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Construction of maximum likelihood estimator in the mixed fractional–fractional Brownian motion model with double long-range dependence

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Abstract We construct an estimator of the unknown drift parameter $\theta \in \mathbb{R}$ in the linear model

$$X_t = \theta t + \sigma_1 B^{H_1}(t) + \sigma_2 B^{H_2}(t),\ \ t \in [0, T],$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions with Hurst indices H_1 and H_2 satisfying the condition $\frac{1}{2} \le H_1 < H_2 < 1$. Actually, we reduce the problem to the solution of the integral Fredholm equation of the 2nd kind with a specific weakly singular kernel depending on two power exponents. It is proved that the kernel can be presented as the product of a bounded continuous multiplier and weak singular one, and this representation allows us to prove the compactness of the corresponding integral operator. This, in turn, allows us to establish an existence–uniqueness result for the sequence of the equations on the increasing intervals, to construct accordingly a sequence of statistical estimators, and to establish asymptotic consistency.

Keywords Fractional Brownian motion, maximum likelihood estimator, integral equation with weakly singular kernel, compact operator, asymptotic consistency

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1 Introduction

Consider the continuous-time linear model

$$X(t) = \theta t + \sigma_1 B^{H_1}(t) + \sigma_2 B^{H_2}(t),\ t \in [0, T], \tag{1}$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions with different Hurst indices H_1 and H_2 defined on some stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F})_t, t \geq 0, \mathsf{P})$. We assume that the filtration is generated by these processes and completed by P-negligible sets of $\mathfrak{F}_0$.

Recall that the fractional Brownian motion (fBm) $B_t^H,\ t \geq 0$, with Hurst index $H \in (0, 1)$ is a centered Gaussian process with the covariance function

$$\mathsf{E}\left[B^H(t)B^H(s)\right] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

From now on we suppose that the Hurst indices in (1) satisfy the inequality

$$\frac{1}{2} \leq H_1 < H_2 < 1,$$

and we consider the continuous modifications of both processes, which exist due to the Kolmogorov theorem. Assuming that the Hurst indices H_1, H_2 and parameters $\sigma_1 \geq 0, \sigma_2 \geq 0$ are known, we aim to estimate the unknown drift parameter θ by the continuous observations of the trajectories of X. Due to the long-range dependence property of fBm with $H > 1/2$, we call our model the model with double long-range dependence.

In the case where $H_1 = \frac{1}{2}$, the problem of drift parameter estimation in the model (1) was solved in [3], and in the case where $\frac{1}{2} < H_1 < H_2 < 1$ and $H_2 - H_1 > 1/4$, the estimator was constructed in [6]. The goal of the present paper is to generalize the results from [6] to arbitrary $\frac{1}{2} \leq H_1 < H_2 < 1$. The problem, more technical than principal, is that in the case where $H_2 - H_1 > 1/4$ and $H_1 > 1/2$, the construction of the estimator is reduced to the question if the solution of the Fredholm integral equation of the 2nd kind with weakly singular kernel from $L_2[0, T]$ exists and is unique, but for $H_2 - H_1 \leq 1/4$, the kernel does not belong to $L_2[0, T]$. Moreover, in this case, we can say that in the literature it is impossible to pick up for this kernel any suitable standard techniques for working with weak singular kernels, and it does not belong to any standard class of weak singular kernels. The matter lies in the fact that the kernel contains two power indices, H_1 and H_2, and they create more complex singularity than it usually happens. So, it is necessary to make many additional efforts in order to prove the compactness of the corresponding integral operator. Immediately after establishing the compactness of the corresponding integral operator, the problem of statistical estimation follows the same steps as in the paper [6], and we briefly present these steps for completeness.

The paper is organized as follows. In Section 2, we describe the model and explain how to reduce the solution of the estimation problem to the existence–uniqueness problem for the integral Fredholm equation of the 2nd kind with some nonstandard weakly singular kernel. In Section 3, we solve the existence–uniqueness problem.

Section 4 is devoted to the basic properties of estimator, that is, we establish its form, consistency, and asymptotic normality. Section A contains the properties of hypergeometric function used in the proof of the existence–uniqueness result for the main Fredhom integral equation.

2 Preliminaries. How to reduce the original problem to the integral equation

Since we suppose that the Hurst parameters H_1, H_2 and scale parameters σ_1, σ_2 are known, for technical simplicity, we consider the case where $\sigma_1 = \sigma_2 = 1$ and, as it was mentioned before, $\frac{1}{2} \le H_1 < H_2 < 1$. If we wish to include the unknown parameter θ into the fractional Brownian motion with the smallest Hurst parameter in order to apply Girsanov's theorem for construction of the estimator, we consider a couple of processes $\{\widetilde{B}^{H_1}(t), B^{H_2}(t),\ t \ge 0\}$, $i = 1, 2$, defined on the space $(\Omega, \mathfrak{F}, (\mathfrak{F})_t)$ and let P_θ be a probability measure under which $\widetilde{B}^{H_1}$ and B^{H_2} are independent, B^{H_2} is a fractional Brownian motion with Hurst parameter H_2, and $\widetilde{B}^{H_1}$ is a fractional Brownian motion with Hurst parameter H_1 and drift θ, that is,

$$\widetilde{B}^{H_1}(t) = \theta t + B^{H_1}(t).$$

The probability measure P_0 corresponds to the case $\theta = 0$. Our main problem is the construction of maximum likelihood estimator for $\theta \in \mathbb{R}$ by the observations of the process $Z(t) = \theta t + B^{H_1}(t) + B^{H_2}(t) = \widetilde{B}^{H_1}(t) + B^{H_2}(t),\ t \in [0, T]$. As in [6], we apply to Z the linear transformation in order to reduce the construction to the sum with one term being the Wiener process. So, we take the kernel $l_H(t, s) = (t - s)^{1/2-H} s^{1/2-H}$ and construct the integral

$$\begin{aligned} Y(t) = \int_0^t l_{H_1}(t, s) dZ(s) = \theta \mathrm{B}\left(\frac{3}{2} - H_1, \frac{3}{2} - H_1\right) t^{2-2H_1} + M^{H_1}(t) \\ + \int_0^t l_{H_1}(t, s) dB^{H_2}(s), \end{aligned} \tag{2}$$

where $\mathrm{B}(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1} dx$ is the beta function, and M^{H_1} is a Gaussian martingale (Molchan martingale), admitting the representations

$$M^H(t) = \int_0^t l_H(t, s) dB^H(s) = \gamma_H \int_0^t s^{1/2-H} dW(s)$$

with $\gamma_H = (2H(\frac{3}{2} - H)\Gamma(3/2 - H)^3 \Gamma(H + \frac{1}{2})\Gamma(3 - 2H)^{-1})^{\frac{1}{2}}$ and a Wiener process W. According to [6], the linear transformation (2) is well defined, and the processes Z and Y are observed simultaneously. This means that we can reduce the original problem to the equivalent problem of the construction of maximum likelihood estimator of $\theta \in \mathbb{R}$ basing on the linear transformation Y. For simplicity, denote $\mathcal{B}_{H_1} := \mathrm{B}(\frac{3}{2} - H_1, \frac{3}{2} - H_1)$. Now the main problem can be formulated as follows. Let $\frac{1}{2} \le H_1 < H_2 < 1$,

$$\left\{ \widetilde{X}_1(t) = \widetilde{M}^{H_1}(t), X_2(t) := \int_0^t l_{H_1}(t, s) dB^{H_2}(s),\ t \ge 0 \right\},$$

$i = 1, 2$, be a couple of processes defined on the space $(\Omega, \mathfrak{F})$, and P_θ be a probability measure under which $\widetilde{X}_1$ and X_2 are independent, B^{H_2} is a fractional Brownian motion with Hurst parameter H_2, and $\widetilde{X}_1$ is a martingale with square characteristics $\langle \widetilde{X}_1 \rangle(t) = \frac{\gamma_{H_1}^2}{2-2H_1} t^{2-2H_1}$ and drift $\theta \mathcal{B}_{H_1} t^{2-2H_1}$, that is,

$$\widetilde{X}_1(t) = \widetilde{M}^{H_1}(t) = \theta \mathcal{B}_{H_1} t^{2-2H_1} + M^{H_1}(t).$$

Also, denote $X_1(t) = M^{H_1}(t)$. Our main problem is the construction of maximum likelihood estimator for $\theta \in \mathbb{R}$ by the observations of the process

$$Y(t) = \theta \mathcal{B}_{H_1} t^{2-2H_1} + X_1(t) + X_2(t) = \widetilde{X}_1(t) + X_2(t).$$

Note that, under the measure P_θ, the process

$$\widetilde{W}(t) := W(t) + \frac{\theta(2-2H_1)\mathcal{B}_{H_1}}{\gamma_{H_1}(\frac{3}{2} - H_1)} t^{\frac{3}{2}-H_1}$$

is a Wiener process with drift. Denote $\delta_{H_1} = \frac{(2-2H_1)\mathcal{B}_{H_1}}{\gamma_{H_1}}$.

By Girsanov's theorem and independence of X_1 and X_2,

$$\begin{aligned}\frac{dP_\theta}{dP_0} &= \exp\left\{\theta \delta_{H_1} \int_0^T s^{\frac{1}{2}-H_1} d\widetilde{W}(s) - \frac{\theta^2 \delta_{H_1}^2}{4(1-H_1)} T^{2-2H_1}\right\} \\ &= \exp\left\{\theta \delta_{H_1} \widetilde{X}_1(T) - \frac{\theta^2 \delta_{H_1}^2}{4(1-H_1)} T^{2-2H_1}\right\}.\end{aligned}$$

As it was mentioned in [3], the derivative of such a form is not the likelihood ratio for the problem at hand because it is not measurable with respect to the observed σ-algebra

$$\mathfrak{F}_T^Y := \sigma\{Y(t), t \in [0, T]\} = \mathfrak{F}_T^X := \sigma\{X(t), t \in [0, T]\},$$

where $X(t) = X_1(t) + X_2(t)$.

We shall proceed as in [3]. Let μ_θ be the probability measure induced by Y on the space of continuous functions with the supremum topology under probability P_θ. Then for any measurable set A, $\mu_\theta(A) = \int_A \Phi(x)\mu_0(dx)$, where $\Phi(x)$ is a measurable functional such that $\Phi(X) = E_0(\frac{dP_\theta}{dP_0}|\mathfrak{F}_T^X)$. This means that $\mu_\theta \ll \mu_0$ for any $\theta \in \mathbb{R}$. Taking into account that $\widetilde{X}_1 = X_1$ under P_0 and the fact that the vector process (X_1, X) is Gaussian, we get that the corresponding likelihood function is given by

$$\begin{aligned}L_T(X, \theta) &= \mathsf{E}_0\left(\frac{d\mathsf{P}_\theta}{d\mathsf{P}_0}|\mathfrak{F}_T^X\right) = \mathsf{E}_0\left(\exp\left\{\theta \delta_{H_1} X_1(T) - \frac{\theta^2 \delta_{H_1}^2}{4(1-H_1)} T^{2-2H_1}\right\}|\mathfrak{F}_T^X\right) \\ &= \exp\left\{\theta \delta_{H_1} \mathsf{E}_0\big(X_1(T)|\mathfrak{F}_T^X\big) + \frac{\theta^2 \delta_{H_1}^2}{2}\left(V(T) - \frac{T^{2-2H_1}}{2-2H_1}\right)\right\},\end{aligned}$$

where $V(t) = \mathsf{E}_0(X_1(t) - \mathsf{E}_0(X_1(t)|\mathfrak{F}_t^X))^2,\ t \in [0, T]$.

The next reasonings repeat the corresponding part of [6]. We have to solve the following problem: to find the projection $P_X X_1(T)$ of $X_1(T)$ onto

$$\{X(t) = X_1(t) + X_2(t),\ t \in [0, T]\}.$$

According to [4], the transformation formula for converting fBm into a Wiener process is of the form

$$W_i(t) = \int_0^t \left(\left(K_{H_i}^*\right)^{-1} \mathbf{1}_{[0,t]}\right)(s) dB^{H_i}(s),\ i = 1, 2,$$

where

$$\left(K_H^* f\right)(s) = \int_s^T f(t)\partial_t K_H(t,s)dt = \beta_H s^{1/2-H} \int_s^T f(t) t^{H-1/2}(t-s)^{H-3/2}dt,$$

$\beta_H = (\frac{H(2H-1)}{\mathrm{B}(H-1/2, 2-2H)})^{\frac{1}{2}}$, and the square-integrable kernel $K_H(t, s)$ is of the form

$$K_H(t,s) = \beta_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2} du.$$

We have that $W_i,\ i = 1, 2$, are standard Wiener processes, which are obviously independent. Also, we have

$$X_1(t) = \gamma_{H_1} \int_0^t s^{1/2-H_1} dW_1(s),\ B^{H_2}(t) = \int_0^t K_{H_2}(t,s) dW_2(s). \tag{3}$$

Then

$$X_2(t) = \int_0^t K_{H_1,H_2}(t,s) dW_2(s),$$

where

$$K_{H_1,H_2}(t,s) = \beta_{H_2} s^{1/2-H_2} \int_s^t (t-u)^{1/2-H_1} u^{H_2-H_1} (u-s)^{H_2-3/2} du. \tag{4}$$

For an interval $[0, T]$, denote by $L_H^2[0, T]$ the completion of the space of simple functions $f : [0, T] \to \mathbb{R}$ with respect to the scalar product

$$\langle f, g\rangle_H^2 := \alpha_H \int_0^T \int_0^T f(t)g(s)|t-s|^{2H-2} ds dt,$$

where $\alpha_H = H(2H - 1)$. Note that this space contains both functions and distributions. For functions from $L_{H_2}^2[0, T]$, we have that

$$\int_0^T f(s) dX_2(s) = \int_0^T \left(K_{H_1,H_2}^* f\right)(s) dW_2(s),$$

where

$$\left(K_{H_1,H_2}^* f\right)(s) = \int_s^T f(t) \partial_t K_{H_1,H_2}(t,s) dt.$$

The projection of $X_1(T)$ onto $\{X(t), t \in [0, T]\}$ is a centered X-measurable Gaussian random variable and, therefore, is of the form

$$P_X X_1(T) = \int_0^T h_T(t) dX(t)$$

with $h_T \in L^2_{H_1}[0, T]$. Note that h_T still may be a distribution. However, as we will further see, it is a continuous function. The projection for all $u \in [0, T]$ must satisfy

$$\mathsf{E}\left[X(u) P_X X_1(T)\right] = \mathsf{E}\left[X(u) X_1(T)\right]. \tag{5}$$

Using (5) together with independency of X_1 and X_2, we arrive at the equation

$$\begin{aligned} &\mathsf{E}\left[X_1(u) \int_0^T h_T(t) dX_1(t) + X_2(u) \int_0^T h_T(t) dX_2(t)\right] \\ &\quad = \mathsf{E}\left[X_1(u) X_1(T)\right] = \varepsilon_{H_1} u^{2-2H_1}, \end{aligned} \tag{6}$$

where $\varepsilon_H = \gamma_H^2/(2-2H)$. Finally, from (3)–(6) we get the prototype of a Fredholm integral equation

$$\varepsilon_{H_1} u^{2-2H_1} = \gamma_{H_1}^2 \int_0^u h_T(s) s^{1-2H_1} ds + \int_0^T h_T(s) r_{H_1,H_2}(s, u) ds, \ \ u \in [0, T], \tag{7}$$

where

$$r_{H_1,H_2}(s, u) = \int_0^{s \wedge u} \partial_s K_{H_1,H_2}(s, v) K_{H_1,H_2}(u, v) dv.$$

Differentiating (7), we get the Fredholm integral equation of the 2nd kind,

$$\gamma_{H_1}^2 h_T(u) u^{1-2H_1} + \int_0^T h_T(s) k(s, u) ds = \gamma_{H_1}^2 u^{1-2H_1}, \quad u \in (0, T], \tag{8}$$

where

$$k(s, u) = \int_0^{s \wedge u} \partial_s K_{H_1,H_2}(s, v) \partial_u K_{H_1,H_2}(u, v) dv \tag{9}$$

with the function K_{H_1,H_2} defined by (4).

We will establish in Remark 2 that for the case $H_1 = \frac{1}{2}$, Eq. (8) can be reduced to the corresponding equation from [3]:

$$h_T(u) + H_2(2H_2 - 1) \int_0^T h_T(s) |s - u|^{2H_2 - 2} ds = 1, \quad u \in [0, T], \tag{10}$$

but the difference between (10) and (8) lies in the fact that (10) can be characterized as the equation with standard kernel, whereas (8) with two different power exponents is more or less nonstandard, and, therefore, it requires an unconventional approach. On the one hand, it is known from the paper [6] that if the conditions $H_2 - H_1 > \frac{1}{4}$ and $H_1 > 1/2$ are satisfied, then Eq. (8) has a unique solution h_{T_n} with $h_{T_n}(t) t^{\frac{1}{2} - H_1} \in L_2[0, T_n]$ on any sequence of intervals $[0, T_n]$ except, possibly, a countable number of T_n connected to eigenvalues of the corresponding integral operator (the meaning of this sentence will be specified later because, finally, we will get a similar result but in more general situation). On the other hand, the existence–uniqueness result for Eq. (10) in [3] is proved without any restriction on Hurst index H_2 while $H_1 = \frac{1}{2}$. The difference between these results can be explained so that in [3] the authors state the existence and uniqueness of the continuous solution, whereas in [6] the solution is established in the framework of L_2-theory.

In this paper, we propose to consider Eq. (8) in the space $C[0, T]$ again. This means that we consider the corresponding integral operator as an operator from $C[0, T]$ into $C[0, T]$ and establish an existence–uniqueness result in $C[0, T]$. This approach has the advantage that we do not need anymore the assumption $H_2 - H_1 > \frac{1}{4}$ and can include the case $H_1 = 1/2$ again into the consideration.

We say that two integral equations are equivalent if they have the same continuous solutions. In this sense, Eqs. (7) and (8) are equivalent, and both are equivalent to the equation

$$h_T(u) + \frac{1}{\gamma_{H_1}^2} \int_0^T h_T(s)\kappa(s, u)ds = 1, \quad u \in [0, T], \tag{11}$$

with continuous right-hand side, where

$$\kappa(s, u) = u^{2H_1-1}k(s, u), \quad s, u \in [0, T]. \tag{12}$$

We get that the main problem (i.e., the MLE construction for the drift parameter) is reduced to the existence–uniqueness result for the integral equation (7).

3 Compactness of integral operator. Existence–uniqueness result for the Fredholm integral equation

Consider the integral operator K generated by the kernel K bearing in mind that the notations of the kernel and of the corresponding operator will always coincide:

$$(Kx)(u) = \int_0^T K(s, u)x(s)ds, \quad x \in C[0, T].$$

Now we are in position to establish the properties of the kernel $\kappa(s, u)$ defined by (12) and (9). Introduce the notation $[0, T]_0^2 = [0, T]^2 \setminus \{(0, 0)\}$.

Lemma 1. *Up to a set of Lebesgue measure zero, the kernel $\kappa(s, u)$, $s, u \in [0, T]$, admits the following representation on* $[0, T]$:

$$\kappa(s, u) = \begin{cases} \kappa_0(s, u)\varphi(s, u), & s \neq u, \\ 0, & s = u, \end{cases} \tag{13}$$

where $\varphi(s, u) = (s \wedge u)^{1-2H_1} u^{2H_1-1} |s - u|^{2H_2-2H_1-1}$, and the function κ_0 is bounded and belongs to $C([0, T]_0^2)$.

Proof. We take (9) and first present the derivative of $K_{H_1,H_2}(t,s)$, defined by (4), in an appropriate form. To start, put $u = s + (t-s)z$. This allows us to rewrite $K_{H_1,H_2}(t,s)$ as

$$K_{H_1,H_2}(t,s) = \beta_{H_2} s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1} \times \int_0^1 (1-z)^{\frac{1}{2}-H_1}\big(s+(t-s)z\big)^{H_2-H_1} z^{H_2-\frac{3}{2}}dz. \tag{14}$$

Differentiating (14) w.r.t. t for $0 < s < t \le T$, we get

$$\begin{aligned}\partial_t K_{H_1,H_2}(t,s) &= (H_2-H_1)\beta_{H_2} s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1-1} \\ &\quad\times \int_0^1 (1-z)^{\frac{1}{2}-H_1}\big(s+(t-s)z\big)^{H_2-H_1} z^{H_2-\frac{3}{2}}dz \\ &\quad+ (H_2-H_1)\beta_{H_2} s^{\frac{1}{2}-H_2} \\ &\quad\times (t-s)^{H_2-H_1}\int_0^1 (1-z)^{\frac{1}{2}-H_1}\big(s+(t-s)z\big)^{H_2-H_1-1} z^{H_2-\frac{1}{2}}dz \\ &= (H_2-H_1)\beta_{H_2} s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1-1} \\ &\quad\times \Bigg(\int_0^1 (1-z)^{\frac{1}{2}-H_1}\big(s+(t-s)z\big)^{H_2-H_1} z^{H_2-\frac{3}{2}}dz \\ &\quad+ (t-s)\int_0^1 (1-z)^{\frac{1}{2}-H_1}\big(s+(t-s)z\big)^{H_2-H_1-1} z^{H_2-\frac{1}{2}}dz\Bigg) \\ &= (H_2-H_1)\beta_{H_2} s^{\frac{1}{2}-H_2}(t-s)^{H_2-H_1-1} \\ &\quad\times \Bigg(s^{H_2-H_1}\int_0^1 z^{H_2-\frac{3}{2}}(1-z)^{\frac{1}{2}-H_1}\left(1-\frac{s-t}{s}z\right)^{H_2-H_1}dz \\ &\quad+ (t-s)s^{H_2-H_1-1}\int_0^1 (1-z)^{\frac{1}{2}-H_1}\left(1-\frac{s-t}{s}z\right)^{H_2-H_1-1} z^{H_2-\frac{1}{2}}dz\Bigg).\end{aligned} \tag{15}$$

Denote for technical simplicity $\alpha_i = H_i - \frac{1}{2}$, $i = 1, 2$. Then, according to the definition and properties of the Gauss hypergeometric function (see Eqs. (31) and (32)), the terms in the right-hand side of (15) can be rewritten as follows. For the first term, thats is, for

$$I_1(t,s) := s^{H_2-H_1}\int_0^1 z^{\alpha_2-1}(1-z)^{-\alpha_1}\left(1-\frac{s-t}{s}z\right)^{H_2-H_1}dz, \tag{16}$$

the values of parameters for the underlying integral equal $a = H_1 - H_2$, $b = \alpha_2$, $c = H_2 - H_1 + 1$, and $x = \frac{s-t}{s} < 1$, respectively; therefore, $\frac{x}{x-1} = \frac{t-s}{t}$, $c - b = 1 - \alpha_1$, and

$$\begin{aligned}I_1(t,s) &= \mathrm{B}(1-\alpha_1,\alpha_2)s^{H_2-H_1}F\left(H_1-H_2,\alpha_2,1-H_1+H_2;\frac{s-t}{s}\right) \\ &= \mathrm{B}(1-\alpha_1,\alpha_2)s^{H_2-H_1}\left(\frac{t}{s}\right)^{H_2-H_1}F\left(H_1-H_2,1-\alpha_1,1-H_1+H_2;\frac{t-s}{t}\right)\end{aligned}$$

$$= \mathrm{B}(1-\alpha_1, \alpha_2) t^{H_2-H_1} F\left(H_1 - H_2, 1-\alpha_1, 1-H_1+H_2; \frac{t-s}{t}\right).$$

Similarly, for the second term, that is, for

$$I_2(t,s) := (t-s) s^{H_2-H_1-1} \int_0^1 z^{\alpha_2} (1-z)^{-\alpha_1} \left(1 - \frac{s-t}{s} z\right)^{H_2-H_1-1} dz, \quad (17)$$

the values of parameters for the underlying integral equal $a = H_1 - H_2 + 1$, $b = \alpha_2 + 1$, $c = H_2 - H_1 + 2$, and $x = \frac{s-t}{s}$, respectively; therefore, $\frac{x}{x-1} = \frac{t-s}{t}$, $c - b = 1 - \alpha_1$, and

$$\begin{aligned}
I_2(t,s) &= (t-s) s^{H_2-H_1-1} \mathrm{B}(1-\alpha_1, \alpha_2+1) \\
&\quad \times F\left(H_1 - H_2 + 1, \alpha_2 + 1, H_2 - H_1 + 2; \frac{s-t}{s}\right) \\
&= (t-s) s^{H_2-H_1-1} \left(\frac{t}{s}\right)^{H_2-H_1-1} \\
&\quad \times B(1-\alpha_1, \alpha_2+1) F\left(H_1 - H_2 + 1, 1-\alpha_1, 2 - H_1 + H_2; \frac{t-s}{t}\right) \\
&= (t-s) t^{H_2-H_1-1} \mathrm{B}(1-\alpha_1, \alpha_2+1) \\
&\quad \times F\left(H_1 - H_2 + 1, 1-\alpha_1, 2 - H_1 + H_2; \frac{t-s}{t}\right).
\end{aligned}$$

It is easy to see from the initial representations (16) and (17) that $I_1(t,s)$ and $I_2(t,s)$ are continuous on the set $0 < s \le t \le T$.

Now, introduce the notations

$$\Psi_1(t,s) = \mathrm{B}(1-\alpha_1, \alpha_2) F\left(H_1 - H_2, 1-\alpha_1, 1-H_1+H_2; \frac{t-s}{t}\right)$$

and

$$\begin{aligned}
\Psi_2(t,s) = \left(\frac{t-s}{t}\right)^{1-H_2+H_1} & \mathrm{B}(1-\alpha_1, \alpha_2+1) \\
& \times F\left(H_1 - H_2 + 1, 1-\alpha_1, 2 - H_1 + H_2; \frac{t-s}{t}\right),
\end{aligned}$$

so that $I_1(t,s) = t^{H_2-H_1} \Psi_1(t,s)$ and $I_2(t,s) = (t-s)^{H_2-H_1} \Psi_2(t,s)$. Note that $\frac{t-s}{t} \in [0,1)$; therefore,

$$\begin{aligned}
F&\left(H_1 - H_2, 1-\alpha_1, 1-H_1+H_2; \frac{t-s}{t}\right) \\
&= \frac{1}{\mathrm{B}(1-\alpha_1, \alpha_2)} \times \int_0^1 z^{-\alpha_1} (1-z)^{\alpha_2-1} \left(1 - \frac{t-s}{t} z\right)^{H_2-H_1} dz \\
&\le \frac{1}{\mathrm{B}(1-\alpha_1, \alpha_2)} \int_0^1 z^{-\alpha_1} (1-z)^{\alpha_2-1} dz = 1,
\end{aligned}$$

whence the function $\Psi_1(t,s)$ is bounded by $\mathrm{B}(1-\alpha_1,\alpha_2)$. In order to establish that $\Psi_2(t,s)$ is bounded, we use Proposition 1. Its conditions are satisfied: $a=H_1-H_2+1\in(0,1)$, $b=1-\alpha_1>0$, $c-b=\alpha_2+1>1$, and $x=\frac{t-s}{t}\in[0,1)$. Therefore,

$$\begin{aligned}
&x^{1-H_2+H_1}F(H_1-H_2+1,1-\alpha_1,2-H_1+H_2;x)\le x^{1-H_2+H_1}\\
&\quad\times\left(1-\frac{1-\alpha_1}{1-H_1+H_2}x\right)^{-1-H_1+H_2}=\left(\frac{1}{x}-\frac{1-\alpha_1}{1-H_1+H_2}\right)^{-1-H_1+H_2}\\
&\quad\le\left(1-\frac{1-\alpha_1}{1-H_1+H_2}\right)^{-1-H_1+H_2}=\left(\frac{1-H_1+H_2}{\alpha_2}\right)^{H_1-H_2+1},
\end{aligned}$$

whence $\Psi_2(t,s)\le\mathrm{B}(1-\alpha_1,\alpha_2+1)(\frac{1-H_1+H_2}{\alpha_2})^{H_1-H_2+1}$. Additionally, both functions are homogeneous:

$$\Psi_i(at,as)=\Psi_i(t,s)\text{ for }a>0,\ i=1,2.$$

Introduce the notation

$$\Phi(t,s)=I_1(t,s)+I_2(t,s)=t^{H_2-H_1}\Psi_1(t,s)+(t-s)^{H_2-H_1}\Psi_2(t,s) \tag{18}$$

and note that $\Phi\in C([0,T]_0^2)$ is bounded and homogeneous:

$$\Phi(at,as)=a^{H_2-H_1}\Phi(t,s),a>0. \tag{19}$$

In terms of notation (18), the representation (15) for $\partial_t K_{H_1,H_2}(t,s)$ can be rewritten as

$$\partial_t K_{H_1,H_2}(t,s)=\beta_{H_2}(H_2-H_1)s^{\frac12-H_2}(t-s)^{H_2-H_1-1}\Phi(t,s). \tag{20}$$

In turn, the kernel $k(s,u)$ from (9) can be rewritten as

$$\begin{aligned}
k(s,u)=&\big(\beta_{H_2}(H_2-H_1)\big)^2\\
&\times\int_0^{s\wedge u}v^{1-2H_2}(s-v)^{H_2-H_1-1}(u-v)^{H_2-H_1-1}\Phi(s,v)\Phi(u,v)dv.
\end{aligned} \tag{21}$$

Consider the kernel $k(s,u)$ for $s>u$. Then it evidently equals

$$\begin{aligned}
k(s,u)=&\big(\beta_{H_2}(H_2-H_1)\big)^2\\
&\times\int_0^{u}v^{1-2H_2}(s-v)^{H_2-H_1-1}(u-v)^{H_2-H_1-1}\Phi(s,v)\Phi(u,v)dv.
\end{aligned}$$

Put $z=\frac{u-v}{s-u}$ and transform $k(s,u)$ to

$$\begin{aligned}
k(s,u)=&\big(\beta_{H_2}(H_2-H_1)\big)^2(s-u)^{2H_2-2H_1-1}\int_0^{\frac{u}{s-u}}z^{H_2-H_1-1}(1+z)^{H_2-H_1-1}\\
&\times\big(u-z(s-u)\big)^{1-2H_2}\Phi\big(s,u-z(s-u)\big)\Phi\big(u,u-z(s-u)\big)dz\\
=:&\frac{k_0(s,u)}{(s-u)^{1-2H_2+2H_1}},
\end{aligned}$$

where

$$k_0(s,u) = \big(\beta_{H_2}(H_2-H_1)\big)^2 \int_0^{\frac{u}{s-u}} z^{H_2-H_1-1}(1+z)^{H_2-H_1-1} \times \big(u-z(s-u)\big)^{1-2H_2}\Phi\big(s,u-z(s-u)\big)\Phi\big(u,u-z(s-u)\big)dz.$$

In turn, transform $k_0(s,u)$ with the change of variables $tu = z$ and apply (19):

$$\begin{aligned} k_0(s,u) &= \big(\beta_{H_2}(H_2-H_1)\big)^2 \int_0^{\frac{1}{s-u}} (tu)^{H_2-H_1-1}(1+tu)^{H_2-H_1-1} \\ &\quad \times \big(u-tu(s-u)\big)^{1-2H_2}\Phi\big(s,u-tu(s-u)\big)\Phi\big(u,u-tu(s-u)\big)udt \\ &= \big(\beta_{H_2}(H_2-H_1)\big)^2 u^{1-2H_1} \int_0^{\frac{1}{s-u}} \big(1-t(s-u)\big)^{1-2H_2}(1+tu)^{H_2-H_1-1} \\ &\quad \times t^{H_2-H_1-1}\Phi\big(s,u-tu(s-u)\big)\Phi\big(1,1-t(s-u)\big)dt. \end{aligned} \tag{22}$$

Introducing the kernel $\kappa_0(s,u) = k_0(s,u)u^{2H_1-1}$, we can present $k(s,u)$ as

$$k(s,u) = \frac{\kappa_0(s,u)}{(s-u)^{1-2H_2+2H_1}u^{2H_1-1}}, \tag{23}$$

where, for $s > u > 0$,

$$\begin{aligned} \kappa_0(s,u) &= \big(\beta_{H_2}(H_2-H_1)\big)^2 \int_0^{\frac{1}{s-u}} \big(1-(s-u)t\big)^{1-2H_2}(1+ut)^{H_2-H_1-1} \\ &\quad \times t^{H_2-H_1-1}\Phi\big(s,u-tu(s-u)\big)\Phi\big(1,1-t(s-u)\big)dt \\ &= \big(\beta_{H_2}(H_2-H_1)\big)^2 \int_0^{\infty} 1_{t\le\frac{1}{s-u}} \big(1-(s-u)t\big)^{1-2H_2}(1+ut)^{H_2-H_1-1} \\ &\quad \times t^{H_2-H_1-1}\Phi\big(s,u-tu(s-u)\big)\Phi\big(1,1-t(s-u)\big)dt. \end{aligned} \tag{24}$$

For the case $u > s > 0$, we can replace s and u in formulas (23) and (24). Substituting formally $u = s$ into (24), for $s > 0$, we get

$$\begin{aligned} \kappa_0(s,s) &= \big(\beta_{H_2}(H_2-H_1)\big)^2\Phi(s,s)\Phi(1,1) \int_0^{\infty} (1+st)^{H_2-H_1-1}t^{H_2-H_1-1}dt \\ &= \big(\beta_{H_2}(H_2-H_1)\big)^2 s^{H_2-H_1}\Phi(1,1)^2 \int_0^{\infty} (1+st)^{H_2-H_1-1}t^{H_2-H_1-1}dt. \end{aligned} \tag{25}$$

Note that $\Phi(1,1) = \mathrm{B}(1-\alpha_1,\alpha_2)$ and $\int_0^\infty (1+st)^{H_2-H_1-1}t^{H_2-H_1-1}dt = s^{H_1-H_2}$ $\mathrm{B}(H_2-H_1, 1-2H_2+H_1)$. The former equation holds due to (34). We get that $\kappa_0(s,s)$ does not depend on s and equals some constant $C_H := (\beta_{H_2}(H_2-H_1)\mathrm{B}(1-\alpha_1,\alpha_2))^2\mathrm{B}(H_2-H_1,1-2H_2+H_1)$. Therefore, we define $\kappa_0(s,s) = C_H,\ s > 0$.

Now the continuity of κ_0 on $(0,T]^2$ follows from the Lebesgue dominated convergence theorem supplied by representation (24), Eq. (25), and its consequence

$\kappa_0(s,s) = C_H,\ s > 0$, together with the facts that $\Phi \in C([0,T]_0^2)$ and is bounded. Consider $\kappa_0(s,u)$ for $u \downarrow 0$ and let $s > 0$ be fixed. Then

$$\lim_{u\downarrow 0}\kappa_0(s,u) = C_H^1 := \left(\beta_{H_2}(H_2-H_1)\right)^2\Phi(1,0) \\ \times\int_0^1 (1-y)^{1-2H_2}y^{H_2-H_1-1}\Phi(1,1-y)dy < \infty,$$

and we can put $\kappa_0(s,0) = \kappa_0(0,u) = C_H^1,\ s > 0, u > 0$, thus extending the continuity of κ_0 to $[0,T]_0^2$.

It is easy to see that the values $\kappa_0(s,s)$ and $\kappa_0(s,0)$ do not depend on $s > 0$ and do not coincide: $C_H \neq C_H^1$. Consequently, the limit

$$\lim_{(s,u)\to(0,0)}\kappa_0(s,u)$$

does not exist and depends on the way the variables s and u tend to zero. We can equate $\kappa_0(0,0)$ to any constant; for example, let $\kappa_0(0,0) = 0$.

In order to prove that κ_0 is bounded, we consider the case $s > u$ (the opposite case is treated similarly) and put $z = (s-u)t$. Then

$$\int_0^{\frac{1}{s-u}}\left(1-(s-u)t\right)^{1-2H_2}(1+ut)^{H_2-H_1-1}t^{H_2-H_1-1}\Phi\left(s,u-tu(s-u)\right) \\ \times\Phi\left(1,1-t(s-u)\right)dt = \frac{1}{(s-u)^{H_2-H_1}}\int_0^1(1-z)^{1-2H_2} \\ \times\left(1+\frac{u}{s-u}z\right)^{H_2-H_1-1}z^{H_2-H_1-1}\Phi\left(s,u(1-z)\right))\Phi(1,1-z)dz =: I_3(s,u). \tag{26}$$

It follows from (19) that, for $s \neq 0$,

$$\Phi\left(s,u(1-z)\right) = s^{H_2-H_1}\Phi\left(1,\frac{u}{s}(1-z)\right).$$

Denote $r = \frac{s}{s-u}$ and put $t = \frac{1-z}{1-(1-r)z}$. Then

$$\frac{u}{s-u} = r-1,\ t<1,\ z = \frac{1-t}{1-t(1-r)} \in (0,1),$$

and the right-hand side of (26) can be rewritten as

$$I_3(s,u) = r^{H_2-H_1}\int_0^1(1-z)^{1-2H_2}\left(1-(1-r)z\right)^{H_2-H_1-1}z^{H_2-H_1-1} \\ \times\Phi\left(1,\frac{u}{s}(1-z)\right)\Phi(1,1-z)dz = r^{1-2H_1}\int_0^1 t^{1-2H_2}(1-t)^{H_2-H_1-1} \\ \times\left(1-(1-r)t\right)^{2H_1-1}\Phi\left(1,\frac{u}{s}\frac{rt}{1-(1-r)t}\right)\Phi\left(1,\frac{rt}{1-(1-r)t}\right)dt. \tag{27}$$

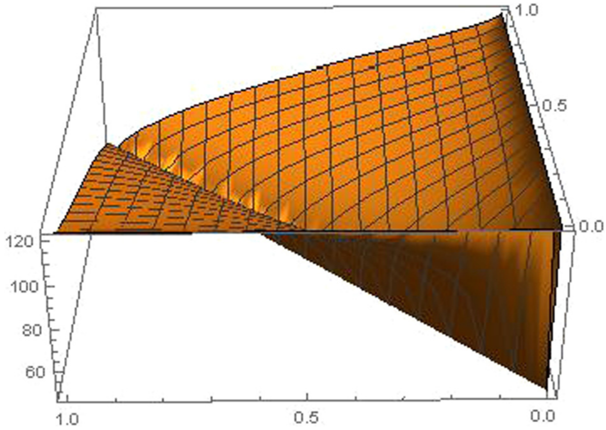

Fig. 1. Function $\kappa_0(s,u)$

Finally, put $y = 1 - t$. Then the right-hand side of (27) is transformed to

$$I_3(s,u) = r^{1-2H_1} r^{2H_1-1} \int_0^1 (1-y)^{1-2H_2} y^{H_2-H_1-1} \left(1 - y\frac{r-1}{r}\right)^{2H_1-1} \times \Phi\left(1, \frac{u}{s}\frac{r(1-y)}{r-y(r-1)}\right) \Phi\left(1, \frac{r(1-y)}{r-y(r-1)}\right) dy.$$

Recall that $r = \frac{s}{s-u}$. Then it follows from the boundedness of Φ that there exists a constant C_H^1 such that, for $s > u$,

$$\begin{aligned} \kappa_0(s,u) &= \left(\beta_{H_2}(H_2 - H_1)\right)^2 \int_0^1 (1-y)^{1-2H_2} \left(1 - \frac{u}{s}y\right)^{2H_1-1} y^{H_2-H_1-1} \\ &\quad \times \Phi\left(1, \frac{u(1-y)}{s-uy}\right) \Phi\left(1, \frac{s(1-y)}{s-uy}\right) dy \\ &\le C_H^1 \int_0^1 (1-y)^{1-2H_2} y^{H_2-H_1-1} dy, \end{aligned} \tag{28}$$

so κ_0 is bounded, and the lemma is proved. $\square$

Remark 1. Figure 1 demonstrates the graph of $\kappa_0(s,u)$ for $H_1 = 0.7$ and $H_2 = 0.9$.

Now, consider the properties of the function

$$\varphi(s,u) = (s \wedge u)^{1-2H_1} u^{2H_1-1} |s-u|^{2H_2-2H_1-1}$$

participating in the kernel representation (13).

Lemma 2. *The function φ has the following properties:*

(i) *for any $u \in [0,T]$, $\varphi(\cdot,u) \in L_1[0,T]$ and* $\sup_{u\in[0,T]} \|\varphi(\cdot,u)\|_{L_1} < \infty$

(ii) *for any* $u_1 \in [0, T]$, $\int_0^T |\varphi(s, u) - \varphi(s, u_1)|ds \to 0$ *as* $u \to u_1$.

Proof. (i) It follows from the evident calculations that

$$\begin{aligned}\int_0^T |\varphi(s, u)|ds &= \int_0^T \varphi(s, u)ds = \int_0^u \frac{u^{2H_1-1}ds}{s^{2H_1-1}(u-s)^{1+2H_1-2H_2}} \\ &+ \int_u^T \frac{ds}{(s-u)^{1+2H_1-2H_2}} = u^{2H_2-2H_1}\mathrm{B}(2-2H_1, 2H_2-2H_1) \\ &+ \frac{(T-u)^{2H_2-2H_1}}{2H_2-2H_1} \le C_{H_1,H_2}T^{2H_2-2H_1} < \infty \text{ for all } u \in [0, T].\end{aligned}$$

(ii) First, let $u_1 = 0$ and $u \downarrow 0$. Note that $\varphi(s, 0) = s^{2H_2-2H_1-1}$. Therefore,

$$\begin{aligned}&\int_0^T |\varphi(s, u) - \frac{1}{s^{1+2H_1-2H_2}}|ds = \int_0^u |\frac{u^{2H_1-1}}{s^{2H_1-1}(u-s)^{1+2H_1-2H_2}} - \frac{1}{s^{1+2H_1-2H_2}}|ds \\ &\quad + \int_u^T \frac{ds}{(s-u)^{1+2H_1-2H_2}} - \int_u^T \frac{ds}{s^{1+2H_1-2H_2}}ds \le \int_0^u \frac{u^{2H_1-1}ds}{s^{2H_1-1}(u-s)^{1+2H_1-2H_2}} \\ &\quad + \int_0^u \frac{ds}{s^{1+2H_1-2H_2}} + \frac{1}{2H_2-2H_1}\left((s-u)^{2H_2-2H_1} - s^{2H_2-2H_1}\right)\Big|_{s=u}^{s=T} \\ &= \mathrm{B}(2-2H_1, 2H_2-2H_1)u^{2H_2-2H_1} \\ &\quad + \frac{1}{2H_2-2H_1}\left(2u^{2H_2-2H_1} + (T-u)^{2H_2-2H_1} - T^{2H_2-2H_1}\right) \to 0, \text{ as } u \to 0.\end{aligned}$$

From now on suppose that $u_1 > 0$ is fixed. Without loss of generality, suppose that $u \uparrow u_1$. Then

$$\begin{aligned}\int_0^T |\varphi(s, u) - \varphi(s, u_1)|ds &= \int_0^u |\varphi(s, u) - \varphi(s, u_1)|ds + \int_u^{u_1} |\varphi(s, u) - \varphi(s, u_1)|ds \\ &+ \int_{u_1}^T |\varphi(s, u) - \varphi(s, u_1)|ds =: I_1(u, u_1) + I_2(u, u_1) + I_3(u, u_1).\end{aligned}$$

Consider the terms separately. First, we establish that $\varphi(s, \cdot)$ is decreasing in the second argument. Indeed, for $0 < s < u < u_1$,

$$\begin{aligned}\varphi(s, u_1) &= \frac{u_1^{2H_1-1}}{s^{2H_1-1}(u_1-s)^{1+2H_1-2H_2}} = \frac{1}{s^{2H_1-1}(1-\frac{s}{u_1})^{1+2H_1-2H_2}u_1^{2-2H_2}} \\ &\le \frac{1}{s^{2H_1-1}(1-\frac{s}{u})^{1+2H_1-2H_2}u^{2-2H_2}} = \varphi(s, u).\end{aligned}$$

Therefore,

$$I_1(u,u_1)=\int_0^u \big(\varphi(s,u)-\varphi(s,u_1)\big)ds=\int_0^u \varphi(s,u)ds-\int_0^{u_1}\varphi(s,u_1)ds$$
$$+\int\limits_u^{u_1}\varphi(s,u_1)ds\le \mathrm{B}(2-2H_1,2H_2-2H_1)\big(u^{2H_2-2H_1}-u_1^{2H_2-2H_1}\big)$$
$$+\frac{u_1^{2H_1-1}(u_1-u)^{2H_2-2H_1}}{2H_2-2H_1}\to 0,\ \text{as } u\uparrow u_1.$$

The second integral vanishes as well:

$$I_2(u,u_1)\le \int\limits_u^{u_1}\varphi(s,u)ds+\int\limits_u^{u_1}\varphi(s,u_1)ds$$
$$\le \left(\frac{1}{2H_2-2H_1}+\left(\frac{u_1}{u}\right)^{2H_1-1}\right)(u_1-u)^{2H_2-2H_1}\to 0$$

as $u\uparrow u_1$. Finally,

$$I_3(u,u_1)=\int_{u_1}^T\frac{ds}{(s-u_1)^{1+2H_1-2H_2}}-\int_{u_1}^T\frac{ds}{(s-u)^{1+2H_1-2H_2}}=\frac{1}{2H_2-2H_1}$$
$$\times\big((T-u_1)^{2H_2-2H_1}-(T-u)^{2H_2-2H_1}+(u_1-u)^{2H_2-2H_1}\big)\to 0$$
$$\text{as } u\uparrow u_1.$$

The lemma is proved. □

Lemma 3. *The kernel κ generates a compact integral operator $\kappa : C[0,T]\to C[0,T]$.*

Proof. According to [2], it suffices to prove that the kernel κ defined by (13) satisfies the following two conditions:

(iii) for any $u\in[0,T]$, $\kappa(\cdot,u)\in L_1[0,T]$ and $\sup\limits_{u\in[0,T]}\|\kappa(\cdot,u)\|_{L_1}<\infty$;

(iv) For any $u_1\in[0,T]$, $\int_0^T|\kappa(s,u)-\kappa(s,u_1)|ds\to 0$ as $u\to u_1$.

The first condition follows directly from fact that $\kappa_0(s,u)$ is bounded (see Lemma 1) and from Lemma 2 (i).

In order to check (iv), consider

$$\int_0^T|\kappa(s,u)-\kappa(s,u_1)|ds=\int_0^T|\kappa_0(s,u)\varphi(s,u)-\kappa_0(s,u_1)\varphi(s,u_1)|ds$$
$$\le\int_0^T\kappa_0(s,u)|\varphi(s,u)-\varphi(s,u_1)|ds+\int_0^T\varphi(s,u_1)|\kappa_0(s,u)-\kappa_0(s,u_1)|ds.$$

Again, Lemma 1 in the part that states that $\kappa_0(s,u)$ is bounded, together with Lemma 2 (ii), guarantees that the first term converges to zero as $u\to u_1$. Furthermore, Lemma 1 in the part that states that $\kappa_0 \in C([0,T]_0^2)$ guarantees that $\kappa_0(s,u)$ converges to $\kappa_0(s,u_1)$ as $u \to u_1$ for a.e. $s \in [0,T]$. Since

$$\varphi(s,u_1)|\kappa_0(s,u)-\kappa_0(s,u_1)| \le C\varphi(s,u_1) \in L_1[0,T],$$

the proof follows from the Lebesgue dominated convergence theorem. □

Remark 2. In the case where $H_1 = \frac{1}{2}$, the kernel $\kappa(s,u)$ can be simplified to

$$\kappa(s,u) = H_2(2H_2-1)|s-u|^{2H_2-2},$$

and Eq. (8) coincides with (10). Indeed, let $H_1 = \frac{1}{2}$. Then the function $\kappa_0(s,u)$ equals $H_2(2H_2-1)$. Consider the function $\Phi(s,v)$ defined by (18):

$$\begin{aligned}\Phi(t,s) &= t^{H_2-\frac{1}{2}}\Bigg(\int_0^1\left(1-\frac{t-s}{t}z\right)^{H_2-\frac{1}{2}}(1-z)^{H_2-\frac{3}{2}}dz \\ &\quad + \frac{t-s}{t}\int_0^1(1-z)^{H_2-\frac{1}{2}}\left(1-\frac{t-s}{t}z\right)^{H_2-\frac{3}{2}}dz\Bigg) \\ &= -\frac{t^{H_2-\frac{1}{2}}}{H_2-\frac{1}{2}}\int_0^1\left(\left(1-\frac{t-s}{t}z\right)^{H_2-\frac{1}{2}}(1-z)^{H_2-\frac{1}{2}}\right)_z' dz = \frac{t^{H_2-\frac{1}{2}}}{H_2-\frac{1}{2}}.\end{aligned} \tag{29}$$

Combining (28) and (29), we get

$$\begin{aligned}&\kappa_0(s,u) \\ &= \left(\beta_{H_2}(H_2-H_1)\right)^2\int_0^1(1-t)^{1-2H_2}t^{H_2-\frac{3}{2}}\Phi\left(1,\frac{u(1-t)}{s-ut}\right)\Phi\left(1,\frac{s(1-t)}{s-ut}\right)dt \\ &= \beta_{H_2}^2\int_0^1(1-t)^{1-2H_2}t^{H_2-\frac{3}{2}}dt = \beta_{H_2}^2\mathrm{B}\left(H_2-\frac{1}{2},2-2H_2\right) = H_2(2H_2-1).\end{aligned}$$

Theorem 1. *There exists a sequence $T_n \to \infty$ such that the integral equation (11) has a unique solution $h_{T_n}(u) \in C[0,T_n]$.*

Proof. We work on the space $C([0,T])$. Recall that (11) is of the form

$$h_T(u) + \frac{1}{\gamma_{H_1}^2}\int_0^T h_T(s)\kappa(s,u)ds = 1, \quad u \in [0,T].$$

The corresponding homogeneous equation is of the form

$$\int_0^T h_T(s)\kappa(s,u)ds = -\gamma_{H_1}^2 h_T(u), \quad u \in [0,T]. \tag{30}$$

Since the integral operator κ is compact, classical Fredholm theory states that Eq. (11) has a unique solution if and only if the corresponding homogeneous equation (30) has only the trivial solution. Now, it is easy to see that, for any $a > 0$, the following equalities hold:

$$\kappa_0(sa,ua) = \kappa_0(s,u),$$

$$\varphi(sa, ua) = a^{2H_2-2H_1-1}\varphi(s, u).$$

Consequently, $\kappa(sa, ua) = a^{2H_2-2H_1-1}\kappa(s, u)$. We can change the variable of integration $s = s'T$ and put $u = u'T$ in (30). Therefore, the equation will be reduced to the equivalent form

$$\int_0^1 h_T(Ts)\kappa(s, u)ds = -\gamma_{H_1}^2 T^{2H_1-2H_2} h_T(Tu), \quad u \in [0, 1].$$

Denote $\lambda = -\gamma_{H_1}^2 T^{2H_1-2H_2}$. Note that λ depends continuously on T. At the same time, the compact operator κ has no more than countably many eigenvalues. Therefore, we can take the sequence $T_n \to \infty$ in such a way that

$$\lambda_n = -\gamma_{H_1}^2 T_n^{2H_1-2H_2}$$

will be not an eigenvalue. Consequently, the homogeneous equation has only the trivial solution, whence the proof follows. $\square$

4 Statistical results: The form of a maximum likelihood estimator, its consistency, and asymptotic normality

The following result establishes the way MLE for the drift parameter θ can be calculated. The proof of the theorem is the same as the proof of the corresponding statement from [6], so we omit it.

Theorem 2. *The likelihood function is of the form*

$$L_{T_n}(X, \theta) = \exp\left\{\theta\delta_{H_1} N(T_n) - \frac{1}{2}\theta^2\delta_{H_1}^2\langle N\rangle(T_n)\right\},$$

and the maximum likelihood estimator is of the form

$$\widehat{\theta}(T_n) = \frac{N(T_n)}{\delta_{H_1}\langle N\rangle(T_n)},$$

where $N(t) = E_0(X_1(t)|\mathfrak{F}_t^X)$ is a square-integrable Gaussian $\mathfrak{F}_t^X$-martingale, $N(T_n) = \int_0^{T_n} h_{T_n}(t)dX(t)$ with $h_{T_n}(t)t^{\frac{1}{2}-H_1} \in L_2[0, T_n]$, $h_{T_n}(t)$ is a unique solution to (11), and $\langle N\rangle(T_n) = \gamma_{H_1}^2 \int_0^{T_n} h_{T_n}(t)t^{1-2H_1}dt$.

The next two results establish basic properties of the estimator; their proofs repeat the proofs of the corresponding statements from [6] and [3].

Theorem 3. *The estimator $\widehat{\theta}_{T_n}$ is strongly consistent, and*

$$\lim_{T_n\to\infty} {T_n}^{2-2H_2} E_\theta(\widehat{\theta}_{T_n} - \theta)^2 = \frac{1}{\int_0^1 h_0(u)u^{\frac{1}{2}-H_1}du},$$

where the function $h_0(u)$ is the solution of the integral equation

$$\kappa h(u) = \gamma_{H_1}^2.$$

Theorem 4. *The estimator $\widehat{\theta}_{T_n}$ is unbiased, and the corresponding estimation error is normal*

$$\widehat{\theta}_{T_n} - \theta \sim N\left(0, \frac{1}{\int_0^{T_n} h_{T_n}(s)s^{1-2H_1}ds}\right).$$

A Appendix. Some properties of the hypergeometric function

Recall the integral representation of the Gauss hypergeometric function and some of its properties.

For $c > b > 0$ and $x < 1$, the Gauss hypergeometric function is defined as the integral (see [1], formula 15.3.1)

$$F(a,b,c;x) = {}_2F_1(a,b,c;x) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt. \tag{31}$$

For the same values of parameters, the following equality holds (see [1], 15.3.4):

$$F(a,b,c;x) = (1-x)^{-a} F\left(a, c-b, c; \frac{x}{x-1}\right), \tag{32}$$

Evidently, $F(a,b,c;x)$ at $x = 1$ is correctly defined for $c - a - b > 1$ and in this case equals

$$F(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{33}$$

Finally, it is easy to check with the help of (31) that

$$F(a,b,c;0) = F(0,b,c;x) = 1. \tag{34}$$

The following result gives upper bounds for the hypergeometric function (see [5] Theorem 4 and 5, respectively).

Proposition 1. (i) *For* $c > b > 1,\ x > 0$*, and* $0 < a \le 1$*, we have the inequality*

$$F(a,b,c;-x) < \frac{1}{(1 + x(b-1)/(c-1))^a}.$$

(ii) *For* $0 < a \le 1,\ b > 0,\ c - b > 1$*, and* $x \in (0,1)$*, we have the inequality*

$$F(a,b,c;x) < \frac{1}{(1 - \frac{b}{c-1}x)^a}.$$

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