta) := \frac{1}{2} \cdot \mathbf{Exp}(1/\Theta_1) + \frac{1}{2} \cdot \mathbf{Exp}(1/\Theta_2),$$

Consider the real-valued function β on $(0,\infty)\times D$ with $\beta(x,\theta) := \gamma(x)+\alpha(\theta)$ for all $(x,\theta)\in(0,\infty)\times D$, where $\gamma(x) := \ln\frac{\mathbb{E}_P[X_1]}{2c} - \ln x + \frac{2(c-1)}{c\cdot\mathbb{E}_P[X_1]}\cdot x$, for a real constant $c>2$, and $\alpha(\theta) := 0$. Applying standard computations, we obtain that $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$, $\mathbb{E}_P[X_1\cdot e^{\gamma(X_1)}] = \frac{c}{\zeta} < \infty$ and $\mathbb{E}_P[X_1^2\cdot e^{\gamma(X_1)}] = \frac{2c^2}{\zeta^2} < \infty$, implying $\beta\in\mathcal{F}^2_{P,\Theta}$. Let $\xi\in\mathfrak{M}_+(D)$ be defined by $\xi(\theta) := \xi(\theta_1,\theta_2) := 1$ for any $\theta\in D$. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying $\xi\in\mathcal{R}_+(D)$, while $P\in\mathcal{M}^{*,2}_{S,\mathbf{K}(\Theta)}$ yields

$$\mathbb{E}_P\left[\left(\frac{1}{\mathbb{E}_P[W_1\mid\Theta]}\right)^2\right] = \mathbb{E}_P\left[\left(\frac{2}{\Theta_1+\Theta_2}\right)^2\right] < \infty,$$

implying, along with $\xi(\Theta)=1$, that $\xi\in\mathcal{R}^{*,2}_+(D)$.

(a) Since $(\beta,\xi)\in\mathcal{F}^2_{P,\Theta}\times\mathcal{R}^{*,2}_+(D)$, it follows by Theorem 1 that there exist a unique pair $(\rho,Q)\in\mathfrak{M}_+(D)\times\mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ, and a P_Θ-null set $\widetilde{L}_{**}\in\mathfrak{B}(D)$ such that for any $\theta\notin\widetilde{L}_{**}$ conditions $Q_\theta\in\mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $\widetilde{(*)}$ and (RPM_θ) hold true. It then follows that

$$Q_{X_1}(A) = \mathbb{E}_P\left[\mathbb{1}_{X_1^{-1}[A]}\cdot e^{\gamma(X_1)}\right] = \int_A \frac{\zeta}{c}\cdot e^{-\frac{\zeta}{c}\cdot x}\,\lambda(dx)\quad\text{for any } A\in\mathfrak{B}(0,\infty),$$

and

$$Q_\Theta(B) = \mathbb{E}_P\left[\mathbb{1}_{\Theta^{-1}[B]}\cdot\xi(\Theta)\right] = P_\Theta(B)\quad\text{for any } B\in\mathfrak{B}(D), \tag{12}$$

while condition $(*)$ yields $\rho(\Theta) = \frac{2}{\Theta_1+\Theta_2}$ $P\restriction\sigma(\Theta)$-a.s. Thus, for any $\theta\notin\widetilde{L}_{**}$ the probability measure Q_θ is a PCP satisfying condition

$$p(P_\theta) = \frac{4}{\zeta\cdot(\theta_1+\theta_2)} < \frac{2\cdot c}{\zeta\cdot(\theta_1+\theta_2)} = p(Q_\theta) < \infty; \tag{13}$$

hence condition (10) holds by Remark 9. Conditions (12) and (13) imply condition (11).

(b) By Theorem 1(v) and (iii), the measure Q is a 2-martingale measure for the process $V(\Theta)$ with

$$V_t(\Theta) = S_t - t\cdot\rho(\Theta)\cdot\mathbb{E}_P\left[X_1\cdot e^{\gamma(X_1)}\right] = S_t - t\cdot\frac{2\cdot c}{\zeta\cdot(\Theta_1+\Theta_2)} \tag{14}$$

for every $t\in\mathbb{R}_+$, while for all $\theta\notin\widetilde{L}_{**}$ the probability measure Q_θ is a 2-martingale measure for the process $V(\theta)$ with $V_t(\theta) = S_t - t\cdot\frac{2\cdot c}{\zeta\cdot(\theta_1+\theta_2)}$ for any $t\in\mathbb{R}_+$, respectively. In particular, for any $T>0$, Theorem 2 asserts that both processes $V_{\mathbb{T}}(\Theta)$ and $V_{\mathbb{T}}(\theta)$ satisfy condition (NFLVR).

(c) Consider the reserve process $R^u(\Theta) := u - V(\Theta)$ $(u\in\mathbb{R}_+)$. First note that the equality $c(\Theta) = p(Q,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., together with (a), implies that the pair (P,Θ) satisfies condition (5); hence by Theorem 3(i) we get that condition (ruin(P)) is valid, implying

$$\psi(u) = \mathbb{E}_Q\left[C_1(N_{T_u(\Theta)},W,X,\Theta)\cdot e^{\frac{\zeta(c-1)}{c}\cdot S_{T_u(\Theta)}+\rho(\Theta)\cdot T_u(\Theta)}\right],$$

where

$$C_1(N_{T_u(\Theta)}, W, X, \Theta) := \prod_{j=1}^{N_{T_u(\Theta)}} \frac{c\cdot\zeta}{\rho(\Theta)} \cdot X_j \cdot \left(\frac{1}{2\Theta_1}\cdot e^{-\frac{W_j}{\Theta_1}} + \frac{1}{2\Theta_2}\cdot e^{-\frac{W_j}{\Theta_2}}\right).$$

In our next example we rediscover Shaun Wang's risk-adjusted premium principle (see [35], for the definition and its properties).

Example 6. Let $D := (0,\infty)$ and assume that $P \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\Theta)}$ such that P_{X_1} is absolutely continuous with respect to the Lebesgue measure λ on $\mathfrak{B}$ restricted to $\mathfrak{B}(D)$. Denote by $\overline{F}_{X_1}(x) := P_{X_1}((x,\infty))$ for any $x \in D$ the corresponding survival function of the random variable X_1, and assume that $\int_0^\infty x\cdot(\overline{F}_{X_1}(x))^{\frac{1}{c}}\,\lambda(dx) < \infty$, where $c > 1$ is a real constant. Recall that the **risk adjusted premium** for X_1 is defined by

$$\pi_c(X_1) := \int_0^\infty \left(\overline{F}_{X_1}(x)\right)^{\frac{1}{c}}\lambda(dx) \quad \text{for any } c \ge 1.$$

(see [35], Definition 2).

Consider the real-valued function β with $\beta(x,\theta) := \gamma(x) + \alpha(\theta)$ for all $(x,\theta) \in D\times D$, where $\gamma(x) := -\ln c + (\frac{1}{c} - 1)\cdot \ln \overline{F}_{X_1}(x)$ and $\alpha(\theta) := 0$. By standard computations, we get $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$, $\mathbb{E}_P[X_1\cdot e^{\gamma(X_1)}] = \pi_c(X_1)$ and $\mathbb{E}_P[X_1^2\cdot e^{\gamma(X_1)}] = 2\cdot\int_0^\infty x\cdot(\overline{F}_{X_1}(x))^{\frac{1}{c}}\,\lambda(dx) < \infty$, implying $\pi_c(X_1) < \infty$ and $\beta \in \mathcal{F}^2_{P,\Theta}$. For any $r \in [0, r_\Theta)$, $r_\Theta := \sup\{r \ge 0 : M_\Theta(r) < \infty\}$ and $M_\Theta := M_{P_\Theta}$ is the moment generating function of Θ, define the function $\xi \in \mathfrak{M}_+(D)$ by means of $\xi(\theta) := \frac{e^{r\cdot\theta}}{M_\Theta(r)}$ for any $\theta \in D$. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying that $\xi \in \mathcal{R}_+(D)$. Since

$$\mathbb{E}_P\left[\xi(\Theta)\cdot\left(\frac{e^{\alpha(\Theta)}}{\mathbb{E}_P[W_1 \mid \Theta]}\right)^2\right] = \mathbb{E}_P\left[\xi(\Theta)\cdot\Theta^2\right] = \frac{\mathbb{E}_P[\Theta^2\cdot e^{r\cdot\Theta}]}{M_\Theta(r)} = \frac{M''_\Theta(r)}{M_\Theta(r)} < \infty$$

for all $r \in [0, r_\Theta)$, we get $\xi \in \mathcal{R}^{*,2}_+(D)$.

(a) Since $(\beta,\xi) \in \mathcal{F}^2_{P,\Theta} \times \mathcal{R}^{*,2}_+(D)$, we may apply Theorem 1 in order to get a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ, and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$ satisfying for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\circledast)$ and (RPM_θ). We then get

$$Q_{X_1}(A) = \mathbb{E}_P\left[\mathbb{1}_{X_1^{-1}[A]}\cdot e^{\gamma(X_1)}\right] = \int_A \frac{1}{c}\cdot\left(\overline{F}(x)\right)^{\frac{1}{c}-1} P_{X_1}(dx)$$

for any $A \in \mathfrak{B}(D)$, and

$$Q_\Theta(B) = \mathbb{E}_P\left[\mathbb{1}_{\Theta^{-1}[B]}\cdot\xi(\Theta)\right] = \int_B \frac{e^{r\cdot\theta}}{M_\Theta(r)}\,P_\Theta(d\theta)$$

for any $B \in \mathfrak{B}(D)$ and $r \in [0, r_\Theta)$, whilst condition $(*)$ implies that $\rho(\Theta) = \Theta$ $P\restriction\sigma(\Theta)$-a.s.; hence for any $\theta \notin \widetilde{L}_{**}$ the corresponding measure Q_θ is a PCP satisfying condition

$$p(P_\theta) = \theta\cdot\int_0^\infty \overline{F}(x)\,\lambda(dx) < \theta\cdot\int_0^\infty \left(\overline{F}(x)\right)^{\frac{1}{c}}\lambda(dx) = \theta\cdot\pi_c(X_1) = p(Q_\theta) < \infty,$$

implying condition (10) by Remark 9. Since for all $\theta \notin \widetilde{L}_{**}$ the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ are monotonic of the same monotonicity, we may apply Proposition 3(ii) to conclude condition (11) with

$$p(Q) = \mathbb{E}_{Q_\Theta}\left[p(Q_\theta)\right] = \pi_c(X_1)\cdot\mathbb{E}_{Q_\Theta}[\theta] = \pi_c(X_1)\cdot\mathbb{E}_P\left[\Theta\cdot\xi(\Theta)\right] = \pi_c(X_1)\cdot\frac{M'_\Theta(r)}{M_\Theta(r)}$$

for all $r \in [0, r_\Theta)$.

(b) By Theorem 1(v) and (iii), the probability measure Q is a 2-martingale measure for the process $V(\Theta)$ with $V_t(\Theta) = S_t - t\cdot\rho(\Theta)\cdot\mathbb{E}_P[X_1\cdot e^{\gamma(X_1)}] = S_t - t\cdot\Theta\cdot\pi_c(X_1)$ for any $t \in \mathbb{R}_+$, and for any $\theta \notin \widetilde{L}_{**}$ the probability measure Q_θ is a 2-martingale measure for the process $V(\theta)$ with $V_t(\theta) = S_t - t\cdot\theta\cdot\pi_c(X_1)$ for any $t \in \mathbb{R}_+$, respectively. In particular, for any $T > 0$, Theorem 2 asserts that both processes $V_{\mathbb{T}}(\Theta)$ and $V_{\mathbb{T}}(\theta)$ satisfy condition (NFLVR).

(c) Consider the reserve process $R^u(\Theta) := u - V(\Theta)$ ($u \in \mathbb{R}_+$). The equality $c(\Theta) = p(Q,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., together with (a), implies that $c(\Theta) > p(P,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., i.e., condition (5) is valid. Thus, we can apply Theorem 3(i) in order to get that $\psi(u)$ admits the representation (ruin(P)), implying

$$\psi(u) = M_\Theta(r)\cdot\mathbb{E}_Q\left[e^{-r\cdot\Theta}\cdot c^{N_{T_u(\Theta)}}\cdot\prod_{j=1}^{N_{T_u(\Theta)}}\left(\overline{F}_{X_1}(X_j)\right)^{\frac{c-1}{c}}\right] \quad \text{for any } r \in [0, r_\Theta).$$

Example 7. Let $D := (0,\infty)$ and $P \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(1/\Theta)}$. Define the real-valued function β on $D\times D$ with $\beta(x,\theta) := \gamma(x) + \alpha(\theta)$ for all $(x,\theta) \in D\times D$, where $\gamma(x) := r\cdot x - \ln M_{X_1}(r)$, with $r \in [0, r_{X_1})$ and r_{X_1} being as in Example 1, and $\alpha(\theta) := 0$. By standard computations, we get $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$,

$$\mathbb{E}_P\left[X_1\cdot e^{\gamma(X_1)}\right] = \frac{\mathbb{E}_P[X_1\cdot e^{r\cdot X_1}]}{\mathbb{E}_P[e^{r\cdot X_1}]} = \frac{M'_{X_1}(r)}{M_{X_1}(r)} < \infty,$$

and

$$\mathbb{E}_P\left[X_1^2\cdot e^{\gamma(X_1)}\right] = \frac{\mathbb{E}_P[X_1^2\cdot e^{r\cdot X_1}]}{\mathbb{E}_P[e^{r\cdot X_1}]} = \frac{M''_{X_1}(r)}{M_{X_1}(r)} < \infty,$$

as $M'_{X_1}(r), M''_{X_1}(r) < \infty$ for all $r \in [0, r_{X_1})$; hence $\beta \in \mathcal{F}^2_{P,\Theta}$. Define the function $\xi \in \mathfrak{M}_+(D)$ by means of $\xi(\theta) := \frac{e^{-r\cdot\theta}}{\mathbb{E}_P[e^{-r\cdot\Theta}]}$ for all $r \in [0, r_{X_1})$ and $\theta \in D$. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying $\mathcal{R}_+(D)$, and

$$\mathbb{E}_P\left[\xi(\Theta)\cdot\left(\frac{e^{\alpha(\Theta)}}{\mathbb{E}_P[W_1\mid\Theta]}\right)^2\right] = \mathbb{E}_P\left[\xi(\Theta)\cdot\left(\frac{1}{\Theta}\right)^2\right] = \frac{\mathbb{E}_P[\frac{1}{\Theta^2}\cdot e^{-r\cdot\Theta}]}{\mathbb{E}_P[e^{-r\cdot\Theta}]}$$
$$\leq \frac{\mathbb{E}_P[\frac{1}{\Theta^2}]}{\mathbb{E}_P[e^{-r\cdot\Theta}]} < \infty,$$

where the last inequality follows by $P \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(1/\Theta)}$; hence $\xi \in \mathcal{R}^{*,2}_+(D)$.

(a) Since $(\beta,\xi) \in \mathcal{F}^2_{P,\Theta}\times\mathcal{R}^{*,2}_+(D)$, applying Theorem 1 we obtain a unique pair $(\rho, Q) \in \mathfrak{M}_+(D)\times\mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions ($*$) and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, an essentially unique rcp

$\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ, and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$ such that for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\circledast)$ and (RPM_θ) hold true. Consequently, we deduce that

$$Q_{X_1}(A) = \mathbb{E}_P\left[\mathbb{1}_{X_1^{-1}[A]} \cdot e^{\gamma(X_1)}\right] = \frac{\mathbb{E}_P[\mathbb{1}_{X_1^{-1}[A]} \cdot e^{r\cdot X_1}]}{M_{X_1}(r)}$$

for every $A \in \mathfrak{B}(D)$ and $r \in [0, r_{X_1})$, and

$$Q_\Theta(B) = \mathbb{E}_P\left[\mathbb{1}_{\Theta^{-1}[B]} \cdot \xi(\Theta)\right] = \frac{\mathbb{E}_P[\mathbb{1}_{\Theta^{-1}[B]} \cdot e^{-r\cdot\Theta}]}{\mathbb{E}_P[e^{-r\cdot\Theta}]}$$

for each $B \in \mathfrak{B}(D)$ and $r \in [0, r_{X_1})$, while condition $(*)$ yields $\rho(\Theta) = \frac{1}{\Theta}$ $P\restriction\sigma(\Theta)$-a.s. Thus, for any $\theta \notin \widetilde{L}_{**}$ the probability measure Q_θ is a PCP satisfying condition

$$p(P_\theta) = \frac{\mathbb{E}_P[X_1]}{\theta} < \frac{M'_{X_1}(r)}{\theta \cdot M_{X_1}(r)} = p(Q_\theta) < \infty, \tag{15}$$

for $r \in (0, r_{X_1})$. The inequalities hold true, since for the function $f : (0, r_{X_1}) \longrightarrow \mathbb{R}$ defined by $f(r) := \ln M_{X_1}(r)$ for all $r \in (0, r_{X_1})$, we have $f''(r) > 0$ for any $r \in (0, r_{X_1})$, or equivalently that f is strictly convex on $r \in (0, r_{X_1})$, which is equivalent to the fact that the function f', with $f'(r) = \frac{M'_{X_1}(r)}{M_{X_1}(r)} < \infty$ for $r \in (0, r_{X_1})$, is strictly increasing; hence $\mathbb{E}_P[X_1] < f'(r) < \infty$ for all $r \in (0, r_{X_1})$. As a result, condition (15), together with Remark 9, yields condition (10). Since for any $\theta \notin \widetilde{L}_{**}$ the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ are monotonic of the same monotonicity, we may apply Proposition 3(ii) in order to conclude condition (11) with

$$\begin{aligned} p(Q) &= \mathbb{E}_{Q_\Theta}\left[p(Q_\theta)\right] = \mathbb{E}_{Q_\Theta}\left[\frac{1}{\theta}\right] \cdot \frac{M'_{X_1}(r)}{M_{X_1}(r)} \\ &= \mathbb{E}_P\left[\frac{\xi(\Theta)}{\Theta}\right] \cdot \frac{M'_{X_1}(r)}{M_{X_1}(r)} = \frac{\mathbb{E}_P[\frac{1}{\Theta} \cdot e^{-r\cdot\Theta}]}{\mathbb{E}_P[e^{-r\cdot\Theta}]} \cdot \frac{M'_{X_1}(r)}{M_{X_1}(r)}. \end{aligned}$$

(b) Again by Theorem 1(v) and (iii), we get that the process $V(\Theta)$, with

$$V_t(\Theta) = S_t - t \cdot \rho(\Theta) \cdot \mathbb{E}_P\left[X_1 \cdot e^{\gamma(X_1)}\right] = S_t - t \cdot \frac{1}{\Theta} \cdot \frac{M'_{X_1}(r)}{M_{X_1}(r)} \quad \text{for any } t \in \mathbb{R}_+,$$

is a martingale in $\mathcal{L}^2(Q)$, and that for any $\theta \notin \widetilde{L}_{**}$ the probability measure Q_θ is a 2-martingale measure for the process $V(\theta)$ with $V_t(\theta) = S_t - t \cdot \frac{1}{\theta} \cdot \frac{M'_{X_1}(r)}{M_{X_1}(r)}$ for any $t \in \mathbb{R}_+$, respectively. In particular, for any $T > 0$, Theorem 2 asserts that both processes $V_{\mathbb{T}}(\Theta)$ and $V_{\mathbb{T}}(\theta)$ satisfy condition (NFLVR).

(c) Consider the reserve process $R^u(\Theta) := u - V(\Theta)$ $(u \in \mathbb{R}_+)$. Since $c(\Theta) = p(Q,\Theta) > p(P,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., where the inequality follows by (a), we deduce that condition (5) is valid; hence we get by Theorem 3(i) that Q is a 2-martingale measure for the reserve process $R^u(\Theta)$, ruin occurs Q-a.s. and condition (ruin(P)) holds true, implying

$$\psi(u) = \mathbb{E}_P\left[e^{-r\cdot\Theta}\right] \cdot \mathbb{E}_Q\left[e^{-r\cdot S_{T_u(\Theta)} + r\cdot\Theta + N_{T_u(\Theta)}\cdot \ln M_{X_1}(r)}\right] \quad \text{for any } r \in (0, r_{X_1}).$$

Example 8. Assume that $D := (0,\infty)$ and $P \in \mathcal{M}^{*,2}_{S,\mathbf{Ga}(\Theta,k)}$ for a real constant $k > 0$, such that $P_{X_1} = \mathbf{Exp}(\eta)$, where $\eta > 0$ is a real constant, and $P_\Theta = \mathbf{Ga}(b_1, a)$, where $b_1, a > 0$ are real constants. Consider the real-valued function β on $D \times D$ with $\beta(x,\theta) = \gamma(x) + \alpha(\theta)$ for all $(x,\theta) \in D \times D$, where $\gamma(x) := \ln(1 - c \cdot \mathbb{E}_P[X_1]) + c \cdot x$ with $c < \eta$ a positive real constant, and $\alpha(\theta) := \ln(\frac{\theta}{b} \cdot \mathbb{E}_{P_\theta}[W_1])$, where $b < k$ is a positive constant. It can be easily seen that $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$, $\mathbb{E}_P[X_1 \cdot e^{\gamma(X_1)}] = \frac{1}{\eta - c} < \infty$ and $\mathbb{E}_P[X_1^2 \cdot e^{\gamma(X_1)}] = \frac{2}{(\eta-c)^2} < \infty$, implying $\beta \in \mathcal{F}^2_{P,\Theta}$. Let $\xi \in \mathfrak{M}_+(D)$ be defined by $\xi(\theta) := (\frac{b_2}{b_1})^a \cdot e^{-(b_2 - b_1)\cdot\theta}$ for any $\theta \in D$, where b_2 is a positive constant such that $b_2 < b_1$. Clearly $\mathbb{E}_P[\xi(\Theta)] = 1$, implying that $\xi \in \mathcal{R}_+(D)$. Applying standard computations we get

$$\mathbb{E}_P\left[\xi(\Theta) \cdot \left(\frac{e^{\alpha(\Theta)}}{\mathbb{E}_P[W_1 \mid \Theta]}\right)^2\right] = \mathbb{E}_P\left[\xi(\Theta) \cdot \left(\frac{e^{\ln(\frac{\Theta}{b} \cdot \mathbb{E}_P[W_1|\Theta])}}{\mathbb{E}_P[W_1 \mid \Theta]}\right)^2\right] = \frac{\mathbb{E}_P[\xi(\Theta) \cdot \Theta^2]}{b^2} < \infty,$$

implying that $\xi \in \mathcal{R}^{*,2}_+(D)$.

(a) Since $(\beta,\xi) \in \mathcal{F}^2_{P,\Theta} \times \mathcal{R}^{*,2}_+(D)$, it follows by Theorem 1 that there exist a unique pair $(\rho, Q) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, an essentially unique rcp $\{Q_\theta\}_{\theta\in D}$ of Q over Q_Θ consistent with Θ, and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$ such that for any $\theta \notin \widetilde{L}_{**}$ conditions $Q_\theta \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $(\widetilde{*})$ and (RPM_θ) hold true. Therefore, we deduce that

$$Q_{X_1}(A) = \mathbb{E}_P\left[\mathbb{1}_{X_1^{-1}[A]} \cdot e^{\gamma(X_1)}\right] = \int_A (\eta - c) \cdot e^{-(\eta-c)\cdot x}\,\lambda(dx) \quad \text{for every } A \in \mathfrak{B}(D)$$

and

$$Q_\Theta(B) = \mathbb{E}_P\left[\mathbb{1}_{\Theta^{-1}[B]} \cdot \xi(\Theta)\right] = \int_B \frac{b_2^a}{\Gamma(a)} \cdot \theta^{a-1} \cdot e^{-b_2\cdot\theta}\,\lambda(d\theta) \quad \text{for every } B \in \mathfrak{B}(D),$$

implying that $Q_{X_1} = \mathbf{Exp}(\eta - c)$ and $Q_\Theta = \mathbf{Ga}(b_2, a)$, respectively. Furthermore, condition $(*)$ yields $\rho(\Theta) = \frac{\Theta}{b}$ $P\restriction\sigma(\Theta)$-a.s.; hence for any $\theta \notin \widetilde{L}_{**}$ the probability measure Q_θ is a PCP satisfying condition

$$p(P_\theta) = \frac{\theta}{k \cdot \eta} < \frac{\theta}{b \cdot (\eta - c)} = p(Q_\theta) < \infty.$$

Thus, we may apply Remark 9 in order to conclude condition (10), and since for all $\theta \notin \widetilde{L}_{**}$ the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ are monotonic of the same monotonicity, we may apply Proposition 3(ii) in order to conclude condition (11) with

$$p(Q) = \mathbb{E}_{Q_\Theta}\left[p(Q_\theta)\right] = \mathbb{E}_{Q_\Theta}\left[\frac{\theta}{b \cdot (\eta - c)}\right] = \frac{a}{b_2 \cdot b \cdot (\eta - c)}.$$

(b) Again by Theorem 1(v) and (iii), the probability measure Q is a 2-martingale measure for the process $V(\Theta)$ with

$$V_t(\Theta) = S_t - t \cdot \rho(\Theta) \cdot \mathbb{E}_P\left[X_1 \cdot e^{\gamma(X_1)}\right] = S_t - t \cdot \frac{\Theta}{b \cdot (\eta - c)}$$

for any $t \in \mathbb{R}_+$, and for any $\theta \notin \widetilde{L}_{**}$ the probability measure Q_θ is a 2-martingale measure for the process $V(\theta)$ with $V_t(\theta) = S_t - t \cdot \frac{\theta}{b\cdot(\eta-c)}$ for any $t \in \mathbb{R}_+$, respectively. In particular, for any $T > 0$, Theorem 2 asserts that both processes $V_{\mathbb{T}}(\Theta)$ and $V_{\mathbb{T}}(\theta)$ satisfy condition (NFLVR).

(c) Consider the reserve process $R^u(\Theta) := u - V(\Theta)$ $(u \in \mathbb{R}_+)$. The equality $c(\Theta) = p(Q,\Theta)$ $P\restriction\sigma(\Theta)$-a.s., together with (a), implies that condition (5) holds true. This allows us to apply Theorem 3(i) to conclude that $\psi(u)$ admits the representation (ruin(P)), implying

$$\psi(u) = \mathbb{E}_Q\left[C_2(N_{T_u(\Theta)}, W, \Theta) \cdot e^{-(b_1-b_2)\cdot\Theta - c\cdot S_{T_u(\Theta)} + T_u(\Theta)\cdot(\Theta-\rho(\Theta))}\right],$$

where

$$C_2(N_{T_u(\Theta)}, W, \Theta) := \left(\frac{b_1}{b_2}\right)^a \cdot \left(\prod_{j=1}^{N_{T_u(\Theta)}} \frac{\eta\cdot\Gamma(k)\cdot\rho(\Theta)}{(\eta-c)\cdot\Theta^k\cdot W_j^{k-1}}\right).$$

The following counter-example shows that the assumption of the same monotonicity of the functions $\theta \mapsto p(Q_\theta)$ and $\theta \mapsto \xi(\theta)$ for all $\theta \notin \widetilde{L}_{**}$ is essential for the validity of the conclusion $p(P) \le p(Q)$ in Proposition 3.

Counter-example 9. In the situation of Example 8, replace $\xi \in \mathfrak{M}_+(D)$ with the function $\widetilde{\xi} \in \mathfrak{M}_+(D)$ defined by $\widetilde{\xi}(\theta) := (\frac{\widetilde{b}_2}{b_1})^a \cdot e^{-(\widetilde{b}_2-b_1)\cdot\theta}$ for any $\theta \in D$, where $\widetilde{b}_2$ is a real constant satisfying $\widetilde{b}_2 > \frac{b_1\cdot k\cdot\eta}{b\cdot(\eta-c)}$. Similarly to Example 8 we get $(\beta,\widetilde{\xi}) \in \mathcal{F}^2_{P,\Theta} \times \mathcal{R}^{*,2}_+(D)$, and so we may apply Theorem 1 in order to obtain a unique pair $(\rho,\widetilde{Q}) \in \mathfrak{M}_+(D) \times \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\Theta))}$ determined by conditions $(*)$ and (RPM_ξ), satisfying the conclusions of the statement (ii) of this theorem, an essentially unique rcp $\{\widetilde{Q}_\theta\}_{\theta\in D}$ of $\widetilde{Q}$ over $\widetilde{Q}_\Theta$ consistent with Θ, and a P_Θ-null set $\widetilde{L}_{**} \in \mathfrak{B}(D)$ such that for any $\theta \notin \widetilde{L}_{**}$ conditions $\widetilde{Q}_\theta \in \mathcal{M}^{*,2}_{S,\mathbf{Exp}(\rho(\theta))}$, $\widetilde{(*)}$ and (RPM_θ) hold true. Similarly to Example 8, get $\rho(\Theta) = \frac{\Theta}{b}$ $P\restriction\sigma(\Theta)$-a.s., $\widetilde{Q}_{X_1} = \mathbf{Exp}(\eta-c)$ and $\widetilde{Q}_\Theta = \mathbf{Ga}(\widetilde{b}_2, a)$; hence for any $\theta \notin \widetilde{L}_{**}$ the probability measure $\widetilde{Q}_\theta$ is a PCP satisfying condition $p(P_\theta) = \frac{\theta}{k\cdot\eta} < \frac{\theta}{b\cdot(\eta-c)} = p(\widetilde{Q}_\theta) < \infty$; hence condition (10) holds by Remark 9. Easy computations show that

$$p(P) = \int_D p(P_\theta)\, P_\Theta(d\theta) = \frac{a}{b_1\cdot k\cdot\eta}$$

and

$$p(\widetilde{Q}) = \int_D p(\widetilde{Q}_\theta)\, Q_\Theta(d\theta) = \frac{a}{\widetilde{b}_2\cdot b\cdot(\eta-c)},$$

implying $p(P) > p(\widetilde{Q})$; hence the conclusions (i) and (ii) of Proposition 3 fail. Note that all assumptions of this proposition except for that of the same monotonicity of the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(\widetilde{Q}_\theta)$ for all $\theta \notin \widetilde{L}_{**}$ are satisfied, since the function $\theta \mapsto \xi(\theta)$ is strictly decreasing, while $\theta \mapsto p(\widetilde{Q}_\theta)$ is strictly increasing. As a consequence, we infer that the assumption of the same monotonicity of the functions $\theta \mapsto \xi(\theta)$ and $\theta \mapsto p(Q_\theta)$ is essential for the validity of the conclusion of Proposition 3.

Note that even if $p(P_\theta) = p(\widetilde{Q}_\theta)$ for any $\theta \notin \widetilde{L}_{**}$, i.e., whenever $c = 0$ and $b = k$, the equality $p(P) = p(\widetilde{Q})$ fails.

A Appendix: A list of symbols

Notations:

P-CMRP$(\mathbf{K}(\Theta), P_{X_1})$ P-compound mixed renewal process with parameters $\mathbf{K}(\Theta)$ and P_{X_1}, **see** Section 2, page 4;

P-CMPP(Θ, P_{X_1}) P-compound mixed Poisson process with parameters Θ and P_{X_1}, **see** Section 2, page 4;

P-CRP$(\mathbf{K}(\theta_0), P_{X_1})$ P-compound renewal process with parameters $\mathbf{K}(\theta_0)$ and P_{X_1}, **see** Section 2, page 4;

P-CPP(θ_0, P_{X_1}) P-compound Poisson process with parameters θ_0 and P_{X_1}, **see** Section 2, page 5;

rcp regular conditional probability, **see** Section 2, page 5;

$\mathcal{F}^S := \{\mathcal{F}_t^S\}_{t\in\mathbb{R}_+}$ the canonical filtration of the aggregate claims process S, **see** Section 2, page 5;

$\mathcal{F} := \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ the canonical filtration of S and Θ, **see** Section 2, page 5;

(NFLVR) no free lunch with vanishing risk, **see** Section 3, page 8;

(PEMM) progressively equivalent martingale measure, **see** Section 3, page 8;

PCP premium calculation principle, **see** Section 5, page 23.

Assumptions (see Section 2, page 5)**:**

Assumption **(a1)** The processes W and X are P-conditionally mutually independent.

Assumption **(a2)** The random vector Θ and the process X are P-(unconditionally) independent.

Classes of functions:

$\mathfrak{M}^k(D)$**:** The class of all $\mathfrak{B}(D)$-$\mathfrak{B}_k$-measurable functions on D ($k \in \mathbb{N}$), **see** Notations 1;

$\mathfrak{M}(D)$**:** The class of all $\mathfrak{B}(D)$-$\mathfrak{B}$-measurable functions on D, **see** Notations 1;

$\mathfrak{M}_+(D)$**:** The class of all $\mathfrak{B}(D)$-$\mathfrak{B}(0,\infty)$-measurable functions on D, **see** Notations 1;

$\mathcal{F}_{P,\Theta}$**:** The class of all real-valued $\mathfrak{B}((0,\infty)\times D)$-measurable functions β on $(0,\infty)\times D$, defined by $\beta(x,\theta) := \gamma(x)+\alpha(\theta)$ for any $(x,\theta)\in(0,\infty)\times D$, where $\alpha \in \mathfrak{M}(D)$ and γ is a real-valued $\mathfrak{B}(0,\infty)$-measurable function satisfying condition $\mathbb{E}_P[e^{\gamma(X_1)}] = 1$, **see** Notations 1(a);

$\mathcal{F}_{P,\Theta}^{\ell}$**:** The class of all $\beta \in \mathcal{F}_{P,\Theta}$ such that $\mathbb{E}_P[X_1^{\ell}\cdot e^{\gamma(X_1)}] < \infty$ ($\ell \in \{1,2\}$), **see** Notations 1(a);

$\mathcal{R}_+(D)$**:** The class of all $\xi \in \mathfrak{M}(D)$ such that $P_\Theta(\{\xi > 0\}) = 1$ and $\mathbb{E}_P[\xi(\Theta)] = 1$, **see** Notations 1(b);

$\mathcal{R}_+^{*,\ell}(D)$**:** The class of all $\xi \in \mathcal{R}_+(D)$ such that $\xi(\Theta) \cdot (\frac{e^{\alpha(\Theta)}}{\mathbb{E}_P[W_1|\Theta]})^\ell \in \mathcal{L}^1(P)$ $(\ell \in \{1,2\})$, **see** Notations 2(a).

Classes of measures:

$\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$**:** The class of all probability measures Q on Σ such that:

(i) conditions (a1) and (a2) holds true,

(ii) are progressively equivalent to P,

(iii) S is a Q-CMRP$(\mathbf{\Lambda}(\rho(\Theta)), Q_{X_1})$,

see Notations 1(c);

$\mathcal{M}^{\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$**:** The class of all $Q \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ with $\mathbb{E}_Q[X_1^\ell] < \infty$ $(\ell \in \{1,2\})$, **see** Notations 1(c);

$\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$**:** The class of all $Q \in \mathcal{M}^{\ell}_{S,\mathbf{\Lambda}(\rho(\Theta))}$ with $(1/\mathbb{E}_Q[W_1 \mid \Theta])^\ell \in \mathcal{L}^1(Q)$ $(\ell \in \{1,2\})$, **see** Notations 2(b);

$\mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$**:** The class of all probability measures Q_θ on Σ, such that $Q_\theta \restriction \mathcal{F}_t \sim P_\theta \restriction \mathcal{F}_t$ for any $t \in \mathbb{R}_+$ and S is a Q_θ-CRP$(\mathbf{\Lambda}(\rho(\theta)), (Q_\theta)_{X_1})$ $(\theta \in D)$, **see** Notations 1(d);

$\mathcal{M}^{\ell}_{S,\mathbf{\Lambda}(\rho(\theta))}$**:** The class of all $Q_\theta \in \mathcal{M}_{S,\mathbf{\Lambda}(\rho(\theta))}$ with $\mathbb{E}_{Q_\theta}[X_1^\ell] < \infty$ $(\ell \in \{1,2\}$ and $\theta \in D)$, **see** Notations 1(d);

$\mathcal{M}^{*,\ell}_{S,\mathbf{\Lambda}(\rho(\theta))}$**:** The class of all $Q_\theta \in \mathcal{M}^{\ell}_{S,\mathbf{\Lambda}(\rho(\theta))}$ such that $(1/\mathbb{E}_{Q_\bullet}[W_1])^\ell \in \mathcal{L}^1(Q_\Theta)$ $(\ell \in \{1,2\}$ and $\theta \in D)$, **see** Notations 2(c).

Conditions:

$(*)$**:** $\alpha(\Theta) = \ln \rho(\Theta) + \ln \mathbb{E}_P[W_1 \mid \Theta]$ $P\restriction\sigma(\Theta)$-a.s., **see** Proposition 1(i);

$(\widetilde{*})$**:** $\rho(\theta) = e^{\alpha(\theta)}/\mathbb{E}_{P_\theta}[W_1]$, **see** Proposition 1(iii);

(RPM_ξ)**:** The formula

$$Q(A) = \int_A M_t^{(\beta)}(\Theta)\, dP \quad \text{for all } 0 \le s \le t \text{ and } A \in \mathcal{F}_s,$$

with

$$M_t^{(\beta)}(\Theta) := \xi(\Theta) \cdot \frac{e^{S_t^{(\gamma)} - \rho(\Theta)\cdot J_t}}{1 - \mathbf{K}(\Theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\Theta))}{d\mathbf{K}(\Theta)}(W_j),$$

where $S_t^{(\gamma)} := \sum_{k=1}^{N_t} \gamma(X_j)$ and $J_t := t - T_{N_t}$, **see** Proposition 1(i);

(RPM_θ)**:** The formula

$$Q_\theta(A) = \int_A \widetilde{M}_t^{(\beta)}(\theta)\, dP_\theta \quad \text{for all } 0 \leq s \leq t \text{ and } A \in \mathcal{F}_s,$$

where

$$\widetilde{M}_t^{(\beta)}(\theta) := \frac{e^{S_t^{(\gamma)} - \rho(\theta)\cdot J_t}}{1 - \mathbf{K}(\theta)(J_t)} \cdot \prod_{j=1}^{N_t} \frac{d\mathbf{Exp}(\rho(\theta))}{d\mathbf{K}(\theta)}(W_j),$$

see Proposition 1(iii);

(ruin(P))**:** The formula for the ruin probability $\psi(u)$ of the reserve process $R^u(\Theta)$, i.e.,

$$\psi(u) = \int \frac{1}{\xi(\Theta)} \cdot e^{-S^{(\gamma)}_{T_u(\Theta)}} \cdot \prod_{j=1}^{N_{T_u(\Theta)}} \frac{d\mathbf{K}(\Theta)}{d\mathbf{Exp}(\rho(\Theta))}(W_j)\, dQ < 1$$

for any $u \in \mathbb{R}_+$,

see Theorem 3(i);

(ruin(P_θ))**:** The formula for the ruin probability $\psi_\theta(u)$ of the reserve process $r^u(\theta)$, i.e.,

$$\psi_\theta(u) = \int e^{-S^{(\gamma)}_{\tau_u(\theta)}} \cdot \prod_{j=1}^{N_{\tau_u(\theta)}} \frac{d\mathbf{K}(\theta)}{d\mathbf{Exp}(\rho(\theta))}(W_j)\, dQ_\theta < 1 \quad \text{for any } u \in \mathbb{R}_+,$$

see Theorem 3(iii).

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