Modern Stochastics: Theory and Applications 9 (4) (2022) 431–452
https://doi.org/10.15559/22-VMSTA211

Gaussian Volterra processes with power-type kernels. Part II

Yuliya Mishura, Sergiy Shklyar*

Department of Probability Theory, Statistics and Actuarial Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska Street 64, Kyiv 01601, Ukraine

myus@univ.kiev.ua (Yu. Mishura), shklyar@univ.kiev.ua (S. Shklyar)

Received: 21 March 2022, Revised: 9 June 2022, Accepted: 17 June 2022, Published online: 5 July 2022

Abstract In this paper the study of a three-parametric class of Gaussian Volterra processes is continued. This study was started in Part I of the present paper. The class under consideration is a generalization of a fractional Brownian motion that is in fact a one-parametric process depending on Hurst index H. On the one hand, the presence of three parameters gives us a freedom to operate with the processes and we get a wider application possibilities. On the other hand, it leads to the need to apply rather subtle methods, depending on the intervals where the parameters fall. Integration with respect to the processes under consideration is defined, and it is found for which parameters the processes are differentiable. Finally, the Volterra representation is inverted, that is, the representation of the underlying Wiener process via Gaussian Volterra process is found. Therefore, it is shown that for any indices for which Gaussian Volterra process is defined, it generates the same flow of sigma-fields as the underlying Wiener process – the property that has been used many times when considering a fractional Brownian motion.

Keywords Gaussian Volterra processes, fractional Brownian motion, sample path differentiability, inversion of the Volterra representation

2010 MSC 60G22, 60G15, 60G17, 60G18, 60H05

*Corresponding author.

1 Introduction and preliminaries

This paper is a continuation of the paper [7], and we adhere to the same basic notation and the same object. Namely, let $T > 0$ and let $\{W_s,\ s \in [0, T]\}$ be a Wiener process. We consider the Gaussian process with Volterra kernel (Gaussian Volterra process) of the form

$$X_t = \int_0^t K(t, s)\, dW_s, \tag{1}$$

with

$$K(t, s) = s^\alpha \int_s^t u^\beta\, (u - s)^\gamma\, du\, \mathbf{1}_{s \le t}, \tag{2}$$

so that

$$X_t = \int_0^t s^\alpha \int_s^t u^\beta\, (u - s)^\gamma\, du\, dW_s. \tag{3}$$

According to [7], condition $\int_0^t K^2(t, s) ds < \infty$ is satisfied whenever

$$\alpha > -\frac{1}{2}, \qquad \gamma > -1, \quad \text{and} \quad \alpha + \beta + \gamma > -\frac{3}{2}. \tag{4}$$

Additionally, we proved in [7] that under condition (4) the process X is Hölder continuous on the interval $[0, T]$ up to order $\min\bigl(1,\ \gamma + \frac{3}{2},\ \alpha + \beta + \gamma + \frac{3}{2}\bigr)$.

Note that in the case where $\alpha = 1/2 - H$, $\beta = H - 1/2$, $\gamma = H - 3/2$, $H \in (1/2, 1)$, process X is a fractional Brownian motion. For integration of a deterministic function with respect to a fractional Brownian motion, we refer to [12]. Various approaches to stochastic integration with respect to a fractional Brownian motion are developed in [2, 3, 11]. Another approach to the integration of a nonrandom functions with respect to the Gaussian processes are considered in [6, Section 3.5]. The question of the inverse representation of the underlying Wiener process via fractional Brownian motion was obtained in [10] and clarified in [9].

Thus, the process X under consideration is a generalization of a fractional Brownian motion, moreover, its study requires rather subtle reasoning and estimates that depend on the values of the parameters α, β and γ. In [7] we proved that the process X satisfies the single-point Hölder condition up to order $\alpha + \beta + \gamma + \frac{3}{2}$ at point 0, the "interval" Hölder condition up to order $\min(\gamma + \frac{3}{2},\ 1)$ on the interval $[t_0, T]$ (where $0 < t_0 < T$), and the Hölder condition up to order $\min(\alpha + \beta + \gamma + \frac{3}{2},\ \gamma + \frac{3}{2},\ 1)$ on the entire interval $[0, T]$. In the present paper we are interested in the integration w.r.t. process X and the inverse representation of W via process X. Thus, quite traditional problems are considered, but their solution requires different analytical approaches depending on the values of the parameters, and this is the interest of this study.

For the factional Brownian motion B^H, the Volterra representation (1) is called the Molchan representation. The Wiener process W in this representation can be expressed with B^H step-by-step, as it is shown in [10]. The inverse representation in the form $W_t = \int_0^t L(t, s)\, dB^H$ is obtained in [5]. (More generally, [5] deals with representations of the form $B_t^{H_1} = \int_0^t L(t, s)\, dB_s^{H_2}$, where B^{H_1} and B^{H_2} are two fractional Brownian motions with different Hurst indices which share the Wiener process in their Molchan representations.)

The inverse Volterra representation of the process of the form

$$X_t = \int_0^t a(s) \int_s^t b(u)\, c(u-s)\, du\, dW_s$$

is obtained in [8]. We reuse this representation for the process (3) whenever possible, i.e., when (4) holds true and $\gamma < 0$. However, the conditions of [8, Proposition 2] are satisfied if $\alpha \le 0$ (in addition to (4) and $\gamma < 0$), and are not satisfied if $\alpha > 0$, while the inverse representation is valid anyway. Therefore, we have to justify the inverse representation for the case where the results of [8] are not applicable.

In Section 2 we develop integration of deterministic functions with respect to the process X. We construct a Banach space $\mathcal{E}_{p,k}$ of functions such that the integration of functions in $\mathcal{E}_{p,k}$ can be obtained by continuous extension of integration of piecewise-constant functions. In reality, the space $\mathcal{E}_{p,k}$ is a weighted $L^p([0,T],\mu)$ space for some measure μ.

In Section 3 we find the set of parameters for which the process X is differentiable. For that end, we define a process $\dot{X}_t = t^\beta \int_0^t s^\alpha (t-s)^\gamma\, dW_s$, and prove that $dX_t/dt = \dot{X}_t$ (under some conditions for α, β and γ; see Corollary 2 for details). We also study under what conditions the process $\dot{X}$ is differentiable, and thus the process X is differentiable several times.

In Section 4 we find an expression for the Wiener process W in the Volterra representation (3). The form of the expression depends on whether $\gamma \in (-1, 0)$ or $\gamma \ge 0$. If $\gamma \in (-1,0)$, then the inverse representation has the form $W_v = \int_0^v L(v,t)\, dX_t$. If $\gamma \ge 0$, we use the process $\dot{X}$ to reconstruct W.

In the appendices we present auxiliary results. In Appendix A we restate the Hölder and Young inequalities in a nonstandard form, and propose a lemma that helps to apply the Young inequality. In Appendix B we present some basics of fractional calculus, which allows us to solve the Abel equation needed to invert the Volterra representation for $\gamma > 0$.

2 Integration with respect to the Volterra process

2.1 Classes of integrable functions

Let $\{X_t,\ t \in [0,T]\}$ be a Volterra process defined in (1) with kernel $K(t,s)$ of the form (2), and let $\phi(t)$ be a nonrandom function. In [11], for linear operator K^* defined as

$$K^*\, \mathbf{1}_{[0,t]}(s) = K(t,s), \tag{5}$$

the integral with respect to the process X is defined as

$$\int_0^T \phi(t)\, dX_t = \int_0^T K^* \phi(s)\, dW_s.$$

Since in the case under consideration the kernel $K(t,s)$ is absolutely continuous in t and $K(s,s) = 0$, we can write

$$K(u,s) = \int_s^u \frac{\partial K(t,s)}{\partial t}\, dt,$$

and so the obvious choice for operator K^* is

$$(K^*\phi)(s) = \int_s^T \phi(t)\, \frac{\partial K(t,s)}{\partial t}\, dt.$$

If it will be necessary to distinguish, operator K^* will be marked as $K^*_{(\cdot,\cdot,\cdot)}$, by the values of the powers in the kernel K. Now,

$$\int_0^T \phi(t)\, dX_t = \int_0^T \int_s^T \phi(t)\, \frac{\partial K(t,s)}{\partial t}\, dt\, dW_s.$$

So, the integral with respect to the Volterra process (3) is formally defined as

$$\int_0^T \phi(t)\, dX_t = \int_0^T s^\alpha \int_s^T \phi(t) t^\beta (t-s)^\gamma\, dt\, dW_s. \tag{6}$$

Now our goal is to define the class of functions ϕ for which the integral $\int_0^T \phi(t)\, dX_t$ is well-defined. To that end, for $k \in \mathbb{R}$ and $p \in [1, +\infty)$ we consider the Banach space

$$\mathcal{E}_{p,k} = \left\{ \phi : [0,T] \to \mathbf{R} \,:\, \phi \text{ is Borel measurable and } \int_0^T |\phi(t)|^p\, t^{pk}\, dt < \infty \right\}$$

of functions taken up to a.e. equivalence, with the norm

$$\|\phi\|_{\mathcal{E}_{p,k}} = \left(\int_0^T |\phi(t)|^p\, t^{pk}\, dt \right)^{1/p}.$$

Under additional condition $pk > -1$ the space $\mathcal{E}_{p,k}$ contains indicators of intervals $\mathbf{1}_{[0,t]}$, $0<t\le T$. Regardless whether or not $pk > -1$, the space $\mathcal{E}_{p,k}$ contains indicators of intervals $\mathbf{1}_{[t_0,t]}$, $0 < t_0 < t \le T$, and the span of these indicators is a dense subset in $\mathcal{E}_{p,k}$.

Proposition 1. *Let* (4) *hold true, let* $p \ge 1$ *and*

$$p > \max\left(\frac{2}{2\gamma + 3}, \frac{2}{3 + 2\alpha + 2\beta + 2\gamma} \right). \tag{7}$$

Then the operator

$$(K^*\phi)(s) = s^\alpha \int_s^T \phi(t)\, t^\beta\, (t-s)^\gamma\, dt \tag{8}$$

is a bounded linear operator $L^p[0,T] \to L^2[0,T]$.

Proof. For $p = \infty$ Proposition 1 follows from the proof of Theorem 1 [7]. So, let p is finite, and let us calculate the norms of functions on the interval $[0,T]$. In what follows, $\|f\|_p$ is a short-hand notation for $\|f\|_{L^p[0,T]}$. With this notation,

$$\|t^\alpha\|_p = (\alpha p + 1)^{-1/p} T^{\alpha + 1/p},$$

as long as $p > 0$ and $\alpha p > -1$.

To prove the well-definedness and boundedness of the operator K^*, it suffices to show that

$$\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma \, dt \, ds \le c(p;\alpha,\beta,\gamma)\, \|\psi\|_2 \, \|\phi\|_p \tag{9}$$

for all $\psi \in L^2[0,T]$ and for some constant $c(p;\alpha,\beta,\gamma)$ not depending on ψ. Consider six cases.

Case 1. Let $\gamma \ge 0$. Notice that under conditions of Proposition 1, inequalities $\frac{1}{2}+\alpha > 0$ and $\frac{1}{p} - 1 - \beta - \gamma < \frac{1}{2} + \alpha$ hold true. Choose an arbitrary $\delta \ge 0$ such that $\frac{1}{p} - 1 - \beta - \gamma < \delta < \frac{1}{2} + \alpha$.

Obviously, in the left-hand integral in (9) we have that $s \le t$ and $t - s \le t$. Therefore

$$\begin{aligned}\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma \, dt \, ds &\le \int_0^T |\psi(s)|\, s^{\alpha-\delta} \int_s^T |\phi(t)|\, t^{\delta+\beta+\gamma} \, dt \, ds \\ &\le \int_0^T |\psi(s)|\, s^{\alpha-\delta} \, ds \int_0^T |\phi(t)|\, t^{\delta+\beta+\gamma} \, dt.\end{aligned}$$

By the Hölder inequality,

$$\int_0^T |\psi(s)|\, s^{\alpha-\delta} \, ds \le \|\psi\|_2 \, \|s^{\alpha-\delta}\|_2, \quad \int_0^T |\phi(t)|\, t^{\delta+\beta+\gamma} \, dt \le \|\phi\|_p \|t^{\delta+\beta+\gamma}\|_{p/(p-1)}.$$

Thus, (9) holds true with

$$c(p;\alpha,\beta,\gamma) = \|s^{\alpha-\delta}\|_2 \|t^{\delta+\beta+\gamma}\|_{p/(p-1)},$$

and the finiteness of the norms $\|s^{\alpha-\delta}\|_2$ and $\|t^{\delta+\beta+\gamma}\|_{p/(p-1)}$ follows from the inequalities $p \ge 1$ and $\frac{1}{p} - 1 - \beta - \gamma < \delta < \frac{1}{2} + \alpha$.

Case 2. Let $-1 < \gamma < 0$ and $\frac{1}{p} < \gamma + 1$. Let us find ϵ from the following equation:

$$\frac{1}{p} + \left(-\left(\alpha + \tfrac{1}{2} - \epsilon\right)^+ - \beta\right)^+ - \gamma + 2\epsilon = 1. \tag{10}$$

It follows from assumption (7) that $\frac{1}{p} < \frac{3}{2} + \alpha + \beta + \gamma$. Together with assumption $\frac{1}{p} < \gamma + 1$, this implies that the left-hand side of (10) is less than 1 for $\epsilon = 0$. In addition, the left-hand side of (10) is obviously greater than 1 for $\epsilon = \frac{1}{2}$. Hence, there exists the unique $\epsilon \in (0, \frac{1}{2})$ that satisfies (10); take it for the rest of the proof.

As before, $s \le t$, and thus

$$s^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+} \le t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+}. \tag{11}$$

Since $\alpha - \left(\alpha + \frac{1}{2} - \epsilon\right)^+ = \min\left(\alpha, \epsilon - \frac{1}{2}\right)$,

$$s^{\alpha - \left(\alpha+\frac{1}{2}-\epsilon\right)^+} t^\beta = s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} t^\beta. \tag{12}$$

Multiplying (11) by (12), we get

$$s^\alpha t^\beta \le s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta}.$$

Hence,

$$\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma \, dt\, ds \le$$
$$\le \int_0^T |\psi(s)|\, s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} \int_s^T |\phi(t)|\, t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} (t-s)^\gamma \, dt\, ds.$$

Define q and r from the equations

$$\frac{1}{q} = \left(-\left(\alpha+\tfrac{1}{2}-\epsilon\right)^+ - \beta\right)^+ + \epsilon, \qquad \frac{1}{r} = \epsilon - \gamma,$$

respectively. Then $q > 0$ and $r > 0$ and, according to (10), $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Denote

$$h_s(t) = \begin{cases} 0 & \text{if } t \in [0, s], \\ (t-s)^\gamma & \text{if } t \in (s, T] \end{cases}$$

and apply the Hölder inequality:

$$\int_s^T |\phi(t)|\, t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} (t-s)^\gamma \, dt = \int_0^T |\phi(t)|\, t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} h_s(t)\, dt \le$$
$$\le \|\phi\|_p \left\| t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} \right\|_q \|h_s\|_r \le \|\phi\|_p \left\| t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} \right\|_q \|h_0\|_r.$$

Again, with the Hölder inequality

$$\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma \, dt\, ds \le$$
$$\le \int_0^T |\psi(s)|\, s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} \, dt\, \|\phi\|_p \left\| t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} \right\|_q \|h_0\|_r \le$$
$$\le \|\psi\|_2 \left\| s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} \right\|_2 \|\phi\|_p \left\| t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} \right\|_q \|h_0\|_r.$$

Inequality (9) holds true with

$$c(p; \alpha, \beta, \gamma) = \left\| s^{\min\left(\alpha,\, \epsilon-\frac{1}{2}\right)} \right\|_2 \left\| t^{\left(\alpha+\frac{1}{2}-\epsilon\right)^+ +\beta} \right\|_q \|h_0\|_r.$$

Case 3. $-1 < \gamma < 0$, $\frac{1}{p} \ge \gamma + 1$, $\alpha \le 0$ and $\beta \le 0$. Due to assumption (7), $\frac{1}{p} < \frac{3}{2} + \alpha + \beta + \gamma$, whence $\alpha + \beta > -\frac{1}{2}$. Apply Lemma 2 for $a_1 = \frac{1}{2} - \alpha - \beta$,

$a_2 = \frac{1}{p}$, and $a_3 = -\gamma$, and find ϵ_1 and ϵ_3 such that

$$\frac{1}{2} - \alpha - \beta + \epsilon_1 < 1, \quad -\gamma + \epsilon_3 < 1,$$
$$\frac{1}{2} - \alpha - \beta + \epsilon_1 + \frac{1}{p} - \gamma + \epsilon_3 = 2.$$

By the Hölder inequality for nonconjugate exponents

$$\|\psi(s)s^{\alpha+\beta}\|_{1/(0.5+\epsilon_1-\alpha-\beta)} \le \|\psi\|_2 \|s^{\alpha+\beta}\|_{1/(\epsilon_1-\alpha-\beta)}.$$

Since $\beta \le 0$ and $s \le t$ on the integration domain of (9), we have that $t^\beta \le s^\beta$.

Then by the Young inequality

$$\begin{aligned}\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma\, dt\, ds &\le \int_0^T |\psi(s)|\, s^{\alpha+\beta} \int_s^T |\phi(t)|\, (t-s)^\gamma\, dt\, ds \\ &\le \|\psi(s)s^{\alpha+\beta}\|_{1/(0.5+\epsilon_1-\alpha-\beta)} \|\phi\|_p \|u^\gamma\|_{1/(-\gamma+\epsilon_3)} \\ &\le \|\psi\|_2\, \|s^{\alpha+\beta}\|_{1/(\epsilon_1-\alpha-\beta)}\, \|\phi\|_p\, \|u^\gamma\|_{1/(-\gamma+\epsilon_3)},\end{aligned}$$

and (9) holds true with $c(p; \alpha, \beta, \gamma) = \|s^{\alpha+\beta}\|_{1/(\epsilon_1-\alpha-\beta)}\, \|u^\gamma\|_{1/(-\gamma+\epsilon_3)}$.

Case 4. $-1 < \gamma < 0$, $\frac{1}{p} \ge \gamma + 1$, $\alpha \le 0$, $\beta \ge 0$, and $\beta + \gamma < 0$. These imply $\beta < 1$. Hence

$$t^\beta < s^\beta + (t-s)^\beta \quad \text{for all } s \in (0, t) \tag{13}$$

due to concavity of $s^\beta + (t-s)^\beta$ and the fact that for $s = 0$ and $s = t$ (13) becomes an equality. Thus,

$$\begin{aligned}&\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma\, dt\, ds \\ &\le \int_0^T |\psi(s)|\, s^{\alpha+\beta} \int_s^T |\phi(t)|\, (t-s)^\gamma\, dt\, ds + \int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, (t-s)^{\beta+\gamma}\, dt\, ds.\end{aligned}$$

Apply Lemma 2 for $a_1 = \frac{1}{2} + (\alpha+\beta)^-$, $a_2 = \frac{1}{p}$ and $a_3 = -\gamma$. Due to the same reason as in Case 3, $\alpha + \beta > -\frac{1}{2}$; hence, $0 < \frac{1}{2} + (\alpha+\beta)^- < 1$. Obviously, $0 < \frac{1}{p} \le 1$ and $0 < -\gamma < 1$. Due to (7),

$$\frac{1}{p} < \frac{3}{2} + \alpha + \beta + \gamma \quad \text{and} \quad \frac{1}{p} < \frac{3}{2} + \gamma,$$

whence

$$\frac{1}{2} + (\alpha+\beta)^- + \frac{1}{p} - \gamma < 2.$$

By Lemma 2, there exist ϵ_1 and ϵ_3 such that

$$\frac{1}{2} + (\alpha+\beta)^- + \epsilon_1 < 1, \quad -\gamma + \epsilon_3 < 1,$$
$$\frac{1}{2} + (\alpha+\beta)^- + \epsilon_1 + \frac{1}{p} - \gamma + \epsilon_3 = 2.$$

By the Young inequality and the Hölder inequality for nonconjugate exponents,

$$\int_0^T |\psi(s)|\, s^{\alpha+\beta} \int_s^T |\phi(t)|\, (t-s)^{\gamma}\, dt\, ds$$
$$\le \|\psi(s)s^{\alpha+\beta}\|_{1/(0.5+(\alpha+\beta)^-+\epsilon_1)}\|\phi\|_p\|u^{\gamma}\|_{1/(-\gamma+\epsilon_3)}$$
$$\le \|\psi\|_2\, \|s^{\alpha+\beta}\|_{1/((\alpha+\beta)^-+\epsilon_1)}\, \|\phi\|_p\, \|u^{\gamma}\|_{1/(-\gamma+\epsilon_3)}.$$

Apply Lemma 2 again, this time for $a_1 = \frac{1}{2} - \alpha$, $a_2 = \frac{1}{p}$ and $a_3 = -\beta - \gamma$. Here conditions of Lemma 2 are easier to check. By Lemma 2, there exist δ_1 and δ_3 such that

$$\frac{1}{2} - \alpha + \delta_1 < 1, \quad -\beta - \gamma + \delta_3 < 1,$$
$$\frac{1}{2} - \alpha + \delta_1 + \frac{1}{p} - \beta - \gamma + \delta_3 = 2.$$

By the Young inequality and the Hölder inequality for nonconjugate exponents,

$$\int_0^T |\psi(s)|\, s^{\alpha} \int_s^T |\phi(t)|\, (t-s)^{\beta+\gamma}\, dt\, ds$$
$$\le \|\psi(s)s^{\alpha}\|_{1/(0.5-\alpha+\delta_1)}\|\phi\|_p\, \|u^{\beta+\gamma}\|_{1/(-\gamma-\beta+\delta_3)}$$
$$\le \|\psi\|_2\, \|s^{\alpha}\|_{1/(\delta_1-\alpha)}\, \|\phi\|_p\, \|u^{\beta+\gamma}\|_{1/(\delta_3-\beta-\gamma)}.$$

Thus, inequality (9) holds true with

$$c(p;\alpha,\beta,\gamma) = \|s^{\alpha+\beta}\|_{1/((\alpha+\beta)^-+\epsilon_1)}\, \|u^{\gamma}\|_{1/(-\gamma+\epsilon_3)}$$
$$+ \|s^{\alpha}\|_{1/(\delta_1-\alpha)}\, \|u^{\beta+\gamma}\|_{1/(\delta_3-\beta-\gamma)}.$$

Case 5. $-1 < \gamma < 0$, $\frac{1}{p} \ge \gamma + 1$, $\alpha \le 0$, $\beta \ge 0$, and $\beta + \gamma \ge 0$. Similarly to (13),

$$t^{-\gamma} < s^{-\gamma} + (t-s)^{-\gamma} \quad \text{for all } s \in (0,t).$$

Thus,

$$\int_0^T |\psi(s)|\, s^{\alpha} \int_s^T |\phi(t)|\, t^{\beta}(t-s)^{\gamma}\, dt\, ds$$
$$\le \int_0^T |\psi(s)|\, s^{\alpha-\gamma} \int_s^T |\phi(t)|\, t^{\beta+\gamma}(t-s)^{\gamma}\, dt\, ds + \int_0^T |\psi(s)|\, s^{\alpha} \int_s^T |\phi(t)|\, t^{\beta+\gamma}\, dt\, ds$$
$$\le \int_0^T |\psi(s)|\, s^{\alpha-\gamma} \int_s^T |\phi(t)|\, t^{\beta+\gamma}(t-s)^{\gamma}\, dt\, ds + \int_0^T |\psi(s)|\, s^{\alpha}\, ds \int_0^T |\phi(t)|\, t^{\beta+\gamma}\, dt.$$

Apply Lemma 2 for $a_1 = \frac{1}{2} + (\alpha-\gamma)^-$, $a_2 = \frac{1}{p}$ and $a_3 = -\gamma$. From (7) it follows that $\frac{1}{2} - \gamma + \frac{1}{p} < 2$. Since $\alpha > -\frac{1}{2}$ and $p \ge 1$, $\frac{1}{2} - \alpha + \frac{1}{p} < 2$. Hence,

$$\frac{1}{2} + (\alpha-\gamma)^- + \frac{1}{p} - \gamma = \frac{1}{2} - \min(\alpha,\gamma) + \frac{1}{p} < 2.$$

By Lemma 2 there exist $\epsilon_1 > 0$ and $\epsilon_3 > 0$ such that

$$\frac{1}{2} + (\alpha-\gamma)^- + \epsilon_1 < 1, \quad \epsilon_3 - \gamma < 1,$$
$$\frac{1}{2} + (\alpha-\gamma)^- + \epsilon_1 + \frac{1}{p} + \epsilon_3 - \gamma = 2.$$

By the Young inequality and the Hölder inequality for nonconjugate exponents,

$$\begin{aligned}&\int_0^T |\psi(s)|\, s^{\alpha-\gamma} \int_s^T |\phi(t)|\, t^{\beta+\gamma} (t-s)^\gamma \, dt\, ds \\ &\quad \le \|\psi(s)s^{\alpha-\gamma}\|_{1/(0.5+(\alpha-\gamma)^-+\epsilon_1)} \|\phi\|_p T^{\beta+\gamma} \, \|u^\gamma\|_{1/(\epsilon_3-\gamma)} \\ &\quad \le \|\psi\|_2 \|s^{\alpha-\gamma}\|_{1/((\alpha-\gamma)^-+\epsilon_1)} \, \|\phi\|_p T^{\beta+\gamma} \, \|u^\gamma\|_{1/(\epsilon_3-\gamma)}.\end{aligned}$$

By the Hölder inequality

$$\int_0^T |\psi(s)| s^\alpha \, ds \le \|\psi\|_2 \|s^\alpha\|_2, \qquad \int_0^T |\phi(t)| t^{\beta+\gamma} \, dt \le \|\phi\|_p \|t^{\beta+\gamma}\|_{p/(p-1)}.$$

Thus, inequality (9) holds true with

$$c(p;\alpha,\beta,\gamma) = \|s^{\alpha-\gamma}\|_{1/((\alpha-\gamma)^-+\epsilon_1)} \, T^{\beta+\gamma} \, \|u^\gamma\|_{1/(\epsilon_3-\gamma)} + \|s^\alpha\|_2 \, \|t^{\beta+\gamma}\|_{p/(p-1)}.$$

Case 6. $\alpha > 0$. We have already proved Proposition 1 for $\alpha \le 0$. We are going to use it for $\alpha = 0$.

On the integration domain of (9) $s \le t$, and so $s^\alpha \le t^\alpha$. We have

$$\begin{aligned}\int_0^T |\psi(s)|\, s^\alpha \int_s^T |\phi(t)|\, t^\beta (t-s)^\gamma \, dt\, ds &\le \int_0^T |\psi(s)| \int_s^T |\phi(t)|\, t^{\alpha+\beta} (t-s)^\gamma \, dt\, ds \\ &\le c(p;0,\alpha+\beta,\gamma)\, \|\psi\|_2 \, \|\phi\|_p.\end{aligned}$$

The inequality (9) holds true for $c(p;\alpha,\beta,\gamma) = c(p;0,\alpha+\beta,\gamma)$. □

Corollary 1. *Let*

$$\alpha > -\frac{1}{2}, \qquad \gamma > -1, \qquad p \ge 1, \qquad \frac{1}{p} < \frac{3}{2} + \gamma, \qquad \frac{1}{p} + k < \frac{3}{2} + \alpha + \beta + \gamma.$$

Then the operator K^* defined in (8) *is a bounded linear operator $\mathcal{E}_{p,k} \to L^2[0,T]$.*

Proof. Recall that the operator K^* for specific α, β and γ is denoted by $K^*_{(\alpha,\beta,\gamma)}$. From the definition of $K^*_{(\alpha,\beta,\gamma)}$, it follows that

$$(K^*_{(\alpha,\beta,\gamma)}(\phi(t)t^k))(s) = s^\alpha \int_s^T \phi(t)\, t^{k+\beta} \, (t-s)^\gamma \, dt = (K^*_{(\alpha,\beta+k,\gamma)}\phi)(s).$$

Then

$$\|K^*_{(\alpha,\beta,\gamma)}\phi\|_2 = \|K^*_{(\alpha,\beta-k,\gamma)}(\phi(t)t^k)\|_2$$

$$\leq c(p;\alpha,\beta-k,\gamma)\,\|\phi(t)t^k\|_p = c(p;\alpha,\beta-k,\gamma)\,\|\phi\|_{\mathcal{E}_{p,k}} \qquad (14)$$

for any function ϕ such that $K^*_{(\alpha,\beta-k,\gamma)}(\phi(t)t^k)$ is well-defined. Under conditions of Corollary 1, the operator $K^*_{(\alpha,\beta-k,\gamma)} : L^p[0,T] \to L^2[0,T]$ is bounded and $c(p;\alpha,\beta-k,\gamma)$ can be chosen as a finite number. Thus, the boundedness of the operator $K^*_{(\alpha,\beta,\gamma)} : \mathcal{E}_{p,k} \to L^2[0,T]$ follows from (14). □

In the following definition conditions (4) are satisfied, X is a process defined by (3), and

$$1 \leq p < \infty, \qquad pk > -1, \qquad \frac{1}{p} < \frac{3}{2} + \gamma, \qquad \frac{1}{p} + k < \frac{3}{2} + \alpha + \beta + \gamma. \quad (15)$$

Definition 1. The integral of a nonrandom function $\phi \in \mathcal{E}_{p,k}$ with respect to the Gaussian process X is defined as

$$\int_0^T \phi(t)\,dX_t = \int_0^T K^*\phi(s)\,dW_s = \int_0^T s^\alpha \int_s^T \phi(t)t^\beta(t-s)^\gamma\,dt\,dW_s. \qquad (16)$$

Particularly, Definition 1 can be applied for $1 \leq p < \infty$, $\gamma > -1$,

$$\alpha > -\frac{1}{2}, \quad \frac{1}{p} < \frac{3}{2} + \gamma, \quad \frac{1}{p} < \frac{3}{2} + \alpha + \beta + \gamma, \quad \text{and} \quad \phi \in L^p[0,T].$$

Remark 1. Under conditions of Corollary 1, (16) provides linear continuous mapping $\mathcal{E}_{p,k} \to L^2(\Omega,\mathcal{F},\mathsf{P})$, where $L^2(\Omega,\mathcal{F},\mathcal{P})$ is a space of square-integrable random variables. Furthermore, if $pk > -1$, then (16) gives $\int_0^T \mathbf{1}_{[a,b]}\,dX_t = X_b - X_a$, which is very natural. If $1 \leq p < \infty$, then the indicators of intervals span a dense subset in $\mathcal{E}_{p,k}$. Thus, in Definition 1, we construct a linear and continuous extension of the integral from the set of indicators of intervals to the entire space $\mathcal{E}_{p,k}$.

2.2 *Traditional approach*

This integration is compared with the one defined in [4, 14].

The covariance function of the process X can be represented as

$$\mathsf{E}\,X_{t_1}X_{t_2} = \int_0^{t_1}\int_0^{t_2} r(u,v)\,dv\,du,$$

$$r(u,v) = \int_0^{\min(t_1,t_2)} s^\alpha(u-s)^\beta(v-s)^\gamma\,ds;$$

this representation follows from [7, Eq. (8)]. The integrand satisfies relations $r(u,v) > 0$ and $r \in L^1([0,T]^2)$, since $\int_0^T\int_0^T r(u,v)\,du\,dv = \mathsf{E}\,X_T^2 < \infty$.

The Hilbert space $\mathcal{H}$ is defined as the space that contains piecewise-constant (staircase) functions, equipped with scalar product $\langle \mathbf{1}_{(0,t_1]}, \mathbf{1}_{(0,t_2]}\rangle$, and then completed.

Let $\mathcal{H}^0$ be the set of functions $\phi : [0,T] \to \mathbb{R}$, for which

$$\int_0^T\int_0^T |\phi(u)r(u,v)\phi(v)|\,dv\,du < \infty.$$

The space $\mathcal{H}^0$ contains all piecewise-constant functions.

The space $\mathcal{H}^0$ in embedded into $\mathcal{H}$. The element $f' \in \mathcal{H}$ corresponds to $f \in \mathcal{H}^0$ if

$$\langle f', g\rangle_{\mathcal{H}} = \int_0^T \int_0^T f(u) r(u,v) g(v)\, dv\, du$$

for every piecewise-constant function g. Every element of $f \in \mathcal{H}^0$ has its counterpart in $\mathcal{H}$ according to [4, Theorem 1.1].

The integration operator is an isometric linear operator $\mathcal{H} \to L^2(\Omega, \mathcal{F}, \mathsf{P})$ such that $\int_0^T \mathbf{1}_{(0,t]}(s)\, dX_s = X_t$. Here $L^2(\Omega, \mathcal{F}, \mathsf{P})$ is the space of square-integrable random variables.

Proposition 2. *Let conditions* (4) *and* (15) *hold true.*

1. *Then $\mathcal{E}_{p,k} \subset \mathcal{H}^0$, and, as the consequence, $\mathcal{E}_{p,k}$ is embedded into $\mathcal{H}$.*
2. *If $\phi \in \mathcal{E}_{p,k}$, then Definition 1 yields the same integral $\int_0^T \phi(t)\, dX_t$ as the one defined in [4].*

Proof. For kernel K defined in (2) and operator K^* defined in (8), Eq. (5) holds true: $K^* \mathbf{1}_{(0,t]}(s) = K(t,s)$ for all $t, s \in (0, T]$.

For functions $f, g : [0,T] \to [0, +\infty)$

$$\int_0^T K^* f(s)\, K^* g(s)\, ds = \int_0^T \int_0^T f(u) r(u,v) g(v)\, dv\, du;$$

this is proved by changing the order of integration. If $\phi \in \mathcal{E}_{p,k}$, then $|\phi| \in \mathcal{E}_{p,k}$. By Corollary 1, $K^*|\phi| \in L^2[0,T]$, whence

$$\int_0^T \int_0^T |\phi(u) r(u,v) \phi(v)|\, dv\, du = \int_0^T (K^*|\phi|(s))^2\, ds < \infty.$$

Thus, $\phi \in \mathcal{H}^0$.

In Definition 1, $\int_0^T \phi(t)\, dX_t = \int_0^T K^*\phi(s)\, dW_s$. This agrees with [14, equation (24)]. □

Remark 2. Inequality $pk > -1$ is not essential for the first statement of Proposition 2.

Remark 3. In Proposition 2, $\mathcal{E}_{p,k}$ is a proper subset of $\mathcal{H}^0$, that is, $\mathcal{E}_{p,k} \neq \mathcal{H}^0$. Indeed, let (4) and (15) hold true. Then there exists k' such that $k < k' < \alpha + \beta + \gamma + \frac{3}{2} - \frac{1}{p}$. Due to Proposition 2, $\mathcal{E}_{p,k'} \subset \mathcal{H}^0$. Since $k < k'$, $\mathcal{E}_{p,k} \subset \mathcal{E}_{p,k'}$ and $\mathcal{E}_{p,k} \neq \mathcal{E}_{p,k'}$. Hence, $\mathcal{E}_{p,k} \neq \mathcal{H}^0$.

3 Differentiability of Volterra process

Consider the stochastic process

$$\dot{X}_t = t^\beta \int_0^t s^\alpha (t-s)^\gamma\, dW_s. \tag{17}$$

It is well-defined if

$$\alpha > -\frac{1}{2} \quad \text{and} \quad \gamma > -\frac{1}{2},$$

and is self-similar with exponent $\alpha + \beta + \gamma + \frac{1}{2}$. Let $\lceil x \rceil$ stand for the least integer that is greater or equal to x.

Proposition 3. *If $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$ and $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, then the process $\dot{X}$ has a modification which is Hölder continuous up to order $\gamma + \frac{1}{2}$ on any interval $[t_0, T]$, $0 < t_0 < T$.*

If $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$ and $\gamma > \frac{1}{2}$, then the process $\dot{X}$ has a modification which is continuously differentiable on $(0, T]$, and, as a result, is Lipschitz continuous on any interval $[t_0, T]$. Moreover, the process $\dot{X}$ is $\lceil \gamma - \frac{1}{2} \rceil$ times continuously differentiable.

Proof. The process $M_t = \int_0^t s^\gamma \, dW_s$ has a modification that satisfies the Hölder condition up to order $\frac{1}{2}$ on any interval $[t_0, T]$, $t_0 \in (0, T)$.

First, consider the case $\gamma \in \left(-\frac{1}{2}, \frac{1}{2}\right]$. Let $\lambda \in \left(0 \vee \gamma, \gamma + \frac{1}{2}\right)$. The process M satisfies the Hölder condition of order $\lambda - \gamma \in \left(0, \frac{1}{2}\right)$ on the interval $[t_0, T]$. Then $\int_0^t s^\alpha (t-s)^\gamma \, dW_s = \int_0^t (t-s)^\gamma \, dM_s$ satisfies the Hölder condition of order λ due to [10, Lemma 2.1]. Thus, the process $\dot{X}$ also satisfies the Hölder condition of order λ on $[t_0, T]$.

Second, consider the case $\gamma > \frac{1}{2}$. We can choose a positive integer $k \in \mathbb{N}$ and a real number $\lambda_0 \in \left(0, \frac{1}{2}\right)$ such that $0 < \gamma - k + \lambda_0 < 1$. (We can choose $k = \lceil \gamma - \frac{1}{2} \rceil$, and $\lceil \gamma - \frac{1}{2} \rceil \vee 0 < \lambda_0 < \frac{1}{2}$, as $\lceil \gamma - \frac{1}{2} \rceil < \gamma + \frac{1}{2}$.) The process M is Hölder continuous of order λ_0 on any interval $[t_0, T]$. Then the process $\int_0^t s^\alpha (t-s)^{\gamma-k} \, dW_s = \int_0^t (t-s)^{\gamma-k} \, dM_s$ is Hölder continuous of order $\gamma - k + \lambda_0$ on any interval $[t_0, T]$, and thus continuous on $(0, T]$. The process

$$\int_0^t s^\alpha (t-s)^\gamma \, dW_s = \frac{\Gamma(\gamma+1)}{(k-1)!\,\Gamma(\gamma-k+1)} \int_0^t (t-u)^{k-1} \int_0^u s^\alpha (u-s)^{\gamma-k} \, dW_s \, du$$

is the kth antiderivative of the process $\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-k+1)} \int_0^t s^\alpha (t-s)^{\gamma-k} \, dW_s$, and thus is k times continuously differentiable on $(0, T]$. Here and hereafter $\Gamma(x)$ is the gamma function. As the result, the process $\dot{X}$ is k times continuously differentiable on $(0, T]$. It satisfies the Lipschitz condition on any interval $[t_0, T]$. $\square$

The next general result is in fact the part of Lemma 2 from [7], and it will be applied in the proof of Proposition 4.

Lemma 1. *Let the process Y be self-similar with exponent $H > 0$ and be mean-square continuous on $[0, T]$. Additionally, let the process Y satisfy the inequality*

$$\mathsf{E}(Y_T - Y_t)^2 \le C\, T^{2H-2\lambda} (T-t)^{2\lambda}, \qquad t_0 < t < T,$$

for some $C > 0$, $\lambda > 0$, and $0 < t_0 < T$. Then there exists $c > 0$ such that

$$\mathsf{E}(Y_t - Y_s)^2 \le c\,(t-s)^{2(\lambda \wedge H)}, \qquad 0 \le s < t \le T.$$

Now let us return to our process X.

Proposition 4. *If $\alpha > -\frac{1}{2}$ and $\gamma > -\frac{1}{2}$, then the process $\dot{X}$ is mean-square continuous and has continuous modification on $(0, T]$. If, in addition, $\alpha+\beta+\gamma > -\frac{1}{2}$, then the process $\dot{X}$ is mean-square continuous and has continuous modification on $[0, T]$.*

Proof. Path continuity of the process $\dot{X}$ on $(0, T)$ follows from the local Hölder or Lipschitz condition. The process $\dot{X}$ satisfies the Hölder condition up to order $(\gamma + \frac{1}{2}) \vee 1$ on any finite interval $[t_0, T]$, $0 < t_0 < T$. According to [1, Theorem 1], for any λ_0, $0 < \lambda_0 < (\gamma + \frac{1}{2}) \vee 1$ the incremental variance satisfies

$$\mathsf{E}(\dot{X}_t - \dot{X}_s)^2 \le C(t_0, T, \lambda_0)\, |t-s|^{2\lambda_0},$$

whence the mean-square continuity on $(0, T]$ follows. Furthermore,

$$\mathsf{E}\,\dot{X}_t^2 = \mathsf{E}(\dot{X}_t - \dot{X}_0)^2 = t^{2\alpha+2\beta+2\gamma+1}\,\mathrm{B}(2\alpha{+}1,\ 2\gamma{+}1),$$

where $\mathrm{B}(x, y)$ is the beta function. Under additional condition $\alpha+\beta+\gamma > -\frac{1}{2}$ the process $\dot{X}$ is mean-square continuous at point 0. Due to Lemma 1, the process $\dot{X}$ satisfies the inequality

$$\mathsf{E}(\dot{X}_t - \dot{X}_s)^2 \le C_1\,(t-s)^{(2\lambda_0)\wedge(2\alpha+2\beta+2\gamma+1)}, \qquad 0 \le s < t \le T.$$

Hence, the process $\dot{X}$ has a modification that is Hölder continuous up to order $\lambda_0 \wedge \big(\alpha+\beta+\gamma+\frac{1}{2}\big)$ on the interval $[0, T]$ and thus is continuous at point 0. $\square$

The continuity of $\dot{X}$ allows to change the order of integration. For the process X defined in (3),

$$\begin{aligned} X_{t_2} - X_{t_1} &= \int_0^{t_2} s^\alpha \int_{s\vee t_1}^{t_2} u^\beta\,(u-s)^\gamma\,du\,dW_s \\ &= \int_{t_1}^{t_2} u^\beta \int_0^u s^\alpha\,(u-s)^\gamma\,dW_s\,du = \int_{t_1}^{t_2} \dot{X}_u\,du. \end{aligned}$$

Corollary 2. *If $\alpha > -\frac{1}{2}$, $\gamma > -\frac{1}{2}$ and $\alpha+\beta+\gamma > -\frac{3}{2}$, then some modifications of the processes X and $\dot{X}$ defined in (3) and (17) satisfy the relation*

$$X_t = X_{t_0} + \int_{t_0}^t \dot{X}_s\,ds.$$

The process X_t is $\lceil \gamma + \frac{1}{2} \rceil$ times continuously differentiable on $(0, T]$, and $dX_t/dt = \dot{X}_t$ for $0 < t \le T$.

If, in addition, $\alpha+\beta+\gamma > -\frac{1}{2}$, then the processes X and $\dot{X}$ satisfy the relation

$$X_t = \int_0^t \dot{X}_s\,ds. \tag{18}$$

Then the process X_t is continuously differentiable on $[0,\ T]$.

Representation (18) is also valid for $\alpha+\beta+\gamma \in \big(-\frac{3}{2}, -\frac{1}{2}\big]$. However, in this case, the integrand $\dot{X}$ may be unbounded in the neighborhood of 0. Thus, the integral here should be understood in improper sense:

$$X_t = \lim_{t_0\to 0+} \int_{t_0}^t \dot{X}_s\,ds.$$

4 Inverse representation

In Volterra representation (1) (or, specifically, (3)) the process X is represented as the integral with respect to the Wiener process W. We construct the representation of the Wiener process W in (3) using the process X. However, the form of the representation depends on whether $\gamma \in (-1, 0)$ or $\gamma \geq 0$, and on whether γ is integer or not.

4.1 Case $\gamma < 0$

Reduction to the integral equation

We are going to find the inverse representation to (1):

$$W_v = \int_0^v L(v, t)\, dX_t\, . \tag{19}$$

Assume that integration with respect to X is performed according to (6). Then the left-hand side and the right-hand side of (19) admit the representations

$$W_v = \int_0^v dW_v,$$
$$\int_0^v L(v, t)\, dX_t = \int_0^v \int_s^v L(v, t)\, \frac{\partial K(t, s)}{\partial t}\, dt\, dW_s.$$

Thus, the sufficient and necessary condition for (19) is

$$\int_s^v L(v, t) \frac{\partial K(t, s)}{\partial t}\, dt = 1 \tag{20}$$

for all $v \in (0, T]$ and for almost all $s \in (0, v)$. For the kernel defined by (2), the integral equation (20) turns into

$$\int_s^v L(v, t)\, s^{\alpha}\, t^{\beta}\, (t - s)^{\gamma}\, dt = 1. \tag{21}$$

The explicit solution

The solution to equation

$$\int_s^v L(v, t) a(s) b(t) c(t - s)\, dt = 1$$

is found in Mishura *et al.* [8, Proposition 3]. In that solution, they use the "Sonine pair" with the function $c(s)$, which is a function $p(s)$ that satisfies the integral equation

$$\int_0^t p(s)\, c(t - s)\, ds = 1, \qquad t \in (0, T].$$

The "Sonine pair" with the power function s^{γ} exists for all $\gamma \in (-1, 0)$; it is equal to $s^{-\gamma-1} / \mathrm{B}(-\gamma, \gamma+1)$.

Now we construct a solution to equation (21) within a class of functions $L(v, t)$, $0 < t < v < T$, that satisfy the following absolute integrability condition

$$\int_s^v |L(v, t)|\, dt < \infty, \qquad 0 < s < v \leq T. \tag{22}$$

Theorem 1. *Let $\alpha > -\frac{1}{2}$, $\beta \in \mathbb{R}$, and $-1 < \gamma < 0$. Then the solution to the integral equation* (21) *for all s and v such that $0 < s < v \le T$ is*

$$\begin{aligned} L(v,t) &= -\frac{t^{-\beta}}{\mathrm{B}(\gamma+1,-\gamma)} \frac{\partial}{\partial t}\left(\int_t^v s^{-\alpha}(s-t)^{-\gamma-1}\,ds\right) \\ &= \frac{1}{\mathrm{B}(-\gamma,\gamma+1)\,t^{\beta}} \left(v^{-\alpha}(v-t)^{-\gamma-1} + \alpha \int_t^v u^{-\alpha-1}(u-t)^{-\gamma-1}\,du\right). \end{aligned} \tag{23}$$

The solution is unique up to equality for all $v \in (0,T]$ and for almost all $t \in (0,v)$.

Proof. *Necessity (if the function $L(v,t)$ is a solution to* (21) *under conditions* (22)*, then it must be defined by* (23) *for almost all $t \in (0,v)$).* Let $L(t,s)$ satisfy both (21) and (22). Let us find the expression for $L(t,s)$.

Let $0 < r < v \le T$. Calculate the double integral

$$\iint\limits_{r<s<t<v} L(v,t)t^{\beta}\,(t-s)^{\gamma}(s-r)^{-\gamma-1}\,dt\,ds. \tag{24}$$

First, proof the existence.

$$\begin{aligned} &\iint\limits_{r<s<t<v} |L(v,t)t^{\beta}\,(t-s)^{\gamma}(s-r)^{-\gamma-1}|\,dt\,ds = \\ &\qquad = \int_r^v t^{\beta}\,|L(v,t)| \int_r^t (t-s)^{\gamma}(s-r)^{-\gamma-1}\,ds\,dt = \\ &\qquad = \mathrm{B}(\gamma+1,-\gamma) \int_r^v t^{\beta}\,|L(v,t)|\,dt \le \\ &\qquad \le \mathrm{B}(\gamma+1,-\gamma)\ \max(r^{\beta},\ v^{\beta}) \int_r^v |L(v,t)|\,dt < \infty. \end{aligned}$$

Thus, the integral in (24) exists absolutely. Calculate it in two ways. On the one hand,

$$\begin{aligned} &\iint\limits_{r<s<t<v} L(v,t)t^{\beta}\,(t-s)^{\gamma}(s-r)^{-\gamma-1}\,dt\,ds = \\ &\qquad = \int_r^v t^{\beta}\,L(v,t) \int_r^t (t-s)^{\gamma}(s-r)^{-\gamma-1}\,ds\,dt = \\ &\qquad = \mathrm{B}(\gamma+1,-\gamma) \int_r^v t^{\beta}\,L(v,t)\,dt. \end{aligned}$$

On the other hand, with (21),

$$\begin{aligned} &\iint\limits_{r<s<t<v} L(v,t)t^{\beta}\,(t-s)^{\gamma}(s-r)^{-\gamma-1}\,dt\,ds = \\ &\qquad = \int_r^v s^{-\alpha} \int_s^v L(v,t)\,s^{\alpha}t^{\beta}(t-s)^{\gamma}\,dt\,(s-r)^{-\gamma-1}\,ds = \end{aligned}$$

$$= \int_r^v s^{-\alpha} (s-r)^{-\gamma-1}\, ds.$$

Thus, $L(v,t)$ satisfies the equation

$$\mathrm{B}(\gamma+1,-\gamma) \int_r^v t^\beta\, L(v,t)\, dt = \int_r^v s^{-\alpha} (s-r)^{-\gamma-1}\, ds.$$

Hence $L(v,t)$ can be expressed explicitly:

$$L(v,t) = -\frac{t^{-\beta}}{\mathrm{B}(\gamma+1,-\gamma)} \frac{\partial}{\partial t}\left(\int_t^v s^{-\alpha} (s-t)^{-\gamma-1}\, ds \right)$$

for almost all $t \in (0,v)$.

Sufficiency (the function $L(v,t)$ defined by (23) *satisfies* (21) *and* (22)*).* First, consider the generic case $\alpha \ne 0$, that is, either $\alpha \in \left(-\frac{1}{2}, 0\right)$ or $\alpha > 0$. Let $0 < s < v \le T$. Then

$$\int_s^v \frac{v^{-\alpha}}{(v-t)^{\gamma+1}}\, (t-s)^\gamma\, dt = \frac{B(-\gamma,\gamma+1)}{v^\alpha},$$

$$\begin{aligned}\int_s^v \int_t^v \frac{u^{-\alpha-1}\, du}{(u-t)^{\gamma+1}}\, (t-s)^\gamma\, dt &= \int_s^v u^{-\alpha-1} \int_s^u \frac{(t-s)^\gamma\, dt}{(u-t)^{\gamma+1}}\, du = \\ &= \int_s^v u^{-\alpha-1} \mathrm{B}(\gamma+1,-\gamma)\, du = \\ &= \frac{1}{\alpha}\, B(-\gamma,\gamma+1)\, (s^{-\alpha} - v^{-\alpha}).\end{aligned}$$

Hence

$$\begin{aligned}&\int_s^v L(v,t)\, s^\alpha t^\beta (t-s)^\gamma\, dt = \\ &\quad = \frac{s^\alpha}{B(-\gamma,\gamma+1)} \int_s^v \left(\frac{v^{-\alpha}}{(v-t)^{\gamma+1}} + \alpha \int_t^v \frac{u^{-\alpha-1}\, du}{(u-t)^{\gamma+1}} \right) (t-s)^\gamma\, dt = \\ &\quad = \frac{s^\alpha}{B(-\gamma,\gamma+1)} \left(\frac{B(-\gamma,\gamma+1)}{v^\alpha} + B(-\gamma,\gamma+1)\, (s^{-\alpha} - v^{-\alpha}) \right) = 1.\end{aligned}$$

For $\alpha \ne 0$, (21) is proved.

Now, consider the case $\alpha = 0$. Then

$$L(v,t) = \frac{1}{B(-\gamma,\gamma+1)\, t^\beta (v-t)^{\gamma+1}}.$$

Then, for all s and v such that $0 < s < v \le T$

$$\int_s^v L(v,t)\, t^\beta (t-s)^\gamma\, dt = \frac{1}{B(-\gamma,\gamma+1)} \int_s^v \frac{(t-s)^\gamma\, dt}{(v-t)^{\gamma+1}} = 1.$$

It remains to verify that the function $L(v,t)$ satisfies condition (22). The factor $t^{-\beta}/\mathrm{B}(-\gamma,\gamma+1)$ is bounded for $t \in (s,v)$. As to the other factor,

$$\int_s^v \left| v^{-\alpha} (v-t)^{-\gamma-1} + \alpha \int_t^v u^{-\alpha-1} (u-t)^{-\gamma-1}\, du \right| dt$$

$$\le \int_s^v v^{-\alpha}(v-t)^{-\gamma-1}\,dt + |\alpha| \int_s^v u^{-\alpha-1} \int_s^u (u-t)^{-\gamma-1}\,dt\,du$$
$$= \frac{v^{-\alpha}(v-s)^{-\gamma}}{-\gamma} + \frac{|\alpha|}{-\gamma} \int_s^v u^{-\alpha-1}(u-s)^{-\gamma}\,du < \infty.$$

The product of bounded continuous function and integrable function is integrable. Thus, $L(v,t)$ satisfies (22). The theorem is proved. □

Theorem 2. *Let $\alpha > -\frac{1}{2}$, $-1 < \gamma < 0$ and $\alpha + \beta + \gamma > -\frac{3}{2}$. Then*

1. *The process X that is well-defined by (3), according to [7, Theorem 1], admits the inverse representation of the form*

$$W_v = \int_0^v L(v,t)\,dX_t$$
$$= \frac{1}{\mathrm{B}(-\gamma,\gamma+1)} \int_0^v t^{-\beta}\left(v^{-\alpha}(v-t)^{-\gamma-1} + \alpha \int_t^v u^{-\alpha-1}(u-t)^{-\gamma-1}\,du\right) dX_t. \tag{25}$$

2. *The integration in (25) can be formally performed according to (6). There exist p and k, more precisely, p and k that satisfy (27), for which $L(v,\,\cdot\,) \in \mathcal{E}_{p,k}$, and integration in (25) can be performed according to Definition 1.*

Proof. For fixed $v > 0$, the kernel function $L(v,t)$ defined in Theorem 1 is continuous in t in the interval $(0,v)$. With [7, Lemma 4], the asymptotics of $L(v,t)$ as $t \to 0$ can be established:

$$L(v,t) = O(t^{-\beta} v^{-\alpha-\gamma-1}) \quad \text{if } \alpha + \gamma < -1,$$
$$L(v,t) \sim \frac{\alpha}{\mathrm{B}(-\gamma,\gamma+1)}\, t^{-\beta} \log(v/t) \quad \text{if } \alpha + \gamma = -1,$$
$$L(v,t) \sim \frac{\alpha\, \mathrm{B}(-\gamma,\alpha+\gamma+1)}{\mathrm{B}(-\gamma,\gamma+1)}\, t^{-\alpha-\beta-\gamma-1} \quad \text{if } \alpha + \gamma > -1.$$

The asymptotics in the other endpoint is

$$L(v,t) \sim \frac{v^{-\beta-\alpha}(v-t)^{-\gamma-1}}{\mathrm{B}(-\gamma,\gamma+1)} \quad \text{as} \quad t \to v-.$$

In order for $L(t,v)$ to be defined for all $t \in (0,T]$ and thus the relation "$L(v,\cdot) \in \mathcal{E}_{p,k}$" make sense, assume $L(v,t) = 0$ for $0 < v \le t$. The relation $L(v,\cdot) \in \mathcal{E}_{p,k}$ holds true if and only if

$$\max(\beta,\ \alpha+\beta+\gamma+1) < \frac{1}{p} + k \quad \text{and} \quad \gamma + 1 < \frac{1}{p}. \tag{26}$$

The conditions of Theorem 2 imply that

$$\max(0,\ \gamma+1) < \min\left(1,\ \gamma + \tfrac{3}{2}\right),$$

$$\max(0,\ \beta,\ \alpha+\beta+\gamma+1) < \alpha+\beta+\gamma+\frac{3}{2}.$$

Therefore, there exist p and k such that

$$\begin{gathered}\max(0,\ \gamma+1) < \frac{1}{p} < \min\left(1,\ \gamma+\tfrac{3}{2}\right), \\ \max(0,\ \beta,\ \alpha+\beta+\gamma+1) < \frac{1}{p}+k < \alpha+\beta+\gamma+\frac{3}{2}.\end{gathered} \tag{27}$$

For these p and k, inequalities (15) and (26) hold true. Then $L(v,\cdot) \in \mathcal{E}_{p,k}$ for all $v \in (0,T]$, and the second part of Theorem 2 is proved.

All functions within $\mathcal{E}_{p,k}$ are integrable with respect to X_t according to Definition 1. Then, with (21),

$$\int_0^v L(v,t)\,dX_t = \int_0^v s^\alpha \int_s^v L(v,t)t^\beta (t-s)^\gamma\,dt\,dW_s = \int_0^v dW_s = W_v.$$

Eq. (25) is proved. □

4.2 Case $\gamma \geq 0$

Theorem 3. *Let $\alpha > -\frac{1}{2}$, $\gamma \geq 0$ and $\alpha+\beta+\gamma > -\frac{3}{2}$. The process X admits a modification which is differentiable on $(0,T]$, and for which, for $\dot{X}_t = dX_t/dt$, the following expression for the inverse representation holds true:*

$$W_v = \frac{1}{\Gamma(\gamma+1)} \int_0^v t^{-\alpha}\, d\big(\mathcal{D}_{0+}^\gamma (t^{-\beta}\,\dot{X}_t)\big),$$

that is, the process W can be obtained from X with four steps:

$$\dot{X}_t = \frac{dX_t}{dt}, \quad Y_t = t^{-\beta}\dot{X}_t, \quad M_t = \frac{1}{\Gamma(\gamma+1)}\mathcal{D}_{0+}^\gamma Y_t, \quad W_v = \int_0^v t^{-\alpha}\,dM_t.$$

Here $\mathcal{D}_{0+}^\gamma$ is the identity operator if $\gamma = 0$. Otherwise, if $\gamma > 0$, then $\mathcal{D}_{0+}^\gamma$ is the fractional differentiation operator defined in (29) or (30).

Proof. Due to Corollary 2, the process X has a modification that is continuously differentiable on $(0,T]$, for which $\dot{X}_t = dX_t/dt$ with the process $\dot{X}$ defined in (17). Denote

$$Y_t = t^{-\beta}\dot{X}_t = \int_0^t s^\alpha (t-s)^\gamma\,dW_s, \qquad M_t = \int_0^t s^\alpha\,dW_s.$$

Now assume that $\gamma > 0$ and use notation from Appendix B. The process Y is continuous on $[0,T]$ by Proposition 4. With changing the order of integration,

$$\begin{aligned}\mathcal{I}_{0+}^\gamma M_t &= \frac{1}{\Gamma(\gamma)}\int_0^t u^{\gamma-1}\int_0^u s^\alpha\,dW_s\,du \\ &= \frac{1}{\Gamma(\gamma)}\int_0^t s^\alpha \int_s^t u^{\gamma-1}\,du\,dW_s = \frac{1}{\Gamma(\gamma+1)}\int_0^t s^\alpha (t-s)^\gamma\,dW_s = \frac{1}{\Gamma(\gamma+1)}Y_t.\end{aligned}$$

Hence, according to the definition of the fractional differentiation in Appendix B,

$$M_t = \frac{1}{\Gamma(\gamma+1)} \mathcal{D}^{\gamma}_{0+} Y_t. \tag{28}$$

Notice that if $\gamma = 0$, then $M_t = Y_t$ and (28) holds true as well, provided that $\mathcal{D}^0_{0+}$ is the identity operator.

Finally, $dM_t = t^{\alpha} dW_t$, whence

$$W_v = \int_0^v t^{-\alpha}\, dM_t.$$

Here we have the integral of a nonstochastic function w.r.t. a martingale. □

Now consider specific cases. Let the conditions of Theorem 3 hold true. If $\gamma = 0$, then

$$W_v = \frac{1}{\Gamma(\gamma+1)} \int_0^v t^{-\alpha}\, d\big(t^{-\beta} \dot{X}_t\big).$$

If γ is a positive integer, then

$$W_v = \frac{1}{\gamma!} \int_0^v t^{-\alpha}\, d\bigg(\frac{d^{\gamma}}{dt^{\gamma}}\big(t^{-\beta} \dot{X}_t\big)\bigg),$$

where $\gamma! = 1 \cdot 2 \cdots \gamma$ is the factorial. Here formula (29) is used.

Consider the general case. Let $\gamma \ge 0$, and let $n > \gamma$ be a positive integer. Then, according to (30),

$$W_v = \frac{1}{\Gamma(\gamma+1)\Gamma(n-\gamma)} \int_0^v t^{-\alpha}\, d\bigg(\frac{d^n}{dt^n}\bigg(\int_0^t (t-s)^{n-\gamma-1} s^{-\beta} \dot{X}_s\, ds\bigg)\bigg).$$

In particular, one can take $n = \lfloor\gamma\rfloor + 1$, where $\lfloor\gamma\rfloor$ is the unique integer of the interval $(\gamma-1,\ \gamma]$. Any $\gamma \ge 0$ can be represented as $\gamma = \lfloor\gamma\rfloor + \{\gamma\}$, with $\lfloor\gamma\rfloor$ nonnegative integer and $0 \le \{\gamma\} < 1$. Then

$$W_v = \frac{1}{\Gamma(\gamma+1)\Gamma(1-\{\gamma\})} \int_0^v t^{-\alpha}\, d\bigg(\frac{d^{\lfloor\gamma\rfloor+1}}{dt^{\lfloor\gamma\rfloor+1}}\bigg(\int_0^t (t-s)^{-\{\gamma\}} s^{-\beta} \dot{X}_s\, ds\bigg)\bigg).$$

A Hölder inequality and Young inequality

In this section, auxiliary results used in the proof of Proposition 1 are presented.

Lemma 2. *Let $a_1 \in [0,1)$, $a_2 \in (0,1]$ and $a_3 \in [0,1)$ be real numbers such that $a_1 + a_2 + a_3 < 2$. Then there exist $\epsilon_1 \in (0,1)$ and $\epsilon_3 \in (0,1)$, such that $a_1 + \epsilon_1 \in (0,1)$, $a_3 + \epsilon_3 \in (0,1)$, and $(a_1+\epsilon_1) + a_2 + (a_3+\epsilon_3) = 2$.*

Proof. We can take

$$\epsilon_i = \frac{(1-a_i)\,(2-a_1-a_2-a_3)}{2-a_1-a_3}, \qquad i = 1,\ 3.$$

Then obviously $\epsilon_i > 0$, $1 - a_i - \epsilon_i > 0$, and $\epsilon_1 + \epsilon_3 = 2 - a_1 - a_2 - a_3$. □

Hölder inequality for nonconjugate exponents. Here we present the Hölder inequality and the Young inequality, both in the nonstandard form, however, both of them follow from the respective inequalities in their classical form.

Let $p \in (0, +\infty]$, $q \in (0, +\infty]$, and let f and g be measurable functions on $[0, T]$. Then

$$\|f(t)g(t)\|_{1/(1/p+1/q)} \le \|f\|_p \|g\|_q.$$

This inequality holds true regardless whether or not $f \in L^p[0, T]$ and $g \in L^q[0, T]$. (If $f \notin L^p[0, T]$, we assume $\|f\|_p = \infty$.) It also holds true regardless whether the exponents p, q and $1/(1/p + 1/q)$ are less, equal or greater than 1. (If $p \in [1, +\infty]$, then $\|\cdot\|_p$ is the norm in $L^p[0, T]$. Otherwise, if $p \in (0, 1)$, then the space $L^p[0, T]$ is not normalizable.)

Hölder inequality for three functions. Let $p, q, r \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, $f \in L^p[0, T]$, $g \in L^q[0, T]$, and $h \in L^r[0, T]$. Then

$$\left|\int_0^T f(t)g(t)h(t)\,dt\right| \le \int_0^T |f(t)g(t)h(t)|\,dt \le \|f\|_p \|g\|_q \|h\|_r.$$

Young inequality for three functions. Let $p, q, r \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, $f \in L^p[0, T]$, $g \in L^q[0, T]$, and $h \in L^r[0, T]$. Then

$$\left|\iint\limits_{0<s<t<T} f(s)g(t)h(t-s)\,ds\,dt\right| \le \iint\limits_{0<s<t<T} |f(s)g(t)h(t-s)|\,ds\,dt \le \|f\|_p \|g\|_q \|h\|_r.$$

B Fractional calculus

Solving the inverse representation problem, in Section 4.2 we need to solve the Abel equation $\Gamma(\alpha) f(t) = \int_0^t s^{\alpha-1} g(s)\,ds$. In this connection, we apply basics of fractional calculus. For details, see [13, §2].

Let f be a continuous function on $[0, T]$, with the standard notation $f \in \mathcal{C}[0, T]$. For $\alpha > 0$, the αth order fractional integral of f is defined as

$$\mathcal{I}^\alpha_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds.$$

The fractional integral has the following semigroup property:

$$\mathcal{I}^\alpha_{0+} \mathcal{I}^\beta_{0+} = \mathcal{I}^{\alpha+\beta}_{0+},$$

whence $\mathcal{I}^n_{0+} = (\mathcal{I}^1_{0+})^n$ for $n \in \mathbb{N}$.

Now we find the inverse operator of $\mathcal{I}^\alpha_{0+}$. We find an operator $\mathcal{D}^\alpha_{0+} : \mathcal{I}^\alpha_{0+}(\mathcal{C}[0, T]) \to \mathcal{C}[0, T]$ such that $\mathcal{D}^\alpha_{0+} \mathcal{I}^\alpha_{0+} f = f$ for all $f \in \mathcal{C}[0, T]$.

As $\mathcal{I}^1_{0+} f(t) = \int_0^t f(s)\,ds$, the inverse operator of $\mathcal{I}^1_{0+}$ is a differentiation operator: $\mathcal{D}^1_{0+} g(t) = \frac{d}{dt} g(t)$. As $\mathcal{I}^n_{0+} = (\mathcal{I}^1_{0+})^n$, the inverse operator of $\mathcal{I}^n_{0+}$ is the iterated differentiation of nth order. Thus,

$$\text{if } g = \mathcal{I}^n_{0+} f, \quad n \in \mathbb{N}, \quad f \in \mathcal{C}[0, T], \quad \text{then} \quad f(t) = \mathcal{D}^n_{0+} g(t) = \frac{d^n}{dt^n} g(t). \tag{29}$$

Now let $\alpha > 0$ be a real number, and let $n \in \mathbb{N}$, $n > \alpha$. As $\mathcal{D}^n_{0+}\mathcal{I}^{n-\alpha}_{0+}\mathcal{I}^{\alpha}_{0+} = \mathcal{D}^n_{0+}\mathcal{I}^n_{0+}$ is an identity operator on $\mathcal{C}[0,T]$, $\mathcal{D}^n_{0+}\mathcal{I}^{n-\alpha}_{0+}$ is an inverse operator of $\mathcal{I}^{\alpha}_{0+}$. Thus,

$$\text{if } g = \mathcal{I}^{\alpha}_{0+} f, \quad 0 < \alpha < n, \quad n \in \mathbb{N}, \quad f \in \mathcal{C}[0,T], \quad \text{then}$$
$$f(t) = \mathcal{D}^{\alpha}_{0+} g(t) = \mathcal{D}^n_{0+}\mathcal{I}^{n-\alpha}_{0+} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n}\left(\int_0^t (t-s)^{n-\alpha-1} g(s)\, ds\right). \tag{30}$$

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