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On the reducibility of affine models with dependent Lévy factors

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Abstract The paper is devoted to the study of the short rate equation of the form

$$\mathrm{d}R(t) = F(R(t))\mathrm{d}t + \sum_{i=1}^{d} G_i(R(t-))\mathrm{d}Z_i(t), \quad R(0) = R_0 \geq 0,\ t > 0,$$

with deterministic functions $F, G_1, \dots, G_d$ and a multivariate Lévy process $Z = (Z_1, \dots, Z_d)$ with possibly dependent coordinates. This equation is assumed to have a nonnegative solution which generates an affine term structure model. Under some mild assumptions on the Lévy measure of Z it is shown that the same term structure is generated by an equation with affine drift term and noise being a one-dimensional α-stable process with index of stability $\alpha \in (1, 2)$. For this case the shape of possible simple forward curves is characterized. A precise description of normal, inverse and humped profiles in terms of the equation coefficients and the stability index α is provided.

The paper generalizes the classical results on the Cox–Ingersoll–Ross (CIR) model [Econometrica 53 (1985), 385–408], as well as on its extended version where Z is a one-dimensional Lévy process [SIAM J. Financ. Math. 11(1) (2020), 131–147, Bond Markets with Lévy Factors, Cambridge University Press, 2020]. It is the starting point for the classification of affine models with dependent Lévy processes, in the spirit of [J. Finance 5 (2000), 1943–1978] and [Classification and calibration of affine models driven by independent Lévy processes, https://arxiv.org/abs/2303.08477].

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1 Introduction

Let us consider a bond market with a family of stochastic processes describing zero coupon bond prices $P(t,T),\ t \in [0,T]$, parametrized by the maturity time $T > 0$ and the short rate process $R(t),\ t \geq 0$. The processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t),\ t \geq 0$. The bond maturing at T pays its owner at time T a nominal value assumed here to be 1, i.e. $P(T,T) = 1$. The discounted value of 1 paid at time $t > 0$ equals $D(t) = e^{-\int_0^t R(s)ds}$. The short rate process R is supposed to satisfy, for each $T > 0$, the condition

$$\mathbb{E}[e^{-\int_t^T R(s)\mathrm{d}s} \mid \mathcal{F}_t] = e^{-A(T-t)-B(T-t)R(t)}, \quad t \in [0,T], \tag{1.1}$$

with some deterministic functions $A(\cdot)$, $B(\cdot)$. Interpreting $\mathbb{P}$ as a risk neutral measure, one recognizes in the left side of (1.1) the price at time t of the bond with maturity T, that is $P(t,T)$. Thus (1.1) means that the short rate R *generates* an *affine term structure*.

The concept of modeling bond prices in the affine fashion was introduced by Filipović [17] and Duffie, Filipović and Schachermeyer [14]. It was motivated by the results of Kawazu and Watanabe [19] on continuous state branching processes with immigration. Further developments on regularity of affine processes are due to Cuchiero, Filipović and Teichmann [11] and Cuchiero and Teichmann [12]. The aforementioned results are settled in the general Markovian setting and the description of affine processes is given in the form of their generators. A class of particular interest are short rates given by stochastic equations. An equation with a solution which generates an affine model is called *a generating equation*. The precursors of generating equations are two classical equations – one due to Cox, Ingersoll and Ross (CIR) [10]

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + c\sqrt{R(t)}\mathrm{d}W(t), \tag{1.2}$$

with $a \in \mathbb{R}$, $b \geq 0$, $c > 0$, and another due to Vasiček [25]

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + c\,\mathrm{d}W(t), \tag{1.3}$$

with $a, b, c \in \mathbb{R}$ – both driven by a one-dimensional Wiener process W. To make the behavior of the short rate process more realistic and to improve the accuracy of calibration to market data more involved equations are considered in the literature. Into account are taken multidimensional noises, including those with jumps, with possibly correlated variates. Passing to more general types of noise offers more flexibility to the arising bond market which is required from the pricing perspective. Dai and Singleton [13] consider factorial models perturbed by correlated Wiener processes and examine the influence of the correlation structure on the resulting affine model. In the case when W is replaced by a Lévy process, it was shown in Barski and Zabczyk [5] that the generalization of (1.2) must be of the form

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + C \cdot (R(t-))^{1/\alpha} dZ^{\alpha}(t), \tag{1.4}$$

with $a \in \mathbb{R}$, $b \geq 0$, $C > 0$, where Z^{α} is an α-stable process with index $\alpha \in (1, 2]$. For a comprehensive study of α-stable processes we refer to Samorodnitsky and Taqqu [24]. It was also shown in [5] that the counterpart of (1.3) in the Lévy setting allows preserving the positivity of R, which clearly lacks in (1.3) like in each Gaussian model. Jiao, Ma and Scotti [18] modify the CIR model by adding an independent α-stable component to the Wiener process. Their α-CIR model reveals better fitting to the European sovereign bond market than the CIR model and the stability index α allows controlling the tail heaviness of the bond prices. Models driven by a multivariate Lévy process with independent coordinates appear, among others, in Duffie and Gârleanu [15], Barndorff-Nielsen and Shephard [3], Barski and Łochowski [4]. Similarly as in [13], it is noticed in [4] that different equations may generate identical affine models. This fact motivated a classification of all generating equations into several classes which are representable by the so-called canonical equations having tractable forms. The case when the coordinates of the multivariate Lévy process are dependent is an unexplored field entered by this paper.

We consider an equation of the form

$$\mathrm{d}R(t) = F(R(t))\mathrm{d}t + \sum_{i=1}^{d} G_i(R(t-))\mathrm{d}Z_i(t), \quad R(0) = x \geq 0, \ t > 0, \tag{1.5}$$

where $F, G := (G_1, \ldots, G_d)$ are deterministic functions and $Z = (Z_1, \ldots, Z_d)$ is a multivariate Lévy process and a martingale. As the coordinates of Z may be dependent, the Lévy measure ν of Z is not necessarily concentrated on axes. Its *polar decomposition*

$$\nu(A) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \mathbf{1}_A(r\xi)\gamma_{\xi}(\mathrm{d}r)\, \lambda(\mathrm{d}\xi), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{1.6}$$

with a finite measure λ on a unit sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ called a *spherical component of* ν and a family of measures $\{\gamma_{\xi}; \xi \in \mathbb{S}^{d-1}\}$ on $(0, +\infty)$ called *radial components* of ν, will play a central role in the sequel. The radial decomposition is known to exist and to be unique for any Lévy measure, see [2], Lemma 2.1 and [22], Proposition 4.2. Let us recall that any α-stable process in $\mathbb{R}^d$ with index $\alpha \in (1, 2)$ admits radial decomposition with identical radial measures given by the density

$$\gamma_{\xi}(\mathrm{d}r) = \gamma(\mathrm{d}r) := \frac{1}{r^{1+\alpha}}\mathrm{d}r, \quad r > 0, \ \xi \in \mathbb{S}^{d-1}, \tag{1.7}$$

and arbitrary spherical measure λ. In fact, the radial decomposition can be explicitly determined in the case when ν has a density with respect to the Lebesgue measure, say g. Then the radial measures are of the form

$$\gamma_{\xi}(\mathrm{d}r) = g(r\xi)r^{d-1}\sqrt{1 - \xi_1^2} \cdot \sqrt{1 - (\xi_1^2 + \xi_2^2)} \cdot \ldots \cdot \sqrt{1 - (\xi_1^2 + \cdots + \xi_{d-2}^2)}\ \mathrm{d}r,$$

for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{S}^{d-1}$, $r > 0$, and the spherical measure λ is the image of the Lebesgue measure by the polar transformation, see Section 3.1.2 for details. We examine the question which affine models can be generated by equation (1.5) and our analysis of this problem is based on the radial decomposition (1.6).

In Example 2.3 we show that if (1.5) is a generating equation and Z is an $\mathbb{R}^d$-valued α-stable process then the resulted affine model is identical with the model generated by (1.4). This means that (1.4) can replace the initial equation, which may be of a complicated form, preserving the bond prices unchanged. In this case we call the initial equation to have the *reducibility property*. This extends the observations from [13] and [4] to the case with dependent noise coordinates. The main result of this paper, Theorem 3.3, provides conditions for (1.5) to have the reducibility property. We prove that if G is a continuous function for which the limit

$$\lim_{x\to 0^+} \frac{G(x)}{|G(x)|}$$

exists and the Laplace exponents associated with the radial measures

$$J_{\gamma_\xi}(b) := \int_0^{+\infty} (e^{-br} - 1 + br)\gamma_\xi(\mathrm{d}r), \quad b \geq 0,\ \xi \in \mathbb{S}^{d-1},$$

satisfy the condition

$$\sup_{\xi \in \mathrm{supp}(\lambda)} J_{\gamma_\xi}(b) \leq K \cdot \inf_{\xi \in \mathrm{supp}(\lambda)} J_{\gamma_\xi}(b), \quad b \geq 0, \tag{1.8}$$

with some $1 \leq K < +\infty$, then any equation with such G and Z can generate only the same affine model as the one generated by (1.4) with some $\alpha \in (1,2)$. Condition (1.8) is shown to be satisfied in the class of *tempered stable distributions* and in the case when Z is $\mathbb{R}^2$-valued and its jump measure has a density g such that the functions

$$\underline{g}(r) := \inf_{|x|=r} g(x), \qquad \bar{g}(r) := \sup_{|x|=r} g(x), \quad r \geq 0,$$

satisfy the integrability conditions

$$0 < \int_0^{+\infty} \min\{r^2, r^3\}\underline{g}(r)\mathrm{d}r \leq \int_0^{+\infty} \min\{r^2, r^3\}\bar{g}(r)\mathrm{d}r < +\infty,$$

and

$$\lim_{\varepsilon\downarrow 0} \frac{\int_\varepsilon^1 r^2 \bar{g}(r)\mathrm{d}r}{\int_\varepsilon^1 r^2 \underline{g}(r)\mathrm{d}r} < +\infty \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \frac{\int_1^{1/\varepsilon} r^3 \bar{g}(r)\mathrm{d}r}{\int_1^{1/\varepsilon} r^3 \underline{g}(r)\mathrm{d}r} < +\infty.$$

For details and extensions in cases where $d > 2$, see Section 3.1.2.

Remark 1.1. Condition (1.8) seems to be fundamental in this regard as there exist Lévy martingales Z for which (1.5) generates an affine short rate model but has no reducibility property; see [4, Theorems 3.1, 3.8] and Section 3.1.

The term structure models are calibrated to market data which may contain, for instance, swap or swaption prices or some spot rates. Empirical curves understood as functions of maturities representing market quotes should be well approximated by the resulting model curves. Therefore, understanding which curve shapes can be generated by the model is of prime importance. We examine possible shapes of the simple spot curve defined by

$$F(x) = \frac{1}{x}\left(\frac{1}{P(0,x)} - 1\right), \quad x \geq 0,$$

in the model (1.4) and provide a precise description of normal (increasing), inverse (decreasing) and humped (possesing one local maximum) shapes in terms of the model parameters. Characterization of yield curves

$$x \mapsto -\frac{1}{x} \ln P(0,x),$$

in affine models is known in the literature, see [20, 21], but it does not imply the shape of simple spot curves. Our characterization in Theorem 4.3 helps to decide if the model can be calibrated to market reference rates like LIBOR or other like SONIA, SOFR, SARON and ESTER. In particular, it helps to decide if the possible simple spot curves produced by the model, which are normal, inverse or humped, fit the shapes of empirical spot curves obtained from these reference rates. Also, comparing (1.4) to the CIR model with a Wiener driving process, we see that the stability index α offers additional fit flexibility.

The paper is organized as follows. In Section 2 we present some basic facts on Lévy processes and the Markovian characterization of generating equations. Section 3 on the reducibility problem contains formulation of the main results including Theorem 3.1 and Theorem 3.3, examples and illustrative analysis of the case when ν has a density, resulting in Theorem 3.8. The proof of Theorem 3.1 and associated auxiliary results are postponed to Section 5. Section 4 is devoted to the description of shapes of simple spot curves.

2 Preliminaries

2.1 *Basic facts on Lévy processes*

Let $Z := (Z_1, Z_2, \ldots, Z_d)$ be a Lévy process in $\mathbb{R}^d$, $d \geq 1$, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, t \geq 0\}$. If Z is a martingale, then it admits the following unique representation

$$Z(t) = W(t) + X(t), \quad t \geq 0,$$

where W is a Wiener process in $\mathbb{R}^d$ with a covariance matrix Q and X is the so-called *jump martingale part* of Z. It is independent of W and can be described in terms of the *jump measure* of Z defined by

$$\pi(t, A) := \sharp\{s \in (0,t] : \triangle Z(s) \in A\}, \quad t \geq 0, \tag{2.1}$$

where $\triangle Z(s) := Z(s) - Z(s-)$ and $A \subset \mathbb{R}^d$ is a set separated from zero, i.e. 0 does not belong to the closure of A. With (2.1) at hand one defines the *Lévy measure* of Z by

$$\nu(A) := \mathbb{E}\big[\pi(1, A)\big].$$

Then X can be written as

$$X(t) := \int_0^t \int_{\mathbb{R}^d} y \,(\pi(\mathrm{d}s, \mathrm{d}y) - \mathrm{d}s\, \nu(\mathrm{d}y)), \quad t \geq 0,$$

and its properties can be formulated in terms of the measure ν. The integrability of X is equivalent to the condition

$$\int_{\mathbb{R}^d} (|y|^2 \wedge |y|)\nu(\mathrm{d}y) < +\infty, \tag{2.2}$$

while the *variation* of paths of X is almost surely locally finite if and only if

$$\int_{|y|<1} |y|\, \nu(\mathrm{d}y) < +\infty. \tag{2.3}$$

In our notation $|\cdot|$ stands for the standard norm in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ for the standard scalar product.

By the independence of X and W we see that, for $\lambda \in \mathbb{R}^d$,

$$\mathbb{E}\left[e^{-\langle \lambda, Z(t)\rangle}\right] = \mathbb{E}\left[e^{-\langle \lambda, W(t)\rangle}\right] \cdot \mathbb{E}\left[e^{-\langle \lambda, X(t)\rangle}\right],$$

so the Laplace exponent J_Z of Z defined by

$$\mathbb{E}\left[e^{-\langle \lambda, Z(t)\rangle}\right] = e^{tJ_Z(\lambda)}$$

exists at λ if and only if $J_X(\lambda)$ is finite. The latter property is equivalent to the condition

$$\int_{|y|>1} e^{-\langle \lambda, y\rangle} \nu(\mathrm{d}y) < +\infty. \tag{2.4}$$

If (2.4) holds, then

$$J_X(\lambda) = \int_{\mathbb{R}^d} (e^{-\langle \lambda, y\rangle} - 1 + \langle \lambda, y\rangle)\nu(\mathrm{d}y), \tag{2.5}$$

and, consequently,

$$\begin{aligned} J_Z(\lambda) &= J_W(\lambda) + J_X(\lambda) \\ &= \frac{1}{2}\langle Q\lambda, \lambda\rangle + \int_{\mathbb{R}^d} (e^{-\langle \lambda, y\rangle} - 1 + \langle \lambda, y\rangle)\nu(\mathrm{d}y). \end{aligned} \tag{2.6}$$

It follows, in particular, that the process Z is uniquely determined by the pair (Q, ν).

2.2 *Markovian characterization of generating equations*

It was shown in [17, Theorem 5.3] that the generator of a general nonnegative Markovian short rate generating an affine model is of the form

$$\begin{aligned} \mathcal{A}f(x) &= cxf''(x) + (\beta x + \gamma)f'(x) \\ &\quad + \int_{(0,+\infty)} \Big(f(x+v) - f(x) - f'(x)(1 \wedge v)\Big)(m(\mathrm{d}v) + x\mu(\mathrm{d}v)), \quad x \geq 0, \end{aligned} \tag{2.7}$$

for $f \in \mathcal{L}(\Lambda) \cup C_c^2(\mathbb{R}_+)$, where $\mathcal{L}(\Lambda)$ is the linear hull of $\Lambda := \{f_\lambda := e^{-\lambda x},\ \lambda \in (0, +\infty)\}$ and $C_c^2(\mathbb{R}_+)$ stands for the set of twice continuously differentiable functions with compact support in $[0,+\infty)$. In the equation above $c,\gamma \geq 0$, $\beta \in \mathbb{R}$ and $m(\mathrm{d}v)$, $\mu(\mathrm{d}v)$ are nonnegative Borel measures on $(0,+\infty)$ satisfying

$$\int_{(0,+\infty)} (1 \wedge v)m(\mathrm{d}v) + \int_{(0,+\infty)} (1 \wedge v^2)\mu(\mathrm{d}v) < +\infty. \tag{2.8}$$

Moreover, the functions $A(\cdot)$, $B(\cdot)$ in (1.1) are uniquely determined by the form of the generator (2.7), for details see [17].

The application of the above given characterization to the case when R is given by (1.5) leads to necessary and sufficient conditions making (1.5) a generating equation, for the proof see Proposition 2.2 in [4]. To formulate these conditions we need to introduce a family of measures related to the pair (G, Z). For $x \geq 0$ we define the measure

$$\nu_{G(x)}(A) := \nu\{y \in \mathbb{R}^d : \langle G(x), y\rangle \in A\}, \quad A \in \mathcal{B}(\mathbb{R}),$$

which is the image of the Lévy measure ν under the linear transformation $y \mapsto \langle G(x), y\rangle$. This measure may have an atom at zero and therefore its restriction $\nu_{G(x)}(\mathrm{d}v) \mid_{v\neq 0}$ is used below. The aforementioned conditions are as follows.

- The drift is affine

$$F(x) = ax + b, \quad \text{where } a \in \mathbb{R},\ b \geq \int_{(1,+\infty)} (v-1)\nu_{G(0)}(\mathrm{d}v). \tag{2.9}$$

- The covariance matrix of the Wiener part of Z satisfies

$$\frac{1}{2}\langle QG(x), G(x)\rangle = cx, \quad x \geq 0, \tag{2.10}$$

  with some $c \geq 0$.

- The jumps of Z and the function G are such that

$$\langle G(x), \triangle Z(t)\rangle \geq 0, \quad x \geq 0,\ t \geq 0, \tag{2.11}$$

$$\nu_{G(0)}(\mathrm{d}v) = m(\mathrm{d}v) \quad \text{and} \quad \int_{(0,+\infty)} v\, \nu_{G(0)}(\mathrm{d}v) < +\infty, \tag{2.12}$$

$$\int_{(0,+\infty)} (v \wedge v^2)\mu(\mathrm{d}v) < +\infty, \tag{2.13}$$

$$\nu_{G(x)}(\mathrm{d}v) \mid_{(0,+\infty)} = \nu_{G(0)}(\mathrm{d}v) \mid_{(0,+\infty)} + x\mu(\mathrm{d}v), \quad x \geq 0. \tag{2.14}$$

Moreover, (2.7) reads

$$\begin{aligned}\mathcal{A}f(x) = cxf''(x) &+ \Big[ax + b + \int_{(1,+\infty)} (1-v)\{\nu_{G(0)}(\mathrm{d}v) + x\mu(\mathrm{d}v)\}\Big] f'(x)\\ &+ \int_{(0,+\infty)} [f(x+v) - f(x) - f'(x)(1 \wedge v)]\{\nu_{G(0)}(\mathrm{d}v) + x\mu(\mathrm{d}v)\}.\end{aligned} \tag{2.15}$$

In particular, with the parameters a, b, c and the measures $\nu_{G(0)}(\mathrm{d}v), \mu(\mathrm{d}v)$ from (2.15) at hand one can determine the zero coupon bond prices, for details see [17].

Note that the integrability requirements (2.12), (2.13) for the measures $m(\mathrm{d}v)$, $\mu(\mathrm{d}v)$ are stronger than in (2.8). They appear due to the fact that Z is a martingale.

Remark 2.1. Conditions (2.10)–(2.14) describe the law of the family of one-dimensional Lévy processes $Z^{G(x)}(t) := \langle G(x), Z(t)\rangle$, $x \geq 0$. Conditions (2.10) and (2.14) can be reformulated in terms of their Laplace exponents

$$J_{Z^{G(x)}}(b) = J_Z(bG(x)) = cb^2 + J_{\nu_{G(0)}}(b) + xJ_\mu(b), \quad b \geq 0, \tag{2.16}$$

where $J_\mu(b) := \int_0^{+\infty}(e^{-bv} - 1 + bv)\mu(\mathrm{d}v)$ and $J_{\nu_{G(0)}}$ is defined analogously.

Remark 2.2. For the equation (1.4) with $\alpha \in (1,2)$ one can show that

$$c = 0, \ \nu_{G(0)} = 0, \ \mu(\mathrm{d}v) = \mathbf{1}_{\{v>0\}}\frac{1}{v^{1+\alpha}}\mathrm{d}v,$$

hence $\mu(\mathrm{d}v)$ is an α-stable measure, for details see [5] or [6].

We start with an example of (1.5) where Z is an α-stable martingale in $\mathbb{R}^d$, $d > 1$, with $\alpha \in (1,2)$. Recall that its radial measure is given by (1.7). Since Z has no Wiener part, the Laplace exponent of the jump part X of Z is identical with the Laplace exponent of Z and admits the following representation:

$$\begin{aligned} J_X(z) &= \int_{\mathbb{S}^{d-1}} \lambda(\mathrm{d}\xi) \int_0^{+\infty} \left(e^{-\langle z, r\xi\rangle} - 1 + \langle z, r\xi\rangle\right) \frac{1}{r^{1+\alpha}}\mathrm{d}r \\ &= \int_{\mathbb{S}^{d-1}} \lambda(\mathrm{d}\xi) \int_0^{+\infty} \left(e^{-r\langle z, \xi\rangle} - 1 + r\langle z, \xi\rangle\right) \frac{1}{r^{1+\alpha}}\mathrm{d}r \\ &= c_\alpha \int_{\mathbb{S}^{d-1}} \langle z, \xi\rangle^\alpha \lambda(\mathrm{d}\xi), \end{aligned} \tag{2.17}$$

where $c_\alpha := \Gamma(2-\alpha)/(\alpha(\alpha-1))$ and Γ stands for the Gamma function. In the above equation, we used the formula

$$\int_0^{+\infty} \left(e^{-uv} - 1 + uv\right)\frac{1}{v^{1+\alpha}}\mathrm{d}v = c_\alpha u^\alpha.$$

In the following example, assuming that Z is an α-stable martingale in $\mathbb{R}^d$, we compute the condition ((2.19)) for the function G in (1.5) so that this equation generates an affine model. This condition is sufficient and necessary when we assume that Z is α-stable and $G(0) = 0$.

Example 2.3. Let Z be an α-stable martingale in $\mathbb{R}^d$ with the Laplace exponent (2.17) and $G : [0, +\infty) \to [0, +\infty)^d$, $G(0) = 0$. Then the equation

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + \langle G(R(t-)), \mathrm{d}Z(t)\rangle, \tag{2.18}$$

with $a \in \mathbb{R}$, $b \geq 0$, generates an affine model if and only if the function G satisfies

$$\int_{\mathbb{S}^{d-1}} \langle G(x), \xi\rangle^\alpha \lambda(\mathrm{d}\xi) = \frac{C}{c_\alpha}x, \quad x \geq 0, \tag{2.19}$$

with $C \geq 0$. To prove this fact we need to show that (2.19) is equivalent to (2.16) with some measure $\mu(\mathrm{d}v)$. Since Z has no Wiener part and $\nu_{G(0)}(\mathrm{d}v) \equiv 0$, we see that (2.16) takes the form

$$J_Z(bG(x)) = J_X(bG(x)) = xJ_\mu(b), \quad x, b \geq 0.$$

By (2.17)

$$J_X(bG(x)) = c_\alpha \int_{\mathbb{S}^{d-1}} \langle bG(x), \xi \rangle^\alpha \lambda(\mathrm{d}\xi) = c_\alpha b^\alpha \int_{\mathbb{S}^{d-1}} \langle G(x), \xi \rangle^\alpha \lambda(\mathrm{d}\xi).$$

Consequently,

$$c_\alpha b^\alpha \int_{\mathbb{S}^{d-1}} \langle G(x), \xi \rangle^\alpha \lambda(\mathrm{d}\xi) = xJ_\mu(b),$$

holds if and only if

$$J_\mu(b) = Cb^\alpha, \qquad \int_{\mathbb{S}^{d-1}} \langle G(x), \xi \rangle^\alpha \lambda(\mathrm{d}\xi) = \frac{C}{c_\alpha} x,$$

for some $C \geq 0$. Hence, μ is an α-stable measure and G can be any function satisfying (2.19). It follows from Remark 2.2 and (2.15) that the generators of equations (2.18) and (1.4) are identical, so are the related bond markets.

To see that already for $d = 2$ there are many possibile forms of the function $G = (G_1, G_2)$, let us take $\lambda = \delta_{(1,0)} + \delta_{(0,1)}$ (δ_a denotes Dirac's delta measure concentrated at the point a). Then condition (2.19) reads

$$G_1(x)^\alpha + G_2(x)^\alpha = \frac{C}{c_\alpha} x$$

and is satisfied, for example, for

$$G_1(x) = \left(\frac{C}{2c_\alpha} (1 + \sin(x))x \right)^{1/\alpha}, \qquad G_2(x) = \left(\frac{C}{2c_\alpha} (1 - \sin(x))x \right)^{1/\alpha}.$$

3 Reducibility of equations with multivariate noise

In this section we specify conditions for the equation (1.5), written now for convenience in the form

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + \langle G(R(t-)), \mathrm{d}Z(t) \rangle, \quad R(0) = x \geq 0, \ t > 0, \tag{3.1}$$

to have the reducibility property. The affine form of the above drift is justified by (2.9). This means that (3.1) is supposed to generate the same bond prices as the equation

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + C \cdot R(t-)^{1/\alpha} \mathrm{d}Z^\alpha(t), \tag{3.2}$$

with $a \in \mathbb{R}$, $b \geq 0$, $C > 0$ and an α-stable real-valued Lévy process Z^α with some $\alpha \in (1, 2)$. Recall that from Example 2.3 we know that each generating equation (3.1) with Z being an α-stable process in $\mathbb{R}^d$ has the reducibility property.

In (3.1), $G : \mathbb{R}_+ \longrightarrow \mathbb{R}^d$ and Z is a Lévy process and martingale in $\mathbb{R}^d$, called a *Lévy martingale* for short. It is characterized by a covariance matrix Q of the Wiener part and a Lévy measure ν with polar decomposition

$$\nu(A) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \mathbf{1}_A(r\xi) \gamma_\xi(\mathrm{d}r) \, \lambda(\mathrm{d}\xi), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{3.3}$$

with a finite spherical measure λ on the unit sphere $\mathbb{S}^{d-1}$ and some radial measures $\{\gamma_\xi; \xi \in \mathbb{S}^{d-1}\}$. To avoid technical complications we can assume, and we do, the nondegeneracy condition for the radial measures, i.e.

$$\xi \in \mathrm{supp}(\lambda) \Longrightarrow \gamma_\xi \neq 0. \tag{3.4}$$

If (3.4) is not satisfied, one can modify λ by cutting off the part of its support where the radial measures disappear. This operation clearly does not affect (3.3). Since Z is a martingale, it follows from (2.2) that

$$\int_{\mathbb{R}^d} (|y|^2 \wedge |y|)\nu(\mathrm{d}y) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} (|r\xi|^2 \wedge |r\xi|)\gamma_\xi(\mathrm{d}r)\lambda(\mathrm{d}\xi) < +\infty,$$

which means that

$$\int_0^{+\infty} (r^2 \wedge r)\gamma_\xi(\mathrm{d}r) < +\infty, \quad \xi \in \mathrm{supp}(\lambda). \tag{3.5}$$

If the jump part of Z has infinite variation, then it follows from (2.3) that

$$\int_{|y|\le 1} |y|\, \nu(dy) = \int_{\mathbb{S}^{d-1}} \int_0^1 r\, \gamma_\xi(\mathrm{d}r)\, \lambda(\mathrm{d}\xi) = +\infty. \tag{3.6}$$

We consider a condition stronger than (3.6), namely, that

$$\lambda(\Gamma_\lambda) > 0, \quad \text{where } \Gamma_\lambda := \left\{\xi \in \mathrm{supp}(\lambda) : \int_0^1 r\gamma_\xi(dr) = +\infty\right\}. \tag{3.7}$$

Consequently, if we assume that Γ_λ is not contained in any proper linear subspace of $\mathbb{R}^d$, i.e.

$$\text{Linear span } (\Gamma_\lambda) = \mathbb{R}^d, \tag{3.8}$$

then we obtain that

$$G(0) = 0. \tag{3.9}$$

To see this, let us notice that, by (2.11) and (3.3), $\lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G(0), \xi\rangle < 0\right\} = 0$ which implies that

$$\langle G(0), \xi\rangle \ge 0 \quad \text{for any } \xi \in \text{supp } (\lambda). \tag{3.10}$$

By (2.12) we have

$$\begin{aligned}\int_0^{+\infty} v\, \nu_{G(0)}(\mathrm{d}v) &= \int_{\mathbb{R}^d} \langle G(0), y\rangle \nu(\mathrm{d}y)\\ &= \int_{\mathbb{S}^{d-1}} \langle G(0), \xi\rangle \int_0^{+\infty} r\, \gamma_\xi(\mathrm{d}r)\lambda(\mathrm{d}\xi) < +\infty,\end{aligned}$$

which, in view of (3.7), (3.8) and (3.10) implies (3.9). Obviously, (3.8) also implies that

$$\text{Linear span (supp } (\lambda)) = \text{Linear span (supp } (\nu)) = \mathbb{R}^d. \tag{3.11}$$

Notice that, for example, the measure $\lambda = \delta_{(1,0)} + \delta_{(0,1)}$ from Example 2.3 with $\gamma_{(1,0)}(\mathrm{d}r) = \gamma_{(0,1)}(\mathrm{d}r) = r^{-1-\alpha}\mathrm{d}r$, $\alpha \in (1,2)$, satisfies (3.7) as well as conditions (3.8) and (3.11).

3.1 Main results

For $\xi \in \operatorname{supp}(\lambda)$ let us consider the Laplace exponent related to the measure γ_ξ, i.e.

$$J_{\gamma_\xi}(b) = \int_0^{+\infty} (e^{-br} - 1 + br)\, \gamma_\xi(\mathrm{d}r), \quad b \geq 0.$$

We need the condition that there exists $K \geq 1$ such that

$$\sup_{\xi \in \operatorname{supp}(\lambda)} J_{\gamma_\xi}(b) \leq K \cdot \inf_{\xi \in \operatorname{supp}(\lambda)} J_{\gamma_\xi}(b), \quad b \geq 0. \tag{3.12}$$

In Section 3.1.1 we show that (3.12) is satisfied in the class of tempered stable distributions, which is of great importance in finance, and formulate some more general sufficient conditions for (3.12) to hold, see Proposition 3.6 and the resulting Example 3.7. Condition (3.12) seems to be fundamental in this regard as there exist Lévy martingales Z for which (3.1) generates an affine short rate model without the reducibility property. Clearly, for such martingales (3.12) is not satisfied. An example of such a martingale and an equation (3.1) is the following:

$$\begin{aligned} \mathrm{d}R(t) &= (aR(t) + b)\mathrm{d}t + (R(t-))^{1/\alpha_1}\mathrm{d}Z_1(t) + (R(t-))^{1/\alpha_2}\mathrm{d}Z_2(t), \\ R(0) &= x \geq 0, \quad t > 0, \end{aligned}$$

where $a \in \mathbb{R}$, $b \geq 0$, $1 < \alpha_1 < \alpha_2 < 2$ and $Z_1(t)$, $Z_2(t)$ are independent, real stable martingales, with stability indices α_1 and α_2, and the Lévy measures $\nu_1(\mathrm{d}x) = \mathbf{1}_{\{x>0\}} x^{-1-\alpha_1}\mathrm{d}x$, $\nu_2(\mathrm{d}x) = \mathbf{1}_{\{x>0\}} x^{-1-\alpha_2}\mathrm{d}x$, respectively; see [4, Theorems 3.1, 3.8].

The main result of the paper is the following theorem.

Theorem 3.1. *Let Z be a Lévy martingale with a covariance matrix Q of the Wiener part and a Lévy measure ν admitting the decomposition (3.3) with a spherical measure λ satisfying (3.11) and radial measures $\{\gamma_\xi; \xi \in \mathbb{S}^{d-1}\}$ satisfying (3.5) and (3.12). Let us also assume that (3.7) and (3.8) are satisfied or that (3.9) holds. Moreover, let $G : [0, +\infty) \to \mathbb{R}^d$ be a continuous function such that*

$$G_0 := \lim_{x \to 0^+} \frac{G(x)}{|G(x)|}, \tag{3.13}$$

exists.

Then if (3.1) generates an affine model, then for any $x \geq 0$ the Laplace exponent of the process $Z^{G(x)} = \langle G(x), Z \rangle$ has the form

$$J_{Z^{G(x)}}(b) = J_Z(bG(x)) = J_{Z^{G(0)}}(b) + cxb^2 + \gamma x b^\alpha, \quad b \geq 0,$$

with $c, \gamma \geq 0$, $\alpha \in (1, 2)$.

The proof of Theorem 3.1 is presented in Subsection 5.2 and is preceded by some auxiliary results presented in Subsection 5.1.

From Theorem 3.1 the following corollaries and theorem follow.

Corollary 3.2. *Let the assumptions of Theorem 3.1 be satisfied. If (3.1) is a generating equation, then for any $x \geq 0$ the Laplace exponent of the process $Z^{G(x)} = \langle G(x), Z \rangle$ has the form*

$$J_{Z^{G(x)}}(b) = J_Z(bG(x)) = \gamma x b^\alpha, \quad b \geq 0,$$

with $\gamma > 0$, $\alpha \in (1, 2)$. This means that the continuous (Wiener) part of the process $Z^{G(x)}$ vanishes for all $x \geq 0$.

Proof. It follows from (2.10) that the Wiener part of $Z^{G(x)}$ satisfies

$$\frac{1}{2}\langle QG(x), G(x)\rangle = cx, \quad x \geq 0 \text{ for some } c \geq 0. \tag{3.14}$$

Either directly by assumption (3.9) or by the assumptions (3.7) and (3.8) we get that $G(0) = 0$. By this and Theorem 3.1, the Laplace transform of the jump part of Z satisfies

$$J_X(bG(x)) = \gamma x b^{\alpha}, \quad x \geq 0 \text{ for some } \gamma \geq 0, \alpha \in (1, 2). \tag{3.15}$$

By (3.11) it follows that $\gamma > 0$. Condition (2.11) guarantees that for G_0 defined by (3.13), $\langle G_0, y\rangle \geq 0$ for any $y \in \operatorname{supp} \nu$ and condition (3.11) guarantees that $y \mapsto \langle G_0, y\rangle, y \in \operatorname{supp} \nu$, does not vanish, hence $J_X(G_0) > 0$. Consequently, from (3.15) we obtain

$$\lim_{x\to 0+} \frac{\gamma x}{|G(x)|^{\alpha}} = \lim_{x\to 0+} J_X\left(\frac{G(x)}{|G(x)|}\right) = J_X(G_0) \in (0, +\infty).$$

From this, $\lim_{x\to 0+} |G(x)| = 0$ and from (3.14) we further have

$$\langle QG_0, G_0\rangle = \lim_{x\to 0+} \frac{\langle QG(x), G(x)\rangle}{|G(x)|^2} = \lim_{x\to 0+} \frac{\gamma x}{|G(x)|^{\alpha}} \frac{2c/\gamma}{|G(x)|^{2-\alpha}} = \begin{cases} 0 & \text{if } c = 0, \\ +\infty & \text{if } c > 0. \end{cases}$$

Since $\langle QG_0, G_0\rangle \neq +\infty$, we necessarily have $c = 0$ which, in view of (3.14), means that the continuous (Wiener) part of $Z^{G(x)}$ vanishes. □

Theorem 3.3. *For each generating equation (3.1) satisfying the assumptions of Theorem 3.1 the generators of (3.1) and of (3.2) are identical for some $C > 0$, so (3.1) has the reducibility property.*

Proof. It follows from Theorem 3.1, Corollary 3.2 and Remark 2.1 that each generating equation (3.1) satisfying assumptions of Theorem 3.1 satisfies conditions (2.10)–(2.14) with

$$c = 0, \;\; \nu_{G(0)} = 0, \;\; \mu(\mathrm{d}v) = \mathbf{1}_{\{v>0\}} \frac{1}{v^{1+\alpha}} \mathrm{d}v, \;\; \alpha \in (1, 2).$$

□

Remark 3.4. In the formulation of Theorem 3.1 the assumption (3.13) can be replaced by the existence of the limit $\lim_{x\to+\infty} \frac{G(x)}{|G(x)|}$. Under the latter condition, however, we were unable to prove Corollary 3.2.

3.1.1 Examples

Here we present some examples concerned with Theorem 3.1 and, in particular, with condition (3.12). We start with a class of tempered stable distributions. Recall that the Lévy measure of a tempered stable distribution has the form

$$\nu(A) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} \mathbf{1}_A(r\xi) \frac{e^{-h(\xi)r}}{r^{1+\alpha}} \lambda(\mathrm{d}\xi), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{3.16}$$

where $h : \mathbb{S}^{d-1} \longrightarrow (0, +\infty)$ is a Borel function called a *tempering exponent* and $\alpha \in (1, 2)$ is the stability index. In fact, the range of values for α can be extended to $(-\infty, 2)$ if one relaxes the requirement for the corresponding process to be a martingale. Tempered stable distributions were introduced in [23], but special cases were known earlier in finance. Of particular interest were one-dimensional processes, for instance, Variance Gamma Process [7] or the CGMY process of Carr, Geman, Madan and Yor, see [8]. The tempering function allows to flexibly control the tail heaviness, with the use of the value $h(\xi)$, passing from light tails of the Gaussian case to the case of heavy-tailed α-stable distribution. The multivariate tempered stable distributions also appear in finance by exponential Lévy models and by pricing basket options, see [27] and [1].

Example 3.5 (Tempered stable distributions)**.** Let $\nu(\mathrm{d}y)$ be given by (3.16) with a bounded tempering function, i.e.

$$0 < A \leq h(\xi) \leq B < +\infty, \quad \xi \in \mathbb{S}^{d-1}. \tag{3.17}$$

We show that then condition (3.12) is satisfied.

By (3.16) we see that for $\xi \in \mathbb{S}^{d-1}$ the radial measure has the form

$$\gamma_\xi(\mathrm{d}r) = \frac{e^{-h(\xi)r}}{r^{1+\alpha}}\,\mathrm{d}r, \quad r > 0,$$

and its Laplace exponent equals

$$\begin{aligned} J_{\gamma_\xi}(b) &= \int_0^{+\infty} (e^{-br} - 1 + br) \cdot \frac{e^{-h(\xi)r}}{r^{1+\alpha}}\,\mathrm{d}r \\ &= \Gamma(-\alpha)\Big[\big(h(\xi)+b\big)^\alpha - h(\xi)^\alpha - \alpha b h(\xi)^{\alpha-1}\Big], \quad b \geq 0, \end{aligned}$$

see [9], p.121. By (3.17) we clearly have

$$F(B,b) \leq J_{\gamma_\xi}(b) \leq F(A,b), \quad b \geq 0, \quad \xi \in \mathbb{S}^{d-1}, \tag{3.18}$$

with $F(A,b) := \Gamma(-\alpha)\Big[\big(A+b\big)^\alpha - A^\alpha - \alpha b A^{\alpha-1}\Big]$, $F(B,b) := \Gamma(-\alpha)\Big[\big(B+b\big)^\alpha - B^\alpha - \alpha b B^{\alpha-1}\Big]$. It follows from (3.18) that

$$\frac{\sup_{\xi \in supp(\lambda)} J_{\gamma_\xi}(b)}{\inf_{\xi \in supp(\lambda)} J_{\gamma_\xi}(b)} \leq \sup_{b \geq 0} \frac{F(A,b)}{F(B,b)},$$

so to show (3.12), it is sufficient to show that the quotient $F(A,b)/F(B,b)$ is bounded. It is however continuous, so we need to show that it has finite positive limits at zero and at infinity. But

$$\lim_{b \to 0} \frac{F(A,b)}{F(B,b)} = \lim_{b \to 0} \frac{\big(A+b\big)^\alpha - A^\alpha - \alpha b A^{\alpha-1}}{\big(B+b\big)^\alpha - B^\alpha - \alpha b B^{\alpha-1}} = \left(\frac{A}{B}\right)^{\alpha-2},$$

and

$$\lim_{b \to \infty} \frac{F(A,b)}{F(B,b)} = \lim_{b \to +\infty} \frac{\big(A+b\big)^\alpha - A^\alpha - \alpha b A^{\alpha-1}}{\big(B+b\big)^\alpha - B^\alpha - \alpha b B^{\alpha-1}} = 1,$$

so the conclusion follows.

The following result provides some sufficient conditions for the condition (3.12) to hold.

Proposition 3.6. *Let* $\gamma(\mathrm{d}r)$ *and* $\Gamma(\mathrm{d}r)$ *be two measures on* $(0, +\infty)$ *such that for any* $\xi \in supp(\lambda)$

$$\gamma(A) \le \gamma_\xi(A) \le \Gamma(A), \quad A \in \mathcal{B}((0,+\infty)), \tag{3.19}$$

and

$$0 < \int_0^{+\infty} (r^2 \wedge r)\, \gamma(\mathrm{d}r) \le \int_0^{+\infty} (r^2 \wedge r)\, \Gamma(\mathrm{d}r) < +\infty. \tag{3.20}$$

If $\left(\int_\varepsilon^1 r\gamma(\mathrm{d}r)\right) \wedge \left(\int_1^{1/\varepsilon} r^2\gamma(\mathrm{d}r)\right) > 0$ *for all* $\varepsilon > 0$ *sufficiently close to* 0 *and there exist the limits*

$$q_0 := \limsup_{\varepsilon \to 0+} \frac{\int_\varepsilon^1 r\Gamma(\mathrm{d}r)}{\int_\varepsilon^1 r\gamma(\mathrm{d}r)}, \qquad q_\infty := \limsup_{\varepsilon \to 0+} \frac{\int_1^{1/\varepsilon} r^2\Gamma(\mathrm{d}r)}{\int_1^{1/\varepsilon} r^2\gamma(\mathrm{d}r)},$$

and both are finite, then (3.12) *is satisfied.*

Proof. Under (3.19) we clearly have

$$J_\gamma(b) \le \inf_{\xi \in \mathrm{supp}(\lambda)} J_{\gamma_\xi}(b) \le \sup_{\xi \in \mathrm{supp}(\lambda)} J_{\gamma_\xi}(b) \le J_\Gamma(b), \quad b \ge 0,$$

where

$$J_\gamma(b) := \int_0^{+\infty} (e^{-br} - 1 + br)\, \gamma(\mathrm{d}r),$$
$$J_\Gamma(b) := \int_0^{+\infty} (e^{-br} - 1 + br)\, \Gamma(\mathrm{d}r), \quad b \ge 0.$$

Therefore (3.12) is implied by the condition

$$J_\Gamma(b) \le K \cdot J_\gamma(b), \quad b \ge 0. \tag{3.21}$$

Since the functions $J_\gamma(\cdot), J_\Gamma(\cdot)$ are continuous, hence bounded on compacts, (3.21) is satisfied with some $K \ge 1$ if and only if

$$p_\infty := \limsup_{b \to +\infty} \frac{J_\Gamma(b)}{J_\gamma(b)} < +\infty, \tag{3.22}$$

and

$$p_0 := \limsup_{b \to 0+} \frac{J_\Gamma(b)}{J_\gamma(b)} < +\infty. \tag{3.23}$$

In what follows we show that (3.22) and (3.23) indeed hold.

Let us notice that for $x \ge 0$

$$e^{-x} - 1 + x \sim x \wedge x^2,$$

where the relation $\sim$ means that there exist universal positive numbers k and K such that

$$k\cdot x\wedge x^2\leq e^{-x}-1+x\leq K\cdot x\wedge x^2.$$

Thus, to prove (3.22) it is sufficient to prove that

$$\begin{aligned}\limsup_{b\to+\infty}&\frac{\int_0^{+\infty}\min(br,b^2r^2)\Gamma(\mathrm{d}r)}{\int_0^{+\infty}\min(br,b^2r^2)\gamma(\mathrm{d}r)}\\&=\limsup_{b\to+\infty}\frac{\int_0^{1/b}b^2r^2\Gamma(\mathrm{d}x)+\int_{1/b}^{+\infty}br\Gamma(\mathrm{d}r)}{\int_0^{1/b}b^2r^2\gamma(\mathrm{d}x)+\int_{1/b}^{+\infty}br\gamma(\mathrm{d}r)}<+\infty.\end{aligned}$$

Let us define the functions

$$G(y):=\int_{(y,+\infty)}r\Gamma(\mathrm{d}r),\qquad g(y):=\int_{(y,+\infty)}r\gamma(\mathrm{d}r),\quad y>0.$$

By integration by parts,

$$\int_0^{1/b}r^2\Gamma(\mathrm{d}r)=\int_0^{1/b}r(-\mathrm{d}G(r))=-r\cdot G(r)|_0^{1/b}+\int_0^{1/b}G(r)\mathrm{d}r. \tag{3.24}$$

We fix $\varepsilon>0$ and for $y\in(0,\varepsilon)$ estimate

$$\begin{aligned}y\cdot G(y)=y\int_y^{+\infty}r\Gamma(\mathrm{d}r)&=\int_y^{\varepsilon}yr\Gamma(\mathrm{d}r)+y\cdot G(\varepsilon)\\&\leq\int_0^{\varepsilon}r^2\Gamma(\mathrm{d}r)+y\cdot G(\varepsilon).\end{aligned}$$

From this it follows

$$\limsup_{y\to0+}y\cdot G(y)\leq\int_0^{\varepsilon}r^2\Gamma(\mathrm{d}r)$$

and by the finiteness of $\int_0^1 r^2\Gamma(\mathrm{d}r)$ and arbitrary choice of ε, we get

$$\limsup_{y\to0+}yG(y)=0.$$

Thus, (3.24) takes the form

$$\int_0^{1/b}r^2\Gamma(\mathrm{d}r)=\int_0^{1/b}r(-\mathrm{d}G(r))=-\frac{1}{b}\cdot G\left(\frac{1}{b}\right)+\int_0^{1/b}G(r)\mathrm{d}r.$$

Similarly,

$$\int_0^{1/b}r^2\gamma(\mathrm{d}r)=-\frac{1}{b}\cdot g\left(\frac{1}{b}\right)+\int_0^{1/b}g(r)\mathrm{d}r.$$

Now we calculate

$$\limsup_{b\to+\infty}\frac{\int_0^{1/b}b^2r^2\Gamma(\mathrm{d}x)+\int_{1/b}^{+\infty}br\Gamma(\mathrm{d}r)}{\int_0^{1/b}b^2r^2\gamma(\mathrm{d}x)+\int_{1/b}^{+\infty}br\gamma(\mathrm{d}r)}$$

$$= \limsup_{b\to+\infty} \frac{b^2\left(-\frac{1}{b}\cdot G\left(\frac{1}{b}\right)+\int_0^{1/b} G(r)\mathrm{d}r\right)+b\cdot G\left(\frac{1}{b}\right)}{b^2\left(-\frac{1}{b}\cdot g\left(\frac{1}{b}\right)+\int_0^{1/b} g(r)\mathrm{d}r\right)+b\cdot g\left(\frac{1}{b}\right)}$$

$$= \limsup_{b\to+\infty} \frac{\int_0^{1/b} G(r)\mathrm{d}r}{\int_0^{1/b} g(r)\mathrm{d}r} < +\infty,$$

where the last estimate follows from the assumption

$$q_0 = \limsup_{y\to 0+} \frac{\int_y^1 r\Gamma(\mathrm{d}r)}{\int_y^1 r\Gamma(\mathrm{d}r)} < +\infty$$

and the finiteness of $\int_1^{+\infty} r\Gamma(\mathrm{d}r)$, which yields that the ratio $G(r)/g(r)$ is separated from $+\infty$ for r sufficiently close to 0.

To prove (3.23) it is sufficient to show that

$$\limsup_{b\to 0+} \frac{\int_0^{+\infty} \min(br, b^2r^2)\Gamma(\mathrm{d}r)}{\int_0^{+\infty} \min(br, b^2r^2)\gamma(\mathrm{d}r)}$$
$$= \limsup_{b\to 0+} \frac{\int_0^{1/b} b^2r^2\Gamma(\mathrm{d}x)+\int_{1/b}^{+\infty} br\Gamma(\mathrm{d}r)}{\int_0^{1/b} b^2r^2\gamma(\mathrm{d}x)+\int_{1/b}^{+\infty} br\gamma(\mathrm{d}r)} < +\infty.$$

We define

$$Q(y) := \int_{(0,y]} r^2\Gamma(\mathrm{d}r), \quad q(y) := \int_{(0,y]} r^2\gamma(\mathrm{d}r), \quad y > 0.$$

By integration by parts,

$$\int_{1/b}^{+\infty} r\Gamma(\mathrm{d}r) = \int_{1/b}^{+\infty} \frac{1}{r}\mathrm{d}Q(r) = \frac{1}{r}Q(r)|_{1/b}^{+\infty} + \int_{1/b}^{+\infty} \frac{Q(r)}{r^2}\mathrm{d}r. \tag{3.25}$$

We fix $M > 0$ and for $y \in (M, +\infty)$ estimate

$$\frac{1}{y}Q(y) = \frac{1}{y}\int_0^y r^2\Gamma(\mathrm{d}r) = \frac{1}{y}\int_0^M r^2\Gamma(\mathrm{d}r) + \int_M^y \frac{r}{y}r\Gamma(\mathrm{d}r)$$
$$\leq \frac{1}{y}Q(M) + \int_M^{+\infty} r\Gamma(\mathrm{d}r).$$

From this it follows

$$\limsup_{y\to+\infty} \frac{1}{y}Q(y) \leq \int_M^{+\infty} r\Gamma(\mathrm{d}r)$$

and by the finiteness of $\int_1^{+\infty} r\Gamma(\mathrm{d}r)$ and arbitrary choice of M, we get

$$\limsup_{y\to+\infty} \frac{1}{y}Q(y) = 0.$$

Thus, (3.25) takes the form

$$\int_{1/b}^{+\infty} r\Gamma(\mathrm{d}r) = -bQ\left(\frac{1}{b}\right) + \int_{1/b}^{+\infty} \frac{Q(r)}{r^2}\mathrm{d}r.$$

Similarly,

$$\int_{0}^{1/b} r^2\gamma(\mathrm{d}r) = -bq\left(\frac{1}{b}\right) + \int_{1/b}^{+\infty} \frac{q(r)}{r^2}\mathrm{d}r.$$

Now we calculate

$$\begin{aligned}
&\limsup_{b\to 0+} \frac{\int_0^{1/b} b^2 r^2\Gamma(\mathrm{d}x) + \int_{1/b}^{+\infty} br\Gamma(\mathrm{d}r)}{\int_0^{1/b} b^2 r^2\gamma(\mathrm{d}x) + \int_{1/b}^{+\infty} br\gamma(\mathrm{d}r)}\\
&= \limsup_{b\to 0+} \frac{b^2 Q\left(\frac{1}{b}\right) + b\left(-bQ\left(\frac{1}{b}\right) + \int_{1/b}^{+\infty} Q(r)\frac{\mathrm{d}r}{r^2}\right)}{b^2 q\left(\frac{1}{b}\right) + b\left(-bq\left(\frac{1}{b}\right) + \int_{1/b}^{+\infty} q(r)\frac{\mathrm{d}r}{r^2}\right)}\\
&= \limsup_{b\to 0+} \frac{\int_{1/b}^{+\infty} Q(r)\frac{\mathrm{d}r}{r^2}}{\int_{1/b}^{+\infty} q(r)\frac{\mathrm{d}r}{r^2}} < +\infty,
\end{aligned}$$

where the last estimate follows from the assumption

$$q_\infty = \limsup_{y\to 0+} \frac{\int_1^{1/y} r^2\Gamma(\mathrm{d}r)}{\int_1^{1/y} r^2\Gamma(\mathrm{d}r)} < +\infty$$

and the finiteness of $\int_0^1 r^2\Gamma(\mathrm{d}r)$, which yields that the ratio $Q(r)/q(r)$ is separated from $+\infty$ for sufficiently large r. □

Example 3.7 (Spherically balanced Lévy measure)**.** Let us consider the case when the radial measures satisfy

$$\gamma(A) \le \gamma_\xi(A) \le K\cdot\gamma(A), \quad A \in \mathcal{B}((0,+\infty)),$$

with some finite constant $K \ge 1$ and a measure γ such that

$$0 < \int_0^{+\infty} (r^2 \wedge r)\,\gamma(\mathrm{d}r) < +\infty$$

and $\left(\int_\varepsilon^1 r\gamma(\mathrm{d}r)\right) \wedge \left(\int_1^{1/\varepsilon} r^2\gamma(\mathrm{d}r)\right) > 0$ for all $\varepsilon > 0$ sufficiently close to 0. Then $q_0 \le K$ and $q_\infty \le K$, so by Proposition 3.6 condition (3.12) is satisfied.

3.1.2 Jump measures with densities

In this subsection we formulate conditions required for the reducibility of (3.1) in the important case when the Lévy measure of Z has a density, i.e. $\nu(\mathrm{d}x) = g(x)\mathrm{d}x$. Let us consider the polar transformation $\xi : P := [0,\pi]^{d-2} \times [0,2\pi] \to \mathbb{S}^{d-1}$ given by

$$\xi_1 = \cos\alpha_1,$$

$$
\begin{aligned}
\xi_2 &= \sin\alpha_1 \cdot \cos\alpha_2, \\
\xi_3 &= \sin\alpha_1 \cdot \sin\alpha_2 \cdot \cos\alpha_3, \\
&\vdots \\
\xi_{d-1} &= \sin\alpha_1 \cdot \sin\alpha_2 \cdot \ldots \cdot \sin\alpha_{d-2} \cdot \cos\alpha_{d-1}, \\
\xi_d &= \sin\alpha_1 \cdot \sin\alpha_2 \cdot \ldots \cdot \sin\alpha_{d-2} \cdot \sin\alpha_{d-1}.
\end{aligned}
$$

The change of variables for polar coordinates $x = r\xi$ yields

$$
\begin{aligned}
\int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) &= \int_{\mathbb{R}^d} f(x)g(x)\mathrm{d}x \\
&= \int_P \int_0^{+\infty} \big(f(r\xi)g(r\xi) \cdot r^{d-1} \sin^{d-2}\alpha_1 \cdot \sin^{d-3}\alpha_2 \\
&\quad \cdot \sin^{d-4}\alpha_3 \cdot \ldots \cdot \sin\alpha_{d-2}\big)\, \mathrm{d}r\, \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{d-1},
\end{aligned}
\tag{3.26}
$$

for a ν-integrable function f. Noting that

$$
\begin{aligned}
&\sin^{d-2}\alpha_1 \cdot \sin^{d-3}\alpha_2 \cdot \sin^{d-4}\alpha_3 \cdot \ldots \cdot \sin\alpha_{d-2} \\
&= \sqrt{1-\xi_1^2} \cdot \sqrt{1-(\xi_1^2+\xi_2^2)} \cdot \ldots \cdot \sqrt{1-(\xi_1^2+\cdots+\xi_{d-2}^2)}
\end{aligned}
$$

we write (3.26) in the form

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}} \int_0^{+\infty} f(r\xi)g(r\xi)r^{d-1}\sqrt{1-\xi_1^2} \\
&\cdot \sqrt{1-(\xi_1^2+\xi_2^2)} \cdot \ldots \cdot \sqrt{1-(\xi_1^2+\cdots+\xi_{d-2}^2)}\, \mathrm{d}r\, \lambda(\mathrm{d}\xi).
\end{aligned}
\tag{3.27}
$$

In the above equation, λ stands for the image of the Lebesgue measure on P under the transformation $\xi : P \to \mathbb{S}^{d-1}$ restricted to the set

$$
\mathcal{G} := \{\xi \in \mathbb{S}^{d-1} : g(r\xi) \not\equiv 0,\ r \geq 0\}. \tag{3.28}
$$

This definition of λ is consistent with (3.4) and implies that

$$
\mathrm{supp}(\lambda) = \mathcal{G}.
$$

Theorem 3.8. *Let Z be a Lévy martingale with a covariance matrix Q of the Wiener part and a Lévy measure with density $\nu(\mathrm{d}x) = g(x)\mathrm{d}x$ satisfying the following conditions:*

a) $\int_{\mathbb{R}^d} (|x|^2 \wedge |x|)\, g(x)\mathrm{d}x < +\infty$,

b) Linear span$(\mathcal{G}) = \mathbb{R}^d$ *with $\mathcal{G}$ given by* (3.28),

c) $\lambda\left(\xi \in \mathcal{G} : \int_0^1 r^d g(r\xi)\mathrm{d}r = +\infty\right) > 0.$

Let us define the functions

$$
\underline{g}(r) := \inf_{|x|=r} g(x)\sqrt{1-\frac{x_1^2}{|x|^2}} \cdot \sqrt{1-\frac{x_1^2+x_2^2}{|x|^2}} \cdot \ldots \cdot \sqrt{1-\frac{x_1^2+x_2^2+\cdots+x_{d-2}^2}{|x|^2}}, \quad r \geq 0,
\tag{3.29}
$$

$$\bar{g}(r) := \sup_{|x|=r} g(x)\sqrt{1-\frac{x_1^2}{|x|^2}}\cdot\sqrt{1-\frac{x_1^2+x_2^2}{|x|^2}}\cdot\ldots\cdot\sqrt{1-\frac{x_1^2+x_2^2+\cdots+x_{d-2}^2}{|x|^2}}, \quad r \geq 0, \tag{3.30}$$

and assume that they satisfy

$$0 < \int_0^{+\infty} (r^d \wedge r^{d+1})\, \underline{g}(r)\mathrm{d}r \leq \int_0^{+\infty} (r^d \wedge r^{d+1})\, \bar{g}(r)\mathrm{d}r < +\infty, \tag{3.31}$$

and

$$\limsup_{\varepsilon\to 0+} \frac{\int_\varepsilon^1 r^d \bar{g}(r)\mathrm{d}r}{\int_\varepsilon^1 r^d \underline{g}(r)\mathrm{d}r} < +\infty \quad \textit{and} \quad \limsup_{\varepsilon\to 0+} \frac{\int_1^{1/\varepsilon} r^{d+1}\bar{g}(r)\mathrm{d}r}{\int_1^{1/\varepsilon} r^{d+1}\underline{g}(r)\mathrm{d}r} < +\infty, \tag{3.32}$$

and the denominators in (3.32) *are positive for all* $\varepsilon > 0$ *sufficiently close to* 0*. Let us also assume that* $G : [0, +\infty) \longrightarrow \mathbb{R}^d$ *is a continuous function such that*

$$G_0 := \lim_{x\to 0^+} \frac{G(x)}{|G(x)|},$$

exists.

Then, if (3.1) *generates an affine model, then it has the reducibility property.*

Proof. The proof is based on Theorem 3.1, so we check the required assumptions. By (a) we see that (3.5) holds while the assumption (b) on the spherical measure λ is equivalent to (3.11). In view of (3.27) the radial measures have the form

$$\gamma_\xi(\mathrm{d}r) = g(r\xi)r^{d-1}\sqrt{1-\xi_1^2}\cdot\sqrt{1-(\xi_1^2+\xi_2^2)}\cdot\ldots\cdot\sqrt{1-(\xi_1^2+\cdots+\xi_{d-2}^2)}\,\mathrm{d}r, \tag{3.33}$$

which implies the following equivalence, for $\xi \in \mathcal{G}$,

$$\int_0^1 r\gamma_\xi(\mathrm{d}r) = +\infty \quad \Longleftrightarrow \quad \int_0^1 r^d g(r\xi)\mathrm{d}r = +\infty.$$

This means that (c) implies (3.7). Now, with the use of Proposition 3.6, we argue that (3.12) is also satisfied. In view of (3.33), (3.29) and (3.30) we have

$$\underline{g}(r)r^{d-1}\mathrm{d}r \leq \gamma_\xi(\mathrm{d}r) \leq \bar{g}(r)r^{d-1}\mathrm{d}r,$$

so we see that the radial measures are bounded from below by the measure $\gamma(\mathrm{d}r) := \underline{g}(r)r^{d-1}\mathrm{d}r$ and from above by the measure and $\Gamma(\mathrm{d}r) := \bar{g}(r)r^{d-1}\mathrm{d}r$ as required in Proposition 3.6. Moreover, by (3.31), these measures satisfy (3.20) and the assumption (3.32) implies that the limits

$$q_0 := \limsup_{\varepsilon\to 0+} \frac{\int_\varepsilon^1 r\Gamma(\mathrm{d}r)}{\int_\varepsilon^1 r\gamma(\mathrm{d}r)}, \qquad q_\infty := \limsup_{\varepsilon\to 0+} \frac{\int_1^{1/\varepsilon} r^2\Gamma(\mathrm{d}r)}{\int_1^{1/\varepsilon} r^2\gamma(\mathrm{d}r)},$$

are finite. It follows from Proposition 3.6 that condition (3.12) is satisfied. □

Remark 3.9. In the case $d = 2$ the functions $\underline{g}(r)$, $\bar{g}(r)$ take a simple form, i.e.

$$\underline{g}(r) := \inf_{|x|=r} g(x), \qquad \bar{g}(r) := \sup_{|x|=r} g(x), \tag{3.34}$$

and therefore (3.31) and (3.32) in Theorem 3.8 provide simple conditions for the reducibility of (3.1).

Example 3.10. Let us consider the following density on the plane

$$g(x,y) := f(x^2) \cdot h(x^2+y^2), \quad \text{with } f(x) := 1 + e^{-x^2},\ h(z) := \frac{1}{z^\beta},\ \beta \in \left(\frac{3}{2}, 2\right).$$

We show that this density meets the assumptions of Theorem 3.8. Denoting $\tilde{x} := (x, y)$ and using the fact that f is bounded by 2 we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} (|\,\tilde{x}\,|^2 \wedge |\,\tilde{x}\,|) g(\tilde{x}) \mathrm{d}\tilde{x} &\le 2\left(\int_{|\tilde{x}|\le 1} |\,\tilde{x}\,|^2 \cdot \frac{1}{|\,\tilde{x}\,|^{2\beta}} \mathrm{d}\tilde{x} + \int_{|\tilde{x}|>1} |\,\tilde{x}\,| \cdot \frac{1}{|\,\tilde{x}\,|^{2\beta}} \mathrm{d}\tilde{x}\right) \\ &= 2\left(\int_{|\tilde{x}|\le 1} \frac{1}{|\,\tilde{x}\,|^{2\beta-2}} \mathrm{d}\tilde{x} + \int_{|\tilde{x}|>1} \frac{1}{|\,\tilde{x}\,|^{2\beta-1}} \mathrm{d}\tilde{x}\right) < +\infty,\end{aligned}$$

because $\beta \in \left(\frac{3}{2}, 2\right)$. Hence (a) is satisfied. Since g is strictly positive, we see that $\mathcal{G} = \mathbb{S}^1$ and that (b) is satisfied. For any $\xi = (\xi_1, \xi_2) \in \mathcal{G}$ we have

$$\int_0^1 r^2 g(r\xi) \mathrm{d}r = \int_0^1 r^2 f(r\xi_1) h(r^2) \mathrm{d}r = \int_0^1 (1 + e^{-r\xi_1}) \frac{1}{r^{2\beta-2}} \mathrm{d}r = +\infty,$$

so condition (c) is satisfied as well. Since f is decreasing we obtain, for $r > 0$,

$$\underline{g}(r) = \inf_{x^2+y^2=r^2} f(x^2)h(x^2+y^2) = \inf_{x^2\in[0,r^2]} f(x^2)h(r^2) = f(r^2)h(r^2) = (1+e^{-r^2})\frac{1}{r^{2\beta}},$$

and

$$\bar{g}(r) = \sup_{x^2+y^2=r^2} f(x^2)h(x^2+y^2) = f(0)h(r^2) = \frac{2}{r^{2\beta}}.$$

Condition (3.31) follows from the estimations

$$\int_0^1 r^3 \bar{g}(r) \mathrm{d}r = \int_0^1 r^3 \frac{2}{r^{2\beta}} \mathrm{d}r = \int_0^1 \frac{2}{r^{2\beta-3}} \mathrm{d}r < +\infty,$$
$$\int_1^{+\infty} r^2 \bar{g}(r) \mathrm{d}r = \int_0^1 \frac{2}{r^{2\beta-2}} \mathrm{d}r < +\infty,$$

which are true because $\beta \in (\frac{3}{2}, 2)$. From the inequality

$$\bar{g}(r) \le 2 \cdot \underline{g}(r), \quad r > 0,$$

we deduce (3.32).

It is clear from the above analysis that f can be replaced by any function separated from 0 and $+\infty$.

4 Shapes of simple forward curves

We would like to characterize the shape of a simple yield curve at time zero defined by

$$F(x) = \frac{1}{x}\left(\frac{1}{P(0,x)} - 1\right), \quad x \geq 0,$$

where

$$P(0,x) = e^{-A(x)-R(0)B(x)}$$

stands for the price at time 0 of a zero-coupon bond maturing at x in the affine model. The short rate $R(\cdot)$ starting from $R(0) > 0$ is a process given by a generalized CIR equation

$$\mathrm{d}R(t) = (aR(t) + b)\mathrm{d}t + C \cdot R(t-)^{1/\alpha}\mathrm{d}Z^{\alpha}(t), \tag{4.1}$$

driven by a one-dimensional stable process with index $\alpha \in (1,2]$. As we already mentioned, the functions $A(\cdot)$, $B(\cdot)$ are uniquely determined by the form of the generator of (4.1) and in our case it follows from [17] that they solve the following differential equations:

$$A'(x) = \mathcal{F}(B(x)), \quad A(0) = 0, \tag{4.2}$$
$$B'(x) = \mathcal{R}(B(x)), \quad B(0) = 0, \tag{4.3}$$

where

$$\mathcal{F}(\lambda) := b\lambda, \tag{4.4}$$
$$\mathcal{R}(\lambda) := 1 + a\lambda - \eta\lambda^{\alpha}, \tag{4.5}$$

with $a \in \mathbb{R}$, $b \geq 0$, $\alpha \in (1,2]$ and $0 < \eta := \frac{1}{2}C^2$ if $\alpha \in (1,2)$ while $\eta := C^{\alpha} \cdot \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}$ for $\alpha = 2$. Note that the function $\mathcal{R}(\cdot)$ starts from 1 and has only one root $\lambda_0 > 0$. Its monotonicity depends on the sign of the parameter a. We have two cases.

- If $a \leq 0$ then

$$\mathcal{R} \text{ is positive on } [0,\lambda_0) \text{ and decreasing with } \mathcal{R}(0) = 1, \tag{4.6}$$
$$\mathcal{R}' \text{ is negative and decreasing with } \mathcal{R}'(0) = a. \tag{4.7}$$

- If $a > 0$ then

$$\mathcal{R} \text{ is positive on } [0,\lambda_0),\ \text{increasing on } [0,\bar{\lambda}_0] \text{ and decreasing on } [\bar{\lambda}_0,\lambda_0), \mathcal{R}(0) = 1, \tag{4.8}$$
$$\mathcal{R}' \text{ is decreasing on } [0,\lambda_0],\ \text{positive on } [0,\bar{\lambda}_0) \text{ and negative on } [\bar{\lambda}_0,\lambda_0), \mathcal{R}'(0) = a, \tag{4.9}$$

  with some point $\bar{\lambda}_0 \in (0,\lambda_0)$.

It follows from the above and (4.3) that the function B is increasing and

$$\lim_{x\to+\infty} B(x) = \lambda_0. \tag{4.10}$$

Our aim is to characterize the shape of the function

$$F(x) = \frac{e^{A(x)+R\cdot B(x)} - 1}{x},$$

where $R := R(0)$ in terms of the parameters $R > 0, a \in \mathbb{R}, b \geq 0, \eta > 0, \alpha \in (1,2]$.

First, by direct computations, one can characterize the behavior of $F(\cdot)$ at zero and at infinity.

Proposition 4.1. *The function $F(\cdot)$ satisfies*

$$\lim_{x\to 0+} F(x) = R, \tag{4.11}$$

$$\lim_{x\to 0+} F'(x) = \frac{1}{2}\left[R^2 + aR + b\right] \tag{4.12}$$

and

$$\lim_{x\to+\infty} F(x) = \begin{cases} 0 & \text{if } b = 0, \\ +\infty & \text{if } b > 0. \end{cases} \tag{4.13}$$

Proof. By L'Hôpital's rule, (4.2) and (4.3) we have

$$\lim_{x\to 0+} F(x) = \lim_{x\to 0+} e^{A(x)+R\cdot B(x)}(bB(x) + R\cdot\mathcal{R}(B(x))) = R. \tag{4.14}$$

For the case $b > 0$, by (4.10), we have

$$A(x) = A(0) + \int_0^x A'(v)\mathrm{d}v = b\cdot\int_0^x B(v)\mathrm{d}v \underset{x\to+\infty}{\longrightarrow} +\infty,$$

and therefore

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} e^{A(x)+R\cdot B(x)}(bB(x) + R\cdot\mathcal{R}(B(x))) = +\infty.$$

If $b = 0$ then (4.10) implies that

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \frac{e^{R\cdot B(x)} - 1}{x} = 0.$$

Since

$$F'(x) = \frac{1}{x^2}\left(e^{A(x)+R\cdot B(x)}[x(A'(x) + R\cdot B'(x)) - 1] + 1\right), \quad x > 0, \tag{4.15}$$

the application of L'Hôpital's rule yields

$$\begin{aligned}\lim_{x\to 0+} F'(x) &= \lim_{x\to 0+} \frac{e^{A(x)+R\cdot B(x)}}{2x}\left[x\left(A'(x) + RB'(x)\right)^2 + x\left(A''(x) + RB''(x)\right)\right] \\ &= \lim_{x\to 0+} \frac{e^{A(x)+R\cdot B(x)}}{2}\left[\left(A'(x) + RB'(x)\right)^2 + \left(A''(x) + RB''(x)\right)\right].\end{aligned}$$

By (4.2)–(4.5) we obtain

$$A''(x) = bB'(x) = b\mathcal{R}(B(x)), \qquad B''(x) = \mathcal{R}'(B(x)) \cdot \mathcal{R}(B(x)),$$

and, consequently,

$$\begin{aligned}\lim_{x\to 0+} F'(x) = \lim_{x\to 0+} \frac{e^{A(x)+R\cdot B(x)}}{2} &\Big[\Big(bB(x) + R\cdot\mathcal{R}(B(x))\Big)^2 \\ &+ b\cdot\mathcal{R}(B(x)) + R\cdot\mathcal{R}'(B(x))\cdot\mathcal{R}(B(x))\Big] \\ &= \frac{1}{2}[R^2 + b + Ra].\end{aligned}$$

□

The monotonicity of $F(\cdot)$ will be studied with the use of the auxiliary function defined by

$$H(x) := e^{-b\int_0^x B(v)\mathrm{d}v - RB(x)} - 1 + x\Big(bB(x) + R\cdot\mathcal{R}(B(x))\Big), \quad x > 0. \tag{4.16}$$

with the following properties.

Proposition 4.2. (a) *For* $x > 0$ *the functions* $F'(\cdot)$ *and* $H(\cdot)$ *have the same roots and*

$$F'(x) > 0 \quad\Longleftrightarrow\quad H(x) > 0. \tag{4.17}$$

(b) *If* $H(x) = 0$, $x > 0$, *then*

$$H'(x) = x\cdot G(B(x)), \tag{4.18}$$

where

$$G(\lambda) := \Big(b\lambda + R\cdot\mathcal{R}(\lambda)\Big)^2 + \mathcal{R}(\lambda)\Big(b + R\cdot\mathcal{R}'(\lambda)\Big), \quad \lambda\in(0,\lambda_0). \tag{4.19}$$

Proof. (a) By (4.15) the condition $F'(x) = 0$, $x > 0$, is equivalent to

$$e^{A(x)+R\cdot B(x)}\Big(x(A'(x) + RB'(x)) - 1\Big) = -1,$$

which, in view of (4.2), (4.3) and (4.4), yields $H(x) = 0$. In the same way one proves (4.17).

(b) It follows from (4.16) that

$$\begin{aligned}H'(x) =& -e^{-b\int_0^x B(v)\mathrm{d}v - R\cdot B(x)}\Big(bB(x) + R\cdot B'(x)\Big) + b\Big(B(x) + xB'(x)\Big) \\ &+ R\Big(\mathcal{R}(B(x)) + x\mathcal{R}'(B(x))B'(x)\Big) \\ =& -\Big[H(x) + 1 - x\Big(bB(x) + R\cdot\mathcal{R}(B(x))\Big)\Big]\cdot\Big(bB(x) + RB'(x)\Big) \\ &+ b\Big(B(x) + xB'(x)\Big) + R\Big(\mathcal{R}(B(x)) + x\mathcal{R}'(B(x))B'(x)\Big), \quad x > 0.\end{aligned}$$

If x is a root of $H(\cdot)$, then we obtain

$$\begin{aligned}
H'(x) = & - bB(x) - RB'(x) + x\Big(bB(x) + RB'(x)\Big)^2 \\
& + bB(x) + xbB'(x) + RB'(x) + xR \cdot \mathcal{R}'(B(x))B'(x) \\
& = x\Big(bB(x) + RB'(x)\Big)^2 + xB'(x)\Big(b + R \cdot \mathcal{R}'(B(x))\Big) \\
& = x \cdot G(B(x)).
\end{aligned}$$

□

Motivated by the asymptotic behavior of F, which depends on the parameter b, see (4.13), we distinguish two cases: $b = 0$ and $b > 0$.

Theorem 4.3. *I) Let $b = 0$.*

(a) If $a \leq 0$ and $R + a \leq 0$ then the curve $F(\cdot)$ is inverse, i.e. decreases in $(0, +\infty)$.

(b) If $a \leq 0$ and $R + a > 0$ then the curve $F(\cdot)$ is humped, i.e. increases in $(0, x_1)$ and decreases in $(x_1, +\infty)$ with some $x_1 > 0$.

(c) If $a > 0$ then the curve $F(\cdot)$ is humped, i.e. increases in $(0, x_1)$ and decreases in $(x_1, +\infty)$ with some $x_1 > 0$.

II) Let $b > 0$. If

$$b + R \cdot \mathcal{R}'(\lambda_0) > 0, \tag{4.20}$$

then $F(\cdot)$ increases in $(0, +\infty)$. In particular, for any fixed model parameters $a \in \mathbb{R}$, $b > 0$, $\eta > 0$, $\alpha \in (1, 2]$ the curve is normal for small initial values $F(0) = R$, i.e. such that $R < -\frac{b}{\mathcal{R}'(\lambda_0)}$.

In view of the result above, we see how the curve shapes depend on the parameters a, b, C in the equation (4.1). They also depend on the initial value of the short rate R and on the noise characteristics, which are hidden in the function $\mathcal{R}$.

Proof. (I) For $b = 0$ the function $G(\cdot)$ given by (4.19) simplifies to

$$G(\lambda) = R \cdot \mathcal{R}(\lambda)\Big[R \cdot \mathcal{R}(\lambda) + \mathcal{R}'(\lambda)\Big], \quad \lambda \in (0, \lambda_0),$$

so it follows that the sign of $G(\lambda)$ is the same as the sign of the factor $R \cdot \mathcal{R}(\lambda) + \mathcal{R}'(\lambda)$. By (4.6), (4.7), (4.8) and (4.9) we see that the function $\lambda \longrightarrow R \cdot \mathcal{R}(\lambda) + \mathcal{R}'(\lambda)$ is

- decreasing in $(0, \lambda_0)$ if $a \leq 0$,
- positive in $(0, \bar{\lambda}_0]$ and decreasing in $(\bar{\lambda}_0, \lambda_0)$ if $a > 0$.

Consequently, the function $G(\cdot)$ may change the sign at most once. Since

$$G(0) = R(R + a),$$

we obtain the following.

If $a \leq 0$ and $R + a \leq 0$, then $G(\lambda) < 0, \lambda \in (0, \lambda_0)$. (4.21)

If $a \leq 0$ and $R + a > 0$, then $G(\lambda) > 0, \lambda \in (0, \lambda^*)$, and $G(\lambda) < 0, \lambda \in (\lambda^*, \lambda_0)$, (4.22)

where λ^* is the unique solution of the equation $R \cdot \mathcal{R}(\lambda) = -\mathcal{R}'(\lambda)$.

If $a > 0$ then $G(\lambda) > 0, \lambda \in (0, \lambda^*)$, and $G(\lambda) < 0, \lambda \in (\lambda^*, \lambda_0)$, (4.23)

where $\lambda^* > \bar{\lambda}_0$ is the unique solution of the equation $R \cdot \mathcal{R}(\lambda) = -\mathcal{R}'(\lambda)$.

By (4.11), (4.13) and (4.12) we have

$$\lim_{x \to 0+} F(x) = R > 0, \quad \lim_{x \to 0+} F'(x) = \frac{1}{2}[R(R + a)], \quad \text{and} \quad \lim_{x \to +\infty} F(x) = 0. \tag{4.24}$$

(a) If $R+a < 0$, then $F(\cdot)$ is decreasing close to zero. If $x_0 > 0$ were a point where the monotonicity of F changes then $F'(x_0) = 0$. By Proposition 4.2, $H(x_0) = 0$. But, by (4.18) we have that then

$$H'(x_0) = x_0 \cdot G(B(x_0)).$$

However, (4.21) implies that $H'(x_0) < 0$, so $H(x) < 0$ for $x > x_0$ close to x_0. Again, by Proposition 4.2, we obtain that $F'(x) < 0$, which means that $F(\cdot)$ does not change its monotonicity.

(b) Since $R + a > 0$, by (4.24), F starts to increase from the value R at zero. The monotonicity of F must change at some point x_0 as F disappears at infinity, due to (4.24). If x_0 is such that $B(x_0) < \lambda^*$ then

$$F'(x_0) = 0 \quad \Longrightarrow \quad H(x_0) = 0 \quad \Longrightarrow \quad H'(x_0) = x_0 \cdot G(B(x_0)) > 0.$$

Consequently, $H(x) > 0$ for $x > x_0$ close to x_0 and thus $F'(x) > 0$, so F does not change the monotonicity at x_0. So we conclude that x_0 is such that $B(x_0) \geq \lambda^*$. Using similar arguments as above but based now on the negativity of G in (λ^*, λ_0) it can be shown that there is no point where F changes the monotonicity again.

(c) One repeats the arguments from (b).

(II) By (4.7) and (4.9) the function $\lambda \longrightarrow b + R \cdot \mathcal{R}'(\lambda)$ is decreasing, so (4.20) implies that

$$b + Ra = b + R \cdot \mathcal{R}'(0) > 0 \tag{4.25}$$

and

$$G(\lambda) = \Big(b\lambda + R \cdot \mathcal{R}(\lambda)\Big)^2 + \mathcal{R}(\lambda)\Big(b + R \cdot \mathcal{R}'(\lambda)\Big) > 0, \quad \lambda \in (0, \lambda_0). \tag{4.26}$$

By (4.12) and (4.25) we see that

$$\lim_{x \to 0} F'(x) = \frac{1}{2}[R^2 + aR + b] > 0,$$

so the function F is increasing in the vicinity of zero. The monotonicity of F cannot change at any point. Assume to the contrary that $F'(x_0) = 0$ for some $x_0 > 0$. Then by Proposition 4.2 and (4.18) we have

$$F'(x_0) = 0 \quad \Longrightarrow \quad H(x_0) = 0 \quad \Longrightarrow \quad H'(x_0) = x_0 \cdot G(B(x_0)) > 0,$$

where the last inequality is a consequence of (4.26). This means that F increases close to x_0. □

5 Proofs of the main results

5.1 Auxiliary results

Let us consider the generating equation (3.1) with some function G and a Lévy martingale Z with the Laplace exponent of its jump part $J_X(\cdot)$, see (2.5) for definition. Our first aim is to estimate the function $J_X(bG(x)), b, x \geq 0$, with the use of the function $J_X(bG_0), b \geq 0$, for x such that $G(x)/|G(x)|$ is close to G_0. The solution of this problem is presented in Lemma 5.1, Proposition 5.3 and Proposition 5.4.

Let $\rho(\mathrm{d}v)$ be an auxiliary Lévy measure on $(0, +\infty)$ satisfying

$$\int_0^{+\infty} (v^2 \wedge v)\rho(\mathrm{d}v) < +\infty, \tag{5.1}$$

and

$$J_\rho(z) := \int_{(0,+\infty)} (e^{-zv} - 1 + zv)\rho(\mathrm{d}v), \quad z \geq 0, \tag{5.2}$$

The second aim of this section is to provide sufficient conditions for J_ρ to be a power function. This problem is solved in Lemma 5.5 and Lemma 5.6.

Lemma 5.1. *The function $H : [0, +\infty) \longrightarrow \mathbb{R}$ given by*

$$H(z) = e^{-z} - 1 + z,$$

is convex, strictly increasing and

$$\min\{1, t^2\} \cdot H(z) \leq H(tz) \leq \max\{1, t^2\} \cdot H(z), \quad z \geq 0, \ t > 0. \tag{5.3}$$

Proof. Since $H'(z) = 1 - e^{-z}$ the monotonicity and convexity of H follows. For $t \geq 1$ it follows from the monotonicity of H that

$$H(tz) \geq H(z) = \min\{1, t^2\}H(z).$$

Let us notice that the function

$$t \mapsto \frac{(1 - e^{-t})t}{e^{-t} - 1 + t}, \quad t \geq 0,$$

is strictly decreasing, with limit 2 at zero and 1 at infinity. This implies that

$$(e^{-t} - 1 + t) < (1 - e^{-t})t < 2(e^{-t} - 1 + t), \quad t \in (0, +\infty). \tag{5.4}$$

From (5.4) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln H(s)=\frac{H'(s)}{H(s)}=\frac{1-e^{-s}}{e^{-s}-1+s}\leq\frac{2}{s},\qquad s>0,$$

and, consequently, we obtain that for $t\geq 1$

$$\ln H(tz)-\ln H(z)\leq\int_z^{tz}\frac{2}{s}\mathrm{d}s=\ln t^2.$$

Thus

$$\min\{1,t^2\}H(z)=H(z)\leq H(tz)\leq t^2H(z)=\max\{1,t^2\}H(z). \tag{5.5}$$

Using the monotonicity of H and (5.5) we see that for $t\in(0,1)$

$$H(tz)\leq H(z)=H\left(\frac{1}{t}tz\right)\leq\frac{1}{t^2}H(tz),$$

so also for $t\in(0,1)$

$$\min\{1,t^2\}H(z)=t^2H(z)\leq H(tz)\leq H(z)=\max\{1,t^2\}H(z).$$

□

Corollary 5.2. *It follows from* (5.3) *and the formula*

$$J_\rho(z):=\int_{(0,+\infty)}H(zv)\rho(\mathrm{d}v)<+\infty$$

that the function J_ρ satisfies

$$\min\left\{1,t^2\right\}\cdot J_\rho(z)\leq J_\rho(tz)\leq\max\left\{1,t^2\right\}\cdot J_\rho(z),\qquad z\geq 0,\ t>0. \tag{5.6}$$

Proposition 5.3. *If* (3.1) *generates an affine model and G_∞ is an arbitraty limit point of the set*

$$\left\{\frac{G(x)}{|G(x)|}:x>0\right\}$$

then

$$\nu\left\{y\in\mathbb{R}^d:\langle G_\infty,y\rangle<0\right\}=0.$$

Proof. Assume that

$$\nu\left\{y\in\mathbb{R}^d:\langle G_\infty,y\rangle<0\right\}=\nu\left\{y\in\mathbb{R}^d\setminus\{0\}:\left\langle G_\infty,\frac{y}{|y|}\right\rangle<0\right\}>0.$$

Then there exists a natural n such that for

$$V_n:=\left\{y\in\mathbb{R}^d\setminus\{0\}:\left\langle G_\infty,\frac{y}{|y|}\right\rangle<-\frac{1}{n}\right\}$$

one has $\nu\left(V_n\right)>0$.

Let x be such that
$$\left|\frac{G(x)}{|G(x)|} - G_\infty\right| \le \frac{1}{2n}.$$
It follows from the Schwarz inequality that, for any $y \in \mathbb{R}^d$,
$$\left|\left\langle \frac{G(x)}{|G(x)|}, y\right\rangle - \langle G_\infty, y\rangle\right| \le \left|\frac{G(x)}{|G(x)|} - G_\infty\right| |y| \le \frac{1}{2n}|y|. \tag{5.7}$$
Let $y \in V_n$. From (5.7) and the definition of V_n we estimate
$$\left\langle \frac{G(x)}{|G(x)|}, y\right\rangle \le \langle G_\infty, y\rangle + \frac{1}{2n}|y| < -\frac{1}{n}|y| + \frac{1}{2n}|y| = -\frac{1}{2n}|y| < 0.$$
Hence
$$\nu\left\{y \in \mathbb{R}^d : \left\langle \frac{G(x)}{|G(x)|}, y\right\rangle < 0\right\} \ge \nu(V_n) > 0$$
which is a contradiction to (2.11). □

Proposition 5.4. *Let us assume that* (3.1) *is a generating equation and that* ν *has the form* (3.3) *where* λ *satisfies* (3.11) *and* $\gamma_\xi(\mathrm{d}r)$ *satisfies* (3.12). *Let* G_∞ *be any limit point of the set*
$$\left\{\frac{G(x)}{|G(x)|} : x > 0\right\}.$$
Define
$$M_{G_\infty}(b) := J_X(b \cdot G_\infty) = \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} H(b\langle G_\infty, r\cdot\xi\rangle)\,\gamma_\xi(\mathrm{d}r)\,\lambda(\mathrm{d}\xi),$$
where $H(z) := e^{-z} - 1 + z$. *There exists a function* $\delta : (0,1) \to (0,+\infty)$ *such that for any* $\varepsilon_0 > 0$, *any* $b \ge 0$ *and* $x > 0$ *such that* $\left|\frac{G(x)}{|G(x)|} - G_\infty\right| \le \delta(\varepsilon_0)$ *we have*
$$(1-\varepsilon_0)\,M_{G_\infty}(b\,|G(x)|) \le J_X(bG(x)) \le (1+\varepsilon_0)\,M_{G_\infty}(b\,|G(x)|). \tag{5.8}$$

Proof. Let $\varepsilon \in (0,1)$ be such that
$$(1+\varepsilon)^2\left(1 + \frac{4K\varepsilon}{(1-\varepsilon)^3}\right) \le 1 + \varepsilon_0, \qquad \frac{(1-\varepsilon)^2}{\left(1 + \frac{K\varepsilon}{1-\varepsilon}\right)} \ge 1 - \varepsilon_0. \tag{5.9}$$
Let us assume that
$$\lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle > 0\right\} = \lambda\left(\mathbb{S}^{d-1}\right) - \lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle = 0\right\} = 1, \tag{5.10}$$
(we can assume this, multiplying λ by a positive constant, provided that
$$\lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle > 0\right\} > 0,$$
otherwise it follows from Proposition 5.3 that we get a degenerated case
$$\lambda\left(\mathbb{S}^{d-1}\right) = \lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle = 0\right\}$$

where (3.11) is broken). Let $\eta \in (0,1)$ be such that

$$\lambda\left\{\xi \in \mathbb{S}^{d-1} : 0 < \langle G_\infty, \xi\rangle < \eta\right\} \le \varepsilon. \tag{5.11}$$

Moreover, by Proposition 5.3,

$$\begin{aligned} 0 = \nu\left\{y \in \mathbb{R}^d : \langle G_\infty, y\rangle < 0\right\} &= \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} r\langle G_\infty, \xi\rangle \gamma_\xi(\mathrm{d}r)\lambda(\mathrm{d}\xi) \\ &\ge \lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle < 0\right\} \cdot \sup_{\xi\in\mathbb{S}^{d-1}} \gamma_\xi\left(\mathbb{R}_+\right), \end{aligned}$$

so it follows that

$$\lambda\left\{\xi \in \mathbb{S}^{d-1} : \langle G_\infty, \xi\rangle < 0\right\} = 0.$$

Let us define

$$\mathbb{V}_\eta = \left\{\xi \in \mathbb{S}^{d-1} : 0 < \langle G_\infty, \xi\rangle < \eta\right\}.$$

Let x be such that

$$\left|\frac{G(x)}{|G(x)|} - G_\infty\right| \le \delta\left(\varepsilon_0\right) := \eta \cdot \varepsilon.$$

From Lemma 5.1, for $b, r \ge 0$ and $\xi \in \mathbb{S}^{d-1}$ such that $\langle G_\infty, \xi\rangle \in [0,\eta)$ we get estimates

$$\begin{aligned} H\left(b \cdot r \langle G(x), \xi\rangle\right) &\le H\left(b \cdot r\,|G(x)|\left(\langle G_\infty, \xi\rangle + \left|\left\langle \frac{G(x)}{|G(x)|} - G_\infty, \xi\right\rangle\right|\right)\right) \\ &\le \max\left\{H\left(b \cdot r\,|G(x)|\,2\langle G_\infty, \xi\rangle\right), H\left(b \cdot r \cdot |G(x)|\,2\left|\left\langle \frac{G(x)}{|G(x)|} - G_\infty, \xi\right\rangle\right|\right)\right\} \\ &\le \max\left\{H\left(b \cdot r\,|G(x)|\,2\eta\right), H\left(b \cdot r \cdot |G(x)|\,2\eta \cdot \varepsilon\right)\right\} \\ &= H\left(b \cdot r\,|G(x)|\,2\eta\right) \\ &\le 4H\left(b \cdot r\,|G(x)|\,\eta\right). \end{aligned} \tag{5.12}$$

It follows from (5.12), (5.11) and (3.12) that

$$\begin{aligned} &\int_{\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r \langle G(x), \xi\rangle\right) \gamma_\xi\left(\mathrm{d}r\right) \lambda(\mathrm{d}\xi) \\ &\quad\le 4 \int_{\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\eta\right) \gamma_\xi\left(\mathrm{d}r\right) \lambda(\mathrm{d}\xi) \\ &\quad\le 4\varepsilon \sup_{\xi\in\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\eta\right) \gamma_\xi\left(\mathrm{d}r\right) \\ &\quad\le 4K\varepsilon \inf_{\xi\in\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\eta\right) \gamma_\xi\left(\mathrm{d}r\right). \end{aligned} \tag{5.13}$$

From Lemma 5.1, for $b, r \ge 0$ and $\xi \in \mathbb{S}^{d-1}$ such that $\langle G_\infty, \xi\rangle \in [\eta, 1]$, we get further estimates

$$H\left(b \cdot r \langle G(x), \xi\rangle\right) \le H\left(b \cdot r\,|G(x)|\left(\langle G_\infty, \xi\rangle + \left|\left\langle \frac{G(x)}{|G(x)|} - G_\infty, \xi\right\rangle\right|\right)\right)$$

$$
\begin{aligned}
&\leq H\left(b \cdot r\,|G(x)|\left(\langle G_\infty, \xi\rangle + \langle G_\infty, \xi\rangle \varepsilon\right)\right) \\
&\leq (1+\varepsilon)^2 H\left(b \cdot r\,|G(x)|\,\langle G_\infty, \xi\rangle\right), \qquad (5.14)
\end{aligned}
$$

and

$$
\begin{aligned}
H\left(b \cdot r\,\langle G(x), \xi\rangle\right) &\geq H\left(b \cdot r\,|G(x)|\left(\langle G_\infty, \xi\rangle - \left|\left\langle \frac{G(x)}{|G(x)|} - G_\infty, \xi\right\rangle\right|\right)\right) \\
&\geq H\left(b \cdot r\,|G(x)|\left(\langle G_\infty, \xi\rangle - \langle G_\infty, \xi\rangle \varepsilon\right)\right) \\
&\geq (1-\varepsilon)^2 H\left(b \cdot r\,|G(x)|\,\langle G_\infty, \xi\rangle\right). \qquad (5.15)
\end{aligned}
$$

Notice that by (5.10) and (5.11), $\lambda\left(\mathbb{S}^{d-1} \setminus \mathbb{V}_\eta\right) \geq 1 - \varepsilon$. From (5.15) and then from $\lambda\left(\mathbb{S}^{d-1} \setminus \mathbb{V}_\eta\right) \geq 1 - \varepsilon$ and (5.13) we obtain

$$
\begin{aligned}
&\int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\geq \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} (1-\varepsilon)^2 H\left(b \cdot r\,|G(x)|\,\langle G_\infty, \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\geq (1-\varepsilon)^2 \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\eta\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\geq (1-\varepsilon)^2 (1-\varepsilon) \inf_{\xi \in \mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\eta\right) \gamma_\xi\,(\mathrm{d}r) \\
&\geq \frac{(1-\varepsilon)^3}{4K\varepsilon} \int_{\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi). \qquad (5.16)
\end{aligned}
$$

From (5.16) and (5.14) we obtain

$$
\begin{aligned}
J_X\left(bG(x)\right) &= \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\quad + \int_{\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\leq \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\quad + \frac{4K\varepsilon}{(1-\varepsilon)^3} \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta} \int_0^{+\infty} H\left(b \cdot r\,\langle G(x), \xi\rangle\right) \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&\leq (1+\varepsilon)^2 \left(1 + \frac{4K\varepsilon}{(1-\varepsilon)^3}\right) \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} H\left(b \cdot r\,|G(x)|\,\langle G_\infty, \xi\rangle\right) \\
&\quad \times \gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi) \\
&= (1+\varepsilon)^2 \left(1 + \frac{4K\varepsilon}{(1-\varepsilon)^3}\right) M_{G_\infty}\left(b \cdot r\,|G(x)|\right).
\end{aligned}
$$

Hence

$$
J_X\left(bG(x)\right) \leq (1+\varepsilon)^2 \left(1 + \frac{4K\varepsilon}{(1-\varepsilon)^3}\right) M_{G_\infty}\left(b \cdot r\,|G(x)|\right). \qquad (5.17)
$$

In order to get the lower bound, let us notice that

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad\geq \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\eta\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad\geq (1-\varepsilon)\inf_{\xi\in\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\eta\right)\gamma_\xi\,(\mathrm{d}r)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad\leq \int_{\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\eta\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad\leq \varepsilon\sup_{\xi\in\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\eta\right)\gamma_\xi\,(\mathrm{d}r)\\
&\quad\leq K\varepsilon\inf_{\xi\in\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\eta\right)\gamma_\xi\,(\mathrm{d}r)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\geq \frac{1-\varepsilon}{K\varepsilon}\int_{\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi),
\end{aligned}\tag{5.18}$$

and from this we obtain

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad= \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\qquad+\int_{\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\quad\leq \left(1+\frac{K\varepsilon}{1-\varepsilon}\right)\int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi).
\end{aligned}\tag{5.19}$$

From (5.15) and (5.19) we get

$$\begin{aligned}
J_X\left(bG(x)\right) &\geq \int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,\langle G(x),\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)\\
&\geq (1-\varepsilon)^2\int_{\mathbb{S}^{d-1}\setminus\mathbb{V}_\eta}\int_0^{+\infty} H\left(b\cdot r\,|G(x)|\,\langle G_\infty,\xi\rangle\right)\gamma_\xi\,(\mathrm{d}r)\,\lambda(\mathrm{d}\xi)
\end{aligned}$$

$$\geq \frac{(1-\varepsilon)^2}{\left(1+\frac{K\varepsilon}{1-\varepsilon}\right)} \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} H\left(b \cdot r\left|G(x)\right| \langle G_\infty, \xi\rangle\right) \gamma_\xi\left(\mathrm{d}r\right) \lambda(\mathrm{d}\xi)$$
$$= \frac{(1-\varepsilon)^2}{\left(1+\frac{K\varepsilon}{1-\varepsilon}\right)} M_{G_\infty}\left(b \cdot r\left|G(x)\right|\right).$$

Hence

$$J_X\left(bG(x)\right) \geq \frac{(1-\varepsilon)^2}{\left(1+\frac{K\varepsilon}{1-\varepsilon}\right)} M_{G_\infty}\left(b \cdot r\left|G(x)\right|\right). \tag{5.20}$$

Now (5.8) follows from (5.17), (5.20) and (5.9). □

Lemma 5.5. *Let J_ρ be given by (5.2) with $\rho(\mathrm{d}v)$ satisfying (5.1). Assume that*

$$J_\rho(\beta b) = \eta J_\rho(b), \quad b \geq 0, \tag{5.21}$$

and

$$J_\rho(\gamma b) = \theta J_\rho(b), \quad b \geq 0, \tag{5.22}$$

for some $\beta > 1$, $\gamma > 1$ such that $\ln\beta/\ln\gamma \notin \mathbb{Q}$ and $\eta > 1$, $\theta > 1$. Then

$$J_\rho(b) = Cb^\alpha, \quad b \geq 0, \tag{5.23}$$

for some $C > 0$ and $\alpha \in (1,2)$.

Proof. By iterative application of (5.21) and (5.22) we see that for any $m, n \in \mathbb{N}$

$$J_\rho(\beta^m\gamma^n b) = \eta^m\theta^n J_\rho(b), \quad b \geq 0,$$

which can be written as

$$J_\rho(be^{m\ln\beta+n\ln\gamma}) = e^{m\ln\eta+n\ln\theta} J_\rho(b), \quad b \geq 0. \tag{5.24}$$

In Lemma 5.6 below we prove that the set

$$D := \{m\ln\beta - n\ln\gamma;\ m, n \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. So, for any $\delta > 0$ there exist $m, n \in \mathbb{Z}$, $m \neq 0$, such that

$$|m\ln\beta - n\ln\gamma| < \delta, \tag{5.25}$$

and then, by (5.6) and (5.24), we obtain that

$$e^{-2\delta} \leq \frac{e^{m\ln\eta}}{e^{n\ln\theta}} = \frac{J_\rho(e^{m\ln\beta})}{J_\rho(e^{n\ln\gamma})} \leq e^{2\delta}. \tag{5.26}$$

It follows from (5.25) that

$$\left|\frac{\ln\beta}{\ln\gamma} - \frac{n}{m}\right| \leq \frac{\delta}{|m|\ln\gamma},$$

and from (5.26) that

$$\left|\frac{\ln\eta}{\ln\theta} - \frac{n}{m}\right| \leq \frac{2\delta}{|m|\ln\theta}.$$

Consequently,

$$\left|\frac{\ln\beta}{\ln\gamma}-\frac{\ln\eta}{\ln\theta}\right| \le \frac{\delta}{|m|\ln\gamma}+\frac{2\delta}{|m|\ln\theta} \le \frac{\delta}{\ln\gamma}+\frac{2\delta}{\ln\theta}.$$

Letting $\delta \longrightarrow 0$ yields

$$\frac{\ln\beta}{\ln\gamma}=\frac{\ln\eta}{\ln\theta}.$$

Let us define

$$\alpha := \frac{\ln\eta}{\ln\beta}=\frac{\ln\theta}{\ln\gamma} > 0,$$

and put $b = 1$ in (5.24). This gives

$$J_\rho(e^{m\ln\beta+n\ln\gamma}) = J_\rho(1)\left(e^{m\ln\beta+n\ln\gamma}\right)^\alpha,$$

which means that $J_\rho(b) = J_\rho(1)b^\alpha$ for b from the set e^D which is dense in $[0,+\infty)$. As J_ρ is continuous, (5.23) follows. Finally, by Proposition 3.4 in [4] it follows that the function $(0,+\infty) \ni b \mapsto J_\rho/b$ is strictly increasing, while the function $(0,+\infty) \ni b \mapsto J_\rho/b^2$ is strictly decreasing on $(0,+\infty)$, hence $\alpha \in (1,2)$. □

The following result is strictly related to Weyl's equidistribution theorem, see [26].

Lemma 5.6. *Let $p,q > 0$ be such that $p/q \notin \mathbb{Q}$. Let us define the set*

$$G := \{mp+nq;\ m,n,=1,2,\ldots\}.$$

Then for each $\delta > 0$ there exists a number $M(\delta) > 0$ such that

$$\forall x \ge M(\delta) \quad \exists\, g \in G \quad \textit{such that } |x-g| \le \delta.$$

Moreover, the set

$$D := \{mp+nq;\ m,n \in \mathbb{Z}\},$$

is dense in $\mathbb{R}$.

Proof. Since $p/q \notin \mathbb{Q}$, at least one of p,q, say q, is irrational. For simplicity assume that $p = 1$ and consider the sequence

$$r(jq),\ j=1,2,\ldots \qquad \text{where } r(x) := x \bmod 1,$$

of fractional parts of the numbers jq, $j = 1,2,\ldots$. Recall that Weyl's equidistribution theorem states that

$$\lim_{N\longrightarrow+\infty}\frac{\sharp\{j\le N : r(jq)\in[a,b]\}}{N} = b-a \tag{5.27}$$

for any $[a,b] \subseteq [0,1)$ if and only if q is irrational.

For fixed $\delta > 0$ and n such that $1/n < \delta$, let us consider a partition of $[0,1)$ of the form

$$[0,1) = \bigcup_{k=0}^{n-1} A_k, \quad A_k := [k/n,(k+1)/n).$$

For a natural number N, let us consider the set $R_N := \{r(jq) : j = 1, 2, \dots, N\}$. By (5.27), for each $k = 0, 1, \dots, n-1$, there exists N_k such that

$$R_{N_k} \cap A_k \neq \emptyset.$$

Then for $\bar{N} := \max\{N_0, N_1, \dots, N_{n-1}\}$ we have

$$R_{\bar{N}} \cap A_k \neq \emptyset, \quad k = 0, 1, \dots, n-1.$$

Let $M = M(\delta) := \bar{N}q$. Then, for $x \geq M$, there exists a number $N_x \leq \bar{N}$ such that

$$|r(N_x q) - r(x)| \leq \frac{1}{n}. \tag{5.28}$$

Then for the number

$$g := \lfloor x \rfloor - \lfloor N_x q \rfloor + N_x g \in G$$

the following holds

$$\begin{aligned} |x - g| &= |x - (\lfloor x \rfloor - \lfloor N_x q \rfloor + N_x q)| \\ &= |\lfloor x \rfloor + r(x) - \lfloor x \rfloor + \lfloor N_x q \rfloor - N_x q| \\ &= |r(x) - r(N_x q)| \leq 1/n < \delta, \end{aligned}$$

where the last inequality follows from (5.28).

The density of D is an immediate consequence of the first part of the Lemma. Indeed, for $x < M(\delta)$ and $g \in G$ such that $x + g > M(\delta)$ there exists $\tilde{g} \in G$ such that $|x + g - \tilde{g}| < \delta$.

The general case with $p \neq 1$ can be proven in the same way but requires a generalized version of Weyl's theorem, which says that the numbers $r_p(nq), n = 1, 2, \dots$, where $r_p(x) := x \bmod p$, are equidistributed on $[0, p)$ if and only if $q/p \notin \mathbb{Q}$. This can be proven by a straightforward modification of (5.27), noticing that

$$x \bmod p = p \cdot \left(\frac{x}{p} \bmod 1 \right).$$

□

5.2 *Proof of Theorem 3.1*

By Remark 2.1 and Remark 2.2 the Laplace transform J_X satisfies

$$J_X(bG(x)) = J_{\nu_{G(0)}}(b) + xJ_\mu(b), \quad b, x \geq 0, \tag{5.29}$$

where $\mu(\mathrm{d}v)$ is the measure satisfying (2.13)–(2.14). By discussion preceding the formulation of Theorem 3.1 we have $G(0) = 0$, hence (5.29) simpilfes to

$$J_X(bG(x)) = xJ_\mu(b), \quad x \geq 0. \tag{5.30}$$

Assumption (3.11) and (5.30) imply that $J_X(y), J_\mu(b) > 0$, $G(x) \neq 0$, for $y \in \mathbb{R}^d \setminus \{0\}$, $b > 0$, $x > 0$.

Let $G_0 = \lim_{x\to 0+} \frac{G(x)}{|G(x)|}$. It follows from Proposition 5.4 that there exists a function $\delta : (0,+\infty) \to (0,+\infty)$, such that for any $\varepsilon > 0$ from the inequality

$$\left| \frac{G(x)}{|G(x)|} - G_0 \right| \le \delta(\varepsilon)$$

follows that for any $b \ge 0$

$$1-\varepsilon \le \frac{J_X\left(b\frac{G(x)}{|G(x)|}\right)}{J_X\left(bG_0\right)} \le 1+\varepsilon.$$

Thus for any $\varepsilon > 0$ there exists $m(\varepsilon) > 0$, such that for $x \in (0, m(\varepsilon))$

$$\left| \frac{G(x)}{|G(x)|} - G_0 \right| \le \delta(\varepsilon),$$

and hence for any $b > 0$

$$1-\varepsilon \le \frac{J_X\left(b\frac{G(x)}{|G(x)|}\right)}{J_X\left(bG_0\right)} \le 1+\varepsilon.$$

Let us fix $\beta > 1$ and take x_1, x_2 satisfying $0 < x_1 \le x_2 < m(\varepsilon), \beta\,|G(x_1)| = |G(x_2)| > 0$ (from the continuity of G it follows that such x_1 and x_2 exist). Then for any $b > 0$ and $i = 1, 2$, by (5.30),

$$1-\varepsilon \le \frac{J_X\left(b\frac{G(x_i)}{|G(x_i)|}\right)}{J_X\left(bG_0\right)} = \frac{x_i J_\mu\left(\frac{b}{|G(x_i)|}\right)}{J_X\left(bG_0\right)} \le 1+\varepsilon.$$

Hence for any $b > 0$, taking $\tilde{b} = \beta\,|G(x_1)|\,b$ we get

$$\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{x_2}{x_1} \le \frac{J_\mu\left(\frac{\tilde{b}}{|G(x_1)|}\right)}{J_\mu\left(\frac{\tilde{b}}{|G(x_2)|}\right)} = \frac{J_\mu\left(\beta b\right)}{J_\mu\left(b\right)} \le \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{x_2}{x_1}$$

which yields

$$\frac{1-\varepsilon}{1+\varepsilon}\cdot\frac{J_\mu\left(\beta b\right)}{J_\mu\left(b\right)} \le \frac{x_2}{x_1} \le \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{J_\mu\left(\beta b\right)}{J_\mu\left(b\right)}.$$

Since $\varepsilon > 0$ is arbitrary, taking $\varepsilon \to 0$ and x_1, x_2 satisfying $0 < x_1 \le x_2 < m(\varepsilon)$, $\beta\,|G(x_1)| = |G(x_2)|$ we obtain that

$$\lim_{\varepsilon\to 0}\frac{x_2}{x_1} = \eta,$$

where $\eta = J_\mu\left(\beta b\right)/J_\mu\left(b\right) > 1$ is independent of $b > 0$. Hence, for all $b \ge 0$ we have

$$J_\mu\left(\beta b\right) = \eta J_\mu\left(b\right).$$

Similarly, take $\gamma > 1$ such that $\ln\beta/\ln\gamma \notin \mathbb{Q}$. Reasoning similarly as before we get that there exists $\theta > 1$, such that for all $b \ge 0$ we have

$$J_\mu\left(\gamma b\right) = \theta J_\mu\left(b\right).$$

Now the thesis follows from Lemma 5.5 and the one to one correspondence between Laplace transforms and measures on $[0,+\infty)$, see [16], p. 233. □

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