Fast $L_2$-approximation of integral-type functionals of Markov processes

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Abstract In this paper, we provide strong $L_2$-rates of approximation of the integral-type functionals of Markov processes by integral sums. We improve the method developed in [2]. Under assumptions on the process formulated only in terms of its transition probability density, we get the accuracy that coincides with that obtained in [3] for a one-dimensional diffusion process.

Keywords Markov processes, integral functional, rates of convergence, strong approximation

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1 Introduction

Let $X_t$, $t \geq 0$, be a Markov process with values in $\mathbb{R}^d$. Consider the following objects:

1) the integral functional

$$I_T(h) = \int_0^T h(X_t) \, dt$$

of this process;

2) the sequence of integral sums

$$I_{T,n}(h) = \frac{T}{n} \sum_{k=0}^{n-1} h(X_{(kT)/n}), \quad n \geq 1.$$
In this paper, we establish strong $L_2$-approximation rates, that is, the bounds for

$$E|I_T(h) - I_{T,n}(h)|^2.$$ 

The current research is mainly motivated by the recent papers [2] and [3]. In [3], strong $L_p$-approximation rates are considered for an important particular case where $X$ is a one-dimensional diffusion. The approach developed in this paper contains both the Malliavin calculus tools and the Gaussian bounds for the transition probability density of the process $X$, and relies substantially on the structure of the process.

Another approach to that problem has been developed in [2]. This approach is, in a sense, a modification of Dynkin’s theory of continuous additive functionals (see [1], Chap. 6) and also involves the technique similar to that used in the proof of the classical Khasminskii lemma (see, e.g., [4, Lemma 2.1]). This approach allows us to obtain strong $L_p$-approximation rates under assumptions on the process $X$ formulated only in terms of its transition probability density.

For a bounded function $h$, the strong $L_p$-rates of approximation of the integral functional $I_T(h)$ obtained in [2] essentially coincide with those established in [3]. However, under additional regularity assumptions on the function $h$ (e.g., when $h$ is Hölder continuous), the rates obtained in [3] are sharper (see [2, Thm. 2.2] and [3, Thm. 2.3]).

In this note, we improve the method developed in [2], so that under the assumption of the Hölder continuity of $h$, the strong $L_2$-approximation rates coincide with those obtained in [3], preserving at the same time the advantage of the method that the assumptions on the process $X$ are quite general and do not essentially rely on the structure of the process.

2 Main result

In what follows, $P_x$ denotes the law of the Markov process $X$ conditioned by $X_0 = x$, and $\mathbb{E}_x$ denotes the expectation with respect to this law. Both the absolute value of a real number and the Euclidean norm in $\mathbb{R}^d$ are denoted by $|\cdot|$.

We make the following assumption on the process $X$.

A. The process $X$ possesses a transition probability density $p_t(x, y)$ that is differentiable with respect to $t$ and satisfies the following estimates:

$$p_t(x, y) \leq C_T t^{-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$

$$|\partial_t p_t(x, y)| \leq C_T t^{-1-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$

$$|\partial_{tt}^2 p_t(x, y)| \leq C_T t^{-2-d/\alpha} Q(t^{-1/\alpha}(x - y)), \quad t \leq T,$$

for some fixed $\alpha \in (0, 2]$ and some distribution density $Q$ such that $
\int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz < \infty$. Without loss of generality, we assume that in (1)–(3) $C_T \geq 1$.

We assume that the function $h$ satisfies the Hölder condition with exponent $\gamma \in (0, \alpha/2]$, that is,

$$\|h\|_\gamma := \sup_{x \neq y} \frac{|h(x) - h(y)|}{|x - y|^{\gamma}} < \infty.$$
Now we formulate the main result of the paper.

**Theorem 1.** Suppose that Assumption A holds. Then

$$\mathbb{E}_x |I_T(h) - I_{T,n}(h)|^2 \leq \begin{cases} DT,\gamma,\alpha,Q C_{\gamma,\alpha} \|h\|^2 \gamma^{-1-2\gamma/\alpha}, & \gamma \neq \alpha/2, \\ DT,\gamma,\alpha,Q \|h\|^2 \gamma^{-2} \ln n, & \gamma = \alpha/2, \end{cases}$$

where

$$DT,\gamma,\alpha,Q = 8C_T^2 T^{2+2\gamma/\alpha} \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz,$$

$$C_{\gamma,\alpha} = \max \left\{ (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1}, \max_{n \geq 1} \left( \frac{(\ln n)^2}{n^{1-2\gamma/\alpha}} \right) \right\}.$$  

We provide the proof of Theorem 1 in Section 3.

**Remark 1.** Any diffusion process satisfies conditions (1)–(3) with $\alpha = 2$, $Q(x) = c_1 e^{-c_2 |x|^2}$, and properly chosen $c_1, c_2$ (see [2]). In the case where $X$ is a one-dimensional diffusion, Theorem 1 provides the same rates of convergence as those obtained in [3] (see Theorem 2.3 in [3]).

**Remark 2.** Similarly to [2], we formulate the assumption on the process $X$ only in terms of its transition probability density. Condition A, compared with condition X (cf. [2]), contains the additional assumption (3).

### 3 Proof of Theorem 1

**Proof.** For $t \in [kT/n, (k+1)T/n)$, denote

$$\eta_n(t) = \frac{kT}{n}, \quad \zeta_n(t) = \frac{(k+1)T}{n},$$

and put $\Delta_n(s) := h(X_s) - h(X_{\eta_n(s)})$, $s \in [0, T]$.

By the Markov property of $X$, for any $r < s$, we have

$$\mathbb{E}_x |X_s - X_r|^{2\gamma} = \mathbb{E}_x \int_{\mathbb{R}^d} p_{s-r}(X_r, z)|X_r - z|^{2\gamma} \, dz$$

$$\leq C_T \mathbb{E}_x \int_{\mathbb{R}^d} (s - r)^{-d/\alpha} Q((s - r)^{-1/\alpha}(X_r - z))|X_r - z|^{2\gamma} \, dz$$

$$= C_T (s - r)^{2\gamma/\alpha} \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz.$$  

Therefore, using the inequality $s - \eta_n(s) \leq T/n$, $s \in [0, T]$ and the Hölder continuity of the function $h$, we obtain:

$$\mathbb{E}_x \left| \Delta_n(s) \right|^2 \leq C_T T^{2\gamma/\alpha} \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) \|h\|^2 \gamma^{-2\gamma/\alpha}. \quad (4)$$

Split

$$\mathbb{E}_x \left| I_T(h) - I_{T,n}(h) \right|^2 = 2\mathbb{E}_x \int_0^T \int_s^T \Delta_n(s) \Delta_n(t) \, dt \, ds = J_1 + J_2 + J_3, \quad (5)$$
where
\[ J_1 = 2\mathbb{E}_x \int_0^T \int_{\xi_n(s) + T/n} \Delta_n(s) \Delta_n(t) \, dt \, ds, \]
\[ J_2 = 2\mathbb{E}_x \int_0^{T/n} \int_{\xi_n(s) + T/n} \Delta_n(s) \Delta_n(t) \, dt \, ds, \]
\[ J_3 = 2\mathbb{E}_x \int_{T/n}^T \int_{\xi_n(s) + T/n} \Delta_n(s) \Delta_n(t) \, dt \, ds. \]

For \(|J_1|\) and \(|J_2|\), the estimates can be obtained in the same way. Indeed, using the Cauchy inequality and (4), we get
\[
|J_1| \leq 2 \int_0^T \int_{\xi_n(s) + T/n} \mathbb{E}_x |\Delta_n(s)|^2 \mathbb{E}_x |\Delta_n(t)|^2 \, dt \, ds
\leq 2C_T T^{2\gamma/\alpha} \|h\|_Y^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-2\gamma/\alpha} \int_0^T (T/n + \xi_n(s) - s) \, ds
\leq 4C_T T^{2+2\gamma/\alpha} \|h\|_Y^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-(1+2\gamma/\alpha)}.
\]

In the last inequality, we have used the inequality \(\xi_n(s) - s \leq T/n, s \in [0, T]\).

Similarly,
\[
|J_2| \leq 2C_T T^{2+2\gamma/\alpha} \|h\|_Y^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) n^{-(1+2\gamma/\alpha)}.
\]

Now we proceed to the estimation of \(|J_3|\), which is the main part of the proof. Observe that the following identities hold:
\[
\int_{\mathbb{R}^d} \partial_{uv}^2 p_u(x, y) p_{v-u}(y, z) \, dz = \partial_{uv}^2 p_u(x, y) = 0, \quad y \in \mathbb{R}^d, \quad (6)
\]
\[
\int_{\mathbb{R}^d} \partial_{uv}^2 p_u(x, y) p_{v-u}(y, z) \, dy = \partial_{uv}^2 p_u(x, y) = 0, \quad z \in \mathbb{R}^d, \quad (7)
\]
where in (6) we used that \(\int_{\mathbb{R}^d} p_r(y, z) \, dz = 1, r > 0, y \in \mathbb{R}^d\), and in (7) we used the Chapman–Kolmogorov equation.

We have:
\[
J_3 = 2 \int_{T/n}^T \int_{\xi_n(s) + T/n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(y) h(z) \left[ p_s(x, y) p_t-s(y, z) - p_{\eta_n(s)}(x, y) p_{t-\eta_n}(y, z) - p_s(x, y) p_{\eta_n(t)-s}(y, z) + p_{\eta_n(s)}(x, y) p_{\eta_n(t)-\eta_n(s)}(y, z) \right] \, dz \, dy \, dt \, ds
\]
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\[ = 2 \int_{T/n}^{T} \int_{\xi_n(s)+T/n}^{T} \int_{\mathbb{R}^d} \int_{\eta_n(s)}^{\eta_n(t)} \int_{\eta_n(t)}^{\eta_n(t)} h(y)h(z) \partial_{uv}^2(p_u(x, y) \times p_{v-u}(y, z)) \, dy \, du \, dz \, dy \, dt \, ds \]

\[ = - \int_{T/n}^{T} \int_{\xi_n(s)+T/n}^{T} \int_{\mathbb{R}^d} \int_{\eta_n(s)}^{\eta_n(t)} \int_{\eta_n(t)}^{\eta_n(t)} \left( h(y) - h(z) \right)^2 \partial_{uv}^2(p_u(x, y) \times p_{v-u}(y, z)) \, dy \, du \, dz \, dy \, dt \, ds, \]

where in the last identity we have used (6) and (7).

Further, we have

\[ \partial_{uv}^2 p_u(x, y) p_{v-u}(y, z) = p_u(x, y) \partial_{rr}^2 p_r(y, z) \bigg|_{r=v-u} + \partial_u p_u(x, y) \partial_r p_r(y, z) \bigg|_{r=v-u}. \]

Then, using condition A and the Hölder continuity of the function $h$, we obtain

\[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( h(y) - h(z) \right)^2 |\partial_{uv}^2(p_u(x, y) p_{v-u}(y, z))| \, dy \, dz \leq C_T^2 \|h\|_2^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) \left( (v-u)^{2\gamma/\alpha - 2} + (v-u)^{2\gamma/\alpha - 1} u^{-1} \right). \]

Therefore, according to (8) and (9),

\[ |J_3| \leq C_T^2 \|h\|_2^2 \left( \int_{\mathbb{R}^d} |z|^{2\gamma} Q(z) \, dz \right) \]

\[ \times \int_{T/n}^{T} \int_{\xi_n(s)+T/n}^{T} \int_{\eta_n(s)}^{\eta_n(t)} \int_{\eta_n(t)}^{\eta_n(t)} \left( (v-u)^{2\gamma/\alpha - 2} + (v-u)^{2\gamma/\alpha - 1} u^{-1} \right) \, dv \, du \, dt \, ds. \]

Denote $a_{\alpha,\gamma}(u, v) := (v-u)^{2\gamma/\alpha - 2} + (v-u)^{2\gamma/\alpha - 1} u^{-1}$. Then

\[ \int_{T/n}^{T} \int_{\xi_n(s)+T/n}^{T} \int_{\eta_n(s)}^{\eta_n(t)} \int_{\eta_n(t)}^{\eta_n(t)} a_{\alpha,\gamma}(u, v) \, dv \, du \, dt \, ds \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} a_{\alpha,\gamma}(u, v) \, dv \, du \, dt \, ds \]

\[ = \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} a_{\alpha,\gamma}(u, v) \, dv \, du \, dt \, ds \]

\[ \leq T^2 n^{-2} \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{jT/n}^{(j+1)T/n} a_{\alpha,\gamma}(u, v) \, dv \, du \]

\[ = T^2 n^{-2} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{(i+1)T/n}^{T} a_{\alpha,\gamma}(u, v) \, dv \, du, \]

where in the fourth line we used that, for $u \in [iT/n, (i+1)T/n)$ and $v \in [jT/n, (j+1)T/n)$, we always have $(i+1)T/n - u \leq T/n$ and $(j+1)T/n - v \leq T/n$. 

Thus, from (10) we obtain

$$|J_3| \leq C_T^2 T^2 \|h\|_2^2 \left( \int_{\mathbb{R}^d} \left|z\right|^{2\gamma} Q(z) \, dz \right) n^{-2}(S_1 + S_2),$$

(11)

where

$$S_1 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{T} (v - u)^{2\gamma/\alpha - 2} \, dv \, du,$$

$$S_2 = \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+2)T/n} (v - u)^{2\gamma/\alpha - 1} u^{-1} \, dv \, du.$$

We estimate each term separately. In what follows, we consider the case $\gamma < \alpha/2$; the case of $\gamma = \alpha/2$ is similar and therefore omitted. We have

$$S_1 \leq (1 - 2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} ((i + 1)T/n - u)^{2\gamma/\alpha - 1} \, du$$

$$= (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} \sum_{i=1}^{n-1} ((i + 1)T/n - iT/n)^{2\gamma/\alpha}$$

$$\leq (1 - 2\gamma/\alpha)^{-1} (2\gamma/\alpha)^{-1} T^{2\gamma/\alpha} n^{1 - 2\gamma/\alpha} \leq C_{\gamma, \alpha} T^{2\gamma/\alpha} n^{1 - 2\gamma/\alpha}. \quad (12)$$

Finally, since $v - u \leq T$ for $0 \leq u < v \leq T$, we have

$$S_2 \leq T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \int_{iT/n}^{(i+1)T/n} \int_{iT/n}^{(i+2)T/n} (v - u)^{-1} u^{-1} \, dv \, du$$

$$\leq T^{2\gamma/\alpha} \sum_{i=1}^{n-1} \left( \int_{iT/n}^{(i+1)T/n} u^{-1} \, du \right) \left( \int_{(i+2)T/n}^{T} (v - (i + 1)T/n)^{-1} \, dv \right)$$

$$\leq T^{2\gamma/\alpha} \ln n \sum_{i=1}^{n-1} \left( \int_{iT/n}^{(i+1)T/n} u^{-1} \, du \right) = T^{2\gamma/\alpha} (\ln n)^2$$

$$\leq C_{\gamma, \alpha} T^{2\gamma/\alpha} n^{1 - 2\gamma/\alpha}. \quad (13)$$

Combining inequality (11) with (12) and (13), we derive

$$|J_3| \leq 2C_{\gamma, \alpha} C_T^2 T^{2 + 2\gamma/\alpha} \|h\|_2^2 \left( \int_{\mathbb{R}^d} \left|z\right|^{2\gamma} Q(z) \, dz \right) n^{-(1 + 2\gamma/\alpha)}. \quad \square$$

References
