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Metatimes, random measures and cylindrical random variables

Fred Espen Benth*, **Iben Cathrine Simonsen**

University of Oslo, Department of Mathematics, P. O. Box 1053 Blindern, N-0316 Oslo, Norway

fredb@math.uio.no (F. E. Benth), ibens@math.uio.no (I. C. Simonsen)

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Abstract Metatimes constitute an extension of time-change to general measurable spaces, defined as mappings between two σ-algebras. Equipping the image σ-algebra of a metatime with a measure and defining the composition measure given by the metatime on the domain σ-algebra, we identify metatimes with bounded linear operators between spaces of square integrable functions. We also analyse the possibility to define a metatime from a given bounded linear operator between Hilbert spaces, which we show is possible for invertible operators. Next we establish a link between orthogonal random measures and cylindrical random variables following a classical construction. This enables us to view metatime-changed orthogonal random measures as cylindrical random variables composed with linear operators, where the linear operators are induced by metatimes. In the paper we also provide several results on the basic properties of metatimes as well as some applications towards trawl processes.

Keywords Metatime, cylindrical random variable, random measure, linear operator, trawl process
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1 Introduction

In this paper we study the connection between metatimes and linear operators on Hilbert spaces on one hand, and orthogonal random measures and cylindrical random

*Corresponding author.

variables on the other hand. Given an orthogonal random measure (or Lévy basis) L on some measurable space $(M, \mathcal{M})$, [1] and [4] introduced a concept of subordination of L on $(M, \mathcal{M})$, generalizing the classical method of time-changing a Lévy process (see [6]). By looking at a class of mappings $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ called *metatimes*, being summable and preserving disjointness of sets, $L \circ \mathcal{T}$ was proposed as a new class of random measures. We show that for an orthogonal random measure L and a metatime $\mathcal{T}$, $L \circ \mathcal{T}$ can be viewed as a combination of a cylindrical random variable $\mathbb{L}$ on a Hilbert space and a linear operator $\widehat{\mathcal{T}}$ on that space. Hence, a metatime-changed orthogonal random measure has an analogue to an operator-changed cylindrical random variable.

In stochastic modelling, time-changing a Brownian motion or Lévy process provides an alternative to amplitude scaling by a volatility, the latter being a stochastic integral with respect to the driving noise. Metatimes extend this flexible modeling device to Lévy bases (see [3]) as an alternative to spatio-temporal stochastic volatility modulation defined, say, by ambit fields (see [3]). Trawl processes, as introduced by [2] and further studied and applied in [5], rest on a particular composition of a Lévy basis with a familiy of metatimes.

In this paper, we first analyse some of the basic properties of metatimes. Next, we establish a link between metatimes and linear operators on some canonically defined Hilbert spaces. Indeed, equipping the image σ-algebra of a metatime with a measure and defining the composition measure given by the metatime on the domain σ-algebra, we identify metatimes with bounded linear operators between spaces of square integrable functions. We also demonstrate that invertible linear operators on general Hilbert spaces can define metatimes, yielding an identification in the opposite direction.

As a second step, we study orthogonal random measures. These measures are closely related to cylindrical random variables, and we show that we can lift the orthogonal random measures to cylindrical variables where orthogonality is preserved. This coincides with more classical studies by [13], and enables us to view metatime-changed orthogonal random measures as operator-changed cylindrical random variables. Moreover, our analysis shows that $L \circ \mathcal{T}$ is an orthogonal random measure which has a lifting to an orthogonal cylindrical variable $\mathbb{L} \circ \widehat{\mathcal{T}}$, where $\mathbb{L}$ is the cylindrical variable induced from L and $\widehat{\mathcal{T}}$ is the linear operator induced from $\mathcal{T}$.

As an application of our results, we extend the class of trawl processes by constructing real-valued trawl processes from cylindrical random variables and curves in a Hilbert space. Next, we define cylindrical trawl processes by looking at operator-changes being time dependent. Some basic properties are derived, in particular for the case of semigroups and Hilbert–Schmidt-valued operators.

2 Metatimes and σ-metatimes

Let $(M_i, \mathcal{M}_i)$, $i = 1, 2$, be two measurable spaces. We define *metatimes* following the definition in [4].

Definition 1. A mapping $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ is called a *metatime* if

(i) For $A, B \in \mathcal{M}_1$ with $A \cap B = \emptyset$, $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$.

(ii) For disjoint sets $A_1, A_2, \ldots \in \mathcal{M}_1$, $\mathcal{T}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathcal{T}(A_i)$.

The next lemma contains some basic properties of metatimes.

Lemma 1. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a metatime. Then the following holds.*

(i) $\mathcal{T}(\emptyset) = \emptyset$.

(ii) For $A, B \in \mathcal{M}_1$ with $A \subset B$, $\mathcal{T}(A) \subset \mathcal{T}(B)$.

Proof.

(i) This follows from (i) in Definition 1. Trivally, $\emptyset \in \mathcal{M}_1$ and $\emptyset \cap \emptyset = \emptyset$. Then $\mathcal{T}(\emptyset) \cap \mathcal{T}(\emptyset) = \emptyset$, which implies that $\mathcal{T}(\emptyset) = \emptyset$.

(ii) This follows from (ii) in Definition 1. If $A, B \in \mathcal{M}_1$ and $A \subset B$, we have the disjoint representation of B as $B = A \cup (B \setminus A)$. Then

$$\mathcal{T}(A) \subset \mathcal{T}(A) \cup \mathcal{T}(B \setminus A) = \mathcal{T}(B).$$

$\square$

The image $\mathcal{T}(\mathcal{M}_1) \subseteq \mathcal{M}_2$ of $\mathcal{M}_1$ under a metatime $\mathcal{T}$ is not necessarily a σ-algebra. By replacing the first condition in the metatime definition with a stricter condition, we get mappings which we show to be preserving the σ-algebra structure (see Lemma 3). We call these mappings *σ-metatimes*.

Definition 2. A mapping $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ is called a *σ-metatime* if

(i) For all $A \in \mathcal{M}_1$, $\mathcal{T}(A)^c = \mathcal{T}(A^c)$.

(ii) For disjoint sets $A_1, A_2, \ldots \in \mathcal{M}_1$, $\mathcal{T}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathcal{T}(A_i)$.

The next lemma states some properties of σ-metatimes.

Lemma 2. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a σ-metatime. Then the following holds.*

(i) $\mathcal{T}(M_1) = M_2$.

(ii) $\mathcal{T}(\emptyset) = \emptyset$.

(iii) For $A, B \in \mathcal{M}_1$ with $A \cap B = \emptyset$, $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$.

Proof.

(i) From (i) in Definition 2 and De Morgan's laws it follows that

$$\begin{aligned} M_2 = \emptyset^c &= (\mathcal{T}(\emptyset) \cap \mathcal{T}(\emptyset)^c)^c = \mathcal{T}(\emptyset)^c \cup (T(\emptyset)^c)^c = \mathcal{T}(\emptyset^c) \cup \mathcal{T}(\emptyset) \\ &= \mathcal{T}(\emptyset^c \cup \emptyset) = \mathcal{T}(M_1). \end{aligned}$$

(ii) From (i) Definition 2 it follows that

$$\emptyset = M_2^c = \mathcal{T}(M_1)^c = \mathcal{T}(M_1^c) = \mathcal{T}(\emptyset).$$

(iii) Let $A, B \in \mathcal{M}_1$ be disjoint. By (i) in Definition 2, (i) and (ii) above and De Morgan's laws,

$$\begin{aligned}(\mathcal{T}(A) \cap \mathcal{T}(B))^c &= \mathcal{T}(A)^c \cup \mathcal{T}(B)^c = \mathcal{T}(A^c) \cup \mathcal{T}(B^c) = \mathcal{T}(A^c \cup B^c) \\ &= \mathcal{T}((A \cap B)^c) = \mathcal{T}(A \cap B)^c = \mathcal{T}(\emptyset)^c = \emptyset^c = M_2.\end{aligned}$$

Then $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$, and property (i) in Definition 1 follows.

□

We show that we have equivalence between σ-metatimes and metatimes having the property $\mathcal{T}(M_1) = M_2$.

Proposition 1. *A σ-metatime is equivalent to a metatime with the property $\mathcal{T}(M_1) = M_2$.*

Proof. If $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ is a σ-metatime, it follows from Lemma 2 that $\mathcal{T}$ is a metatime with the property $\mathcal{T}(M_1) = M_2$.

Let $\mathcal{T}$ be a metatime, and assume that $\mathcal{T}(M_1) = M_2$. To show that $\mathcal{T}$ is a σ-metatime, we must show that for all $A \in \mathcal{M}_1$, $\mathcal{T}(A)^c = \mathcal{T}(A^c)$. Pick any $A \in \mathcal{M}_1$. From (i) in Definition 1, $\mathcal{T}(A)$ and $\mathcal{T}(A^c)$ are disjoint. Hence

$$M_2 = \mathcal{T}(M_1) = \mathcal{T}(A \cup A^c) = \mathcal{T}(A) \cup \mathcal{T}(A^c),$$

where we used the assumption together with (ii) in Definition 1. By the definition of the complement of $\mathcal{T}(A)$, we must then have $\mathcal{T}(A^c) = \mathcal{T}(A)^c$, and hence $\mathcal{T}$ is a σ-metatime. □

In view of this result one may redefine any metatime $\mathcal{T}$ into a σ-metatime by considering the image space M_2 to be $M_2 := \mathcal{T}(M_1)$. Additionally, the $\mathcal{M}_2$ is changed to the smallest σ-algebra containing $\mathcal{M}_2 \cap \mathcal{T}(M_2)$. We keep the distinction between metatimes and σ-metatimes in our exposition.

The next lemma gives an equivalent characterization of injective metatimes and σ-metatimes.

Lemma 3. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a metatime. Then $\mathcal{T}$ is injective if and only if the only element that maps to the empty set is the empty set itself.*

Proof. Assume that $\mathcal{T}$ is injective. Let $A \in \mathcal{M}_1$ be such that $\mathcal{T}(A) = \emptyset$. Since $\mathcal{T}(\emptyset) = \emptyset$, we then have that $\mathcal{T}(A) = \mathcal{T}(\emptyset)$. Since $\mathcal{T}$ is injective, $A = \emptyset$.

To show implication in the other direction, assume that

$$\mathcal{T}(A) = \emptyset \implies A = \emptyset. \tag{1}$$

Suppose that $\mathcal{T}(A) = \mathcal{T}(B)$ for some $A, B \in \mathcal{M}_1$. If $\mathcal{T}(A) = \mathcal{T}(B) = \emptyset$, then $A = B = \emptyset$ by the assumption. Suppose that $\mathcal{T}(A) = \mathcal{T}(B) \neq \emptyset$. If $A \cap B = \emptyset$, we would have $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$ from (i) in Definition 1. So we must have $A \cap B \neq \emptyset$. Since $A \setminus B \subset A$, we have $\mathcal{T}(A \setminus B) \subset \mathcal{T}(A) = \mathcal{T}(B)$ by Lemma 1(ii). Since also $A \setminus B \subset B^c$, we have $\mathcal{T}(A \setminus B) \subset \mathcal{T}(B^c)$ by Lemma 1(ii). Hence $\mathcal{T}(A \setminus B) \subset \mathcal{T}(B) \cap \mathcal{T}(B^c)$. Since $B \cap B^c = \emptyset$, we have $\mathcal{T}(B) \cap \mathcal{T}(B^c) = \emptyset$. Hence $\mathcal{T}(A \setminus B) = \emptyset$, and by the assumption we have that $A \setminus B = \emptyset$. By the same argument one can show that $B \setminus A = \emptyset$. Hence $A = B$, and $\mathcal{T}$ is injective. □

Lemma 4. *A bijective metatime is a σ-metatime.*

Proof. Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a bijective metatime. Since $\mathcal{T}$ is surjective, $\mathcal{T}(\mathcal{M}_1) = \mathcal{M}_2$, i.e. for every $B \in \mathcal{M}_2$ there is an $A \in \mathcal{M}_1$ such that $\mathcal{T}(A) = B$. Since $M_2 \in \mathcal{M}_2$, there is an $A \in \mathcal{M}_1$ such that $\mathcal{T}(A) = M_2$. Then $\mathcal{T}(A)^c = M_2^c = \emptyset$. Since $\mathcal{T}$ is injective, it then follows from Lemma 3 that $A^c = \emptyset$. Hence $A = M_1$, and we have that $\mathcal{T}(M_1) = M_2$, which means that $\mathcal{T}$ is a σ-metatime. □

The following result shows that metatimes and σ-metatimes are "continuous at zero". This result is needed to prove the next proposition.

Lemma 5. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a metatime, and let $A_1, A_2, \ldots \in \mathcal{M}_1$ be a nonincreasing sequence of sets with $\lim_{n\to\infty} A_n = \emptyset$. Then $\lim_{n\to\infty} \mathcal{T}(A_n) = \emptyset$.*

Proof. Define $B_n := A_n \setminus A_{n+1}$, $n = 1, \ldots, \infty$, which form a sequence of disjoint sets in $\mathcal{M}$. Then $A_n = \cup_{i=n}^{\infty} B_i$, and from Definition 1(ii),

$$\mathcal{T}(A_n) = \mathcal{T}\left(\cup_{i=n}^{\infty} B_i\right) = \cup_{i=n}^{\infty} \mathcal{T}(B_i).$$

Since the sets $\mathcal{T}(B_i)$ are disjoint by Definition 1(i), we have that

$$\mathcal{T}(A_n) = \cup_{i=n}^{\infty} \mathcal{T}(B_i) = \cup_{i=1}^{\infty} \mathcal{T}(B_i) \setminus \cup_{i=1}^{n-1} \mathcal{T}(B_i) = \mathcal{T}(A_1) \setminus \cup_{i=1}^{n-1} \mathcal{T}(B_i).$$

Letting $n \to \infty$ the result follows. □

Next we show that for both metatimes and σ-metatimes, the second condition in the definitions holds for all sets $A_1, A_2, \ldots \in \mathcal{M}_1$, not only disjoint sets.

Proposition 2. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a metatime (or σ-metatime). For $A_1, A_2, \ldots \in \mathcal{M}_1$, it holds that*

$$\mathcal{T}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathcal{T}(A_i).$$

Proof. For $A, B \in \mathcal{M}_1$ we have the disjoint representation

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B).$$

By (ii) in Definition 1 (or (ii) in Definition 2) used twice, we find that

$$\begin{aligned}
\mathcal{T}(A \cup B) &= \mathcal{T}(A \setminus B) \cup \mathcal{T}(B \setminus A) \cup \mathcal{T}(A \cap B) \\
&= \mathcal{T}(A \setminus B) \cup \mathcal{T}(A \cap B) \cup \mathcal{T}(B \setminus A) \cup \mathcal{T}(A \cap B) \\
&= \mathcal{T}(A) \cup \mathcal{T}(B).
\end{aligned}$$

By induction it follows that $\mathcal{T}(\cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \mathcal{T}(A_i)$ for a finite collection of sets $A_1, \ldots A_n \in \mathcal{M}_1$. Suppose we have a countable sequence of sets $A_1, A_2, \ldots \in \mathcal{M}_1$. Notice that since $\mathcal{T}(A_i) \in \mathcal{M}_2$ for all $i \in \mathbb{N}$, it follows that $\cup_{i=1}^{\infty} \mathcal{T}(A_i) \in \mathcal{M}_2$. Obviously, we have that $\cup_{i=1}^{n} A_i \subset \cup_{i=1}^{\infty} A_i$, and by Lemma 1(ii) it follows that $\mathcal{T}(\cup_{i=1}^{n} A_i) \subset \mathcal{T}(\cup_{i=1}^{\infty} A_i)$. Therefore

$$\cup_{i=1}^{\infty} \mathcal{T}(A_i) = \lim_{n\to\infty} \cup_{i=1}^{n} \mathcal{T}(A_i) = \lim_{n\to\infty} \mathcal{T}(\cup_{i=1}^{n} A_i) \subseteq \mathcal{T}(\cup_{i=1}^{\infty} A_i),$$

which shows the inclusion one way.

Let us express $\cup_{i=1}^{\infty} A_i$ as a disjoint union of two sets,

$$\cup_{i=1}^{\infty} A_i = (\cup_{i=1}^{n} A_i) \cup (\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{n} A_i).$$

Then, from (ii) in Definition 1 (or (ii) in Definition 2),

$$\mathcal{T}(\cup_{i=1}^{\infty} A_i) = \mathcal{T}(\cup_{i=1}^{n} A_i) \cup \mathcal{T}(\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{n} A_i) = \cup_{i=1}^{n} \mathcal{T}(A_i) \cup \mathcal{T}(\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{n} A_i).$$

We have that $\cup_{i=1}^{\infty} \mathcal{T}(A_i) \in \mathcal{M}_2$ is the set-theoretic limit of $\cup_{i=1}^{n} \mathcal{T}(A_i)$. Moreover, we see that $A_{(n)} := \cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{n} A_i$ is a nonincreasing sequence of measurable sets, which has the set-theoretic limit

$$\lim_{n\to\infty} A_{(n)} = \cap_{n=1}^{\infty} A_{(n)} = \cap_{n=1}^{\infty} (\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{n} A_i) = \emptyset.$$

From Lemma 5 it follows that

$$\lim_{n\to\infty} \mathcal{T}(A_{(n)}) = \emptyset.$$

Hence $\mathcal{T}(\cup_{i=1}^{\infty} A_i) \subseteq \cup_{i=1}^{\infty} \mathcal{T}(A_i)$ and the claim follows. □

Corollary 1. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a σ-metatime. For $A_1, A_2, \ldots \in \mathcal{M}_1$, it holds that*

$$\mathcal{T}(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} \mathcal{T}(A_i).$$

Proof. By Proposition 2, Definition 2 of a σ-metatime and De Morgan's laws, we have that

$$\begin{aligned}\left(\cap_{i=1}^{\infty} \mathcal{T}(A_i)\right)^c &= \cup_{i=1}^{\infty} \mathcal{T}(A_i)^c = \cup_{i=1}^{\infty} \mathcal{T}(A_i^c) \\ &= \mathcal{T}\left(\cup_{i=1}^{\infty} A_i^c\right) = \mathcal{T}\left((\cap_{i=1}^{\infty} A_i)^c\right) = \mathcal{T}\left(\cap_{i=1}^{\infty} A_i\right)^c,\end{aligned}$$

and the result follows. □

We are now ready to show that σ-metatimes preserve σ-algebras.

Proposition 3. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a σ-metatime. Then $\mathcal{T}(\mathcal{M}_1) \subseteq \mathcal{M}_2$ is a σ-algebra.*

Proof. For any $A \in \mathcal{M}_1$, we have $\mathcal{T}(A)^c = \mathcal{T}(A^c)$ by (i) in Definition 2. Thus, we see that $\mathcal{T}(\mathcal{M}_1)$ is closed under complements. Also, as $\mathcal{T}(\emptyset) = \emptyset$ by Lemma 2(ii), we find that $\emptyset \in \mathcal{T}(\mathcal{M}_1)$. Let $B_1, B_2, \ldots \in \mathcal{T}(\mathcal{M}_1)$. Then we can find $A_1, A_2, \ldots \in \mathcal{M}_1$ such that $\mathcal{T}(A_i) = B_i$ for all $i \in \mathbb{N}$. By Proposition 2, we find that

$$\cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} \mathcal{T}(A_i) = \mathcal{T}(\cup_{i=1}^{\infty} A_i) \in \mathcal{T}(\mathcal{M}_1).$$

Hence, $\mathcal{T}(\mathcal{M}_1)$ is also closed under countable unions. The proposition follows. □

Let us consider a simple, canonical example of a σ-metatime.

Example 1 (Metatimes induced by measurable functions [4])**.** Consider a measurable function $f : M_2 \to M_1$. A mapping $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ is defined in [4] by

$$\mathcal{T}(A) = f^{-1}(A) = \{x \in M_2 \; : \; f(x) \in A\}$$

for every $A \in \mathcal{M}_1$. The mapping $\mathcal{T}$ defines a metatime. Since $\mathcal{T}(M_1) = M_2$, $\mathcal{T}$ is also a σ-metatime.

In many applications one typically chooses M_1 and M_2 to be two (Borel) subsets of Euclidean spaces. In that case, one defines $\mathcal{M}_1$ and $\mathcal{M}_2$ as the corresponding Borel σ-algebras of subsets of M_1 and M_2, resp. To introduce a random metatime, one can consider an M_1-valued random field F on M_2, that is, a measurable mapping

$$F : M_2 \times \Omega \to M_1,$$

where $(\Omega, \mathcal{F}, P)$ is a probability space and $M_2 \times \Omega$ is equipped with the product σ-algebra $\mathcal{M}_2 \otimes \mathcal{F}$. Then we have

$$\mathcal{T} : \mathcal{M}_1 \times \Omega \ni (A, \omega) \to \{x \in M_2 \,:\, F(x, \omega) \in A\} \in \mathcal{M}_2.$$

We could for example take $M_1 = \mathbb{R}^n$, $n \in \mathbb{N}$, and consider an $\mathbb{R}^n$-valued random field F on $M_2 = \mathbb{R}^d$, $d \in \mathbb{N}$, say. This induces a random σ-metatime. Fixing $A \in \mathcal{M}_1$, $\mathcal{T}(A) : \Omega \to \mathcal{M}_2$ defines a mapping from the probability space to the σ-algebra of subsets of M_2. Thus, under additional assumptions on $\mathcal{T}$, $\mathcal{T}(A)$ defines a random set (see [9]). To have a random set, we must equip $\mathcal{M}_2$ with an appropriate σ-algebra.

The next example of translation metatimes will be a guiding case in the sequel of this paper.

Example 2 (Translation metatimes)**.** Let $M_1 = M_2 = M$ be a topological vector space equipped with the Borel σ-algebra $\mathcal{M}$. For a fixed $x \in M$, define the translation map $\mathcal{T}_x$ on $\mathcal{M}$ by

$$\mathcal{T}_x(A) := A + \{x\} = \{y \in M \,:\, y - x \in A\} \tag{2}$$

for $A \in \mathcal{M}$. As the translation operator on M (slightly abusing the notation), $\mathcal{T}_x(y) = x + y$, is continuous with a continuous inverse $\mathcal{T}_{-x}$, $\mathcal{T}_x(A) \in \mathcal{M}$.

Let us show that $\mathcal{T}_x$ is a metatime on $\mathcal{M}$. Suppose $A \cap B = \emptyset$. Consider $y \in \mathcal{T}_x(A) \cap \mathcal{T}_x(B)$. Then $y - x \in A \cap B$, but by disjointness this is impossible. Hence, $T_x(A) \cap T_x(B) = \emptyset$. Next, let $A_1, A_2, \ldots \in \mathcal{M}$ be a sequence of sets. If $y \in \cup_{i=1}^{\infty} \mathcal{T}_x(A_i)$, this is equivalent to $y - x \in A_i$ for at least one i. But this is equivalent to $y - x \in \cup_{i=1}^{\infty} A_i$, which in turn is equivalent to $y \in \mathcal{T}_x(\cup_{i=1}^{\infty} A_i)$. Hence $\mathcal{T}_x$ is a metatime on $\mathcal{M}$. We notice that $\mathcal{T}_x(\emptyset) = \{y \in M \,:\, y - x \in \emptyset\} = \emptyset$. Since $\mathcal{T}_x(M) = \{y \in M \,:\, y - x \in M\} = M$, it follows from Proposition 1 that $\mathcal{T}_x$ is a σ-metatime.

Trawl processes, first introduced in [2], have gained significant attention (see [5]). A so-called *ambit set* plays a crucial role in the construction of trawl processes: Let $M = \mathbb{R}^{d+1}$, $A \subseteq \mathbb{R}^d \times (-\infty, 0]$ and $x := x(t) = (\mathbf{0}, t)$, $t \geq 0$. Hence, we consider a moving x which yields a time-dependent metatime, i.e., a mapping $t \mapsto \mathcal{T}_{x(t)}(A)$. We will return to trawl processes in later sections.

The following proposition shows a natural algebraic property of metatimes, namely that they are closed under concatenation.

Proposition 4. *Let $\mathcal{T}_1 : \mathcal{M}_1 \to \mathcal{M}_2$ and $\mathcal{T}_2 : \mathcal{M}_2 \to \mathcal{M}_3$ be two (σ-)metatimes. Then $\mathcal{T}_2 \circ \mathcal{T}_1 : \mathcal{M}_1 \to \mathcal{M}_3$ is a (σ-)metatime.*

Proof. Obviously $\mathcal{T}_1(A) \in \mathcal{M}_2$ for any $A \in \mathcal{M}_1$, and $\mathcal{T}_2 \circ \mathcal{T}_1$ is a well-defined mapping from $\mathcal{M}_1$ into $\mathcal{M}_3$. Let $A, B \in \mathcal{M}_1$ be disjoint. By property (i) in Definition 1 it follows that $\mathcal{T}_1(A) \cap \mathcal{T}_1(B) = \emptyset$. By using property (i) in Definition 1 again, it follows $\mathcal{T}_2(\mathcal{T}_1(A)) \cap \mathcal{T}_2(\mathcal{T}_1(B)) = \emptyset$. Hence, $\mathcal{T}_2 \circ \mathcal{T}_1$ satisfies property (i) in Definition 1.

Let $A_1, A_2, \ldots, \in \mathcal{M}_1$ be disjoint. From property (ii) in Definition 1 we have that $\mathcal{T}_1(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathcal{T}_1(A_i)$. Also, as the A_i's are disjoint, $\mathcal{T}_1(A_i) \cap \mathcal{T}_1(A_j) = \emptyset$ for all $i \neq j$. Hence, by using property (ii) in Definition 1 again,

$$\mathcal{T}_2(\mathcal{T}_1(\cup_{i=1}^{\infty} A_i)) = \mathcal{T}_2(\cup_{i=1}^{\infty} \mathcal{T}_1(A_i)) = \cup_{i=1}^{\infty} \mathcal{T}_2(\mathcal{T}_1(A_i)).$$

Hence, $\mathcal{T}_2 \circ \mathcal{T}_1$ satisfies property (ii) in Definition 1.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are σ-metatimes, $\mathcal{T}_1(A^c) = \mathcal{T}_1(A)^c$ for $A \in \mathcal{M}_1$ by property (i) in Definition 2. Thus $\mathcal{T}_2(\mathcal{T}_1(A^c)) = \mathcal{T}_2(\mathcal{T}_1(A)^c) = \mathcal{T}_2(\mathcal{T}_1(A))^c$ from the same property. Hence, $\mathcal{T}_2 \circ \mathcal{T}_1$ satiesfies property (i) in Definition 2, and is therefore a σ-metatime. $\square$

To end this section, we show that given a measure on $(M_2, \mathcal{M}_2)$, we can use a metatime to define a measure on $(M_1, \mathcal{M}_1)$. Note that this is a generalization of the push-forward measure. If the metatime is defined from a function, it corresponds to the push-forward measure.

Proposition 5. *Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a metatime and let μ_2 be a measure on $(M_2, \mathcal{M}_2)$. Define $\mu_{\mathcal{T}} := \mu_2(\mathcal{T}\cdot)$. Then*

(i) $\mu_{\mathcal{T}}$ is a measure on $(M_1, \mathcal{M}_1)$.

(ii) If μ_2 is finite, then $\mu_{\mathcal{T}}$ is also finite.

(iii) If $\mathcal{T}$ is bijective and μ_2 is σ-finite, then $\mu_{\mathcal{T}}$ is σ-finite.

Proof.

(i) As $\mathcal{T}(\emptyset) = \emptyset$ by Lemma 1(ii), it follows that $\mu_{\mathcal{T}}(\emptyset) = \mu_2(\emptyset) = 0$. If $A_1, A_2, \ldots \in \mathcal{M}_1$ is a sequence of disjoint sets, we find from (ii) in Definition 1 that

$$\begin{aligned}\mu_{\mathcal{T}}(\cup_{i=1}^{\infty} A_i) &= \mu_2(\mathcal{T}(\cup_{i=1}^{\infty} A_i)) = \mu_2(\cup_{i=1}^{\infty} \mathcal{T}(A_i)) \\ &= \sum_{i=1}^{\infty} \mu_2(\mathcal{T}(A_i)) = \sum_{i=1}^{\infty} \mu_{\mathcal{T}}(A_i).\end{aligned}$$

Hence, the $\mu_{\mathcal{T}}$ is a measure on $(M_1, \mathcal{M}_1)$.

(ii) Since $\mathcal{T}(M_1) \subseteq M_2$, it follows that

$$\mu_{\mathcal{T}}(M_1) = \mu_2(\mathcal{T}(M_1)) \leq \mu_2(M_2) < \infty.$$

(iii) If μ_2 is σ-finite, there exists $\{B_i\}_{i=1}^{\infty} \subset \mathcal{M}_2$ such that $\mu_2(B_i) < \infty$ for all $i \in \mathbb{N}$ and $\cup_{i=1}^{\infty} B_i = M_2$. If $\mathcal{T}$ is surjective, there exists $\{A_i\}_{i=1}^{\infty} \subset \mathcal{M}_1$ such that $\mathcal{T}(A_i) = B_i$. Then $\mu_{\mathcal{T}}(A_i) = \mu_2(\mathcal{T}(A_i)) = \mu_2(B_i) < \infty$, and by Proposition 2,

$$\mathcal{T}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} \mathcal{T}(A_i) = \cup_{i=1}^{\infty} B_i = M_2.$$

If $\mathcal{T}$ is also injective, it follows that $\mathcal{T}(M_1) = M_2$ (see Lemma 4). From the injectivity of $\mathcal{T}$ we also have $M_1 = \cup_{i=1}^{\infty} A_i$, and σ-finiteness follows.

$\square$

3 Metatimes and bounded linear operators

Let $(M_1, \mathcal{M}_1)$ and $(M_2, \mathcal{M}_2)$ be measurable spaces, and let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ be a σ-metatime. Given a measure μ_2 on $(M_2, \mathcal{M}_2)$, we denote by $L^2(\mu_2) := L^2(M_2, \mathcal{M}_2, \mu_2)$ the space of square integrable functions on M_2 with values in $\mathbb{R}$. We define the measure $\mu_{\mathcal{T}}$ on $(M_1, \mathcal{M}_1)$ as in Proposition 5, and let $L^2(\mu_{\mathcal{T}}) := L^2(M_1, \mathcal{M}_1, \mu_{\mathcal{T}})$. We will in this section lift the σ-metatime $\mathcal{T}$ to an isometric linear operator $\widehat{\mathcal{T}} : L^2(\mu_{\mathcal{T}}) \to L^2(\mu_2)$.

We first define the operator $\widehat{\mathcal{T}}$ for elementary functions in $L^2(\mu_{\mathcal{T}})$, and then extend it by a standard limiting argument to general functions in $L^2(\mu_{\mathcal{T}})$. To this end, let $\langle \cdot, \cdot \rangle_2$ denote the inner product in $L^2(\mu_2)$ and $\|\cdot\|_2$ denote the norm induced by this inner product. Let $\langle \cdot, \cdot \rangle_{\mathcal{T}}$ and $\|\cdot\|_{\mathcal{T}}$ denote the inner product and norm in $L^2(\mu_{\mathcal{T}})$.

We say that $\phi \in L^2(\mu_{\mathcal{T}})$ is *elementary* if $\phi = \sum_{i=1}^n \phi_i 1_{A_i}$ where $\phi_i \in \mathbb{R}$, $n \in \mathbb{N}$ and $\{A_i\}_{i=1,\dots,n}$ are disjoint subsets in $\mathcal{M}_1$. Notice that if ϕ has such a representation for nondisjoint sets A_i, one can always re-express it into a representation with disjoint sets. A straightforward calculation shows that

$$\begin{aligned}
\|\phi\|_{\mathcal{T}}^2 &= \int_{M_1} \phi^2(x)\, \mu_{\mathcal{T}}(dx) \\
&= \int_{M_1} \Big| \sum_{i=1}^n \phi_i 1_{A_i}(x) \Big|^2 \mu_{\mathcal{T}}(dx) \\
&= \int_{M_1} \sum_{i,j=1}^n \phi_i \phi_j 1_{A_i}(x) 1_{A_j}(x)\, \mu_{\mathcal{T}}(dx) \\
&= \sum_{i=1}^n \phi_i^2 \mu_{\mathcal{T}}(A_i).
\end{aligned}$$

We remark in passing that since $\phi \in L^2(\mu_{\mathcal{T}})$, $\mu_{\mathcal{T}}(A_i) < \infty$ for all $i = 1, \dots, n$. We denote the set of elementary functions in $L^2(\mu_{\mathcal{T}})$ by $\mathcal{E}_{\mathcal{T}}$. Define the operator $\widehat{\mathcal{T}} : \mathcal{E}_{\mathcal{T}} \to L^2(\mu_2)$ by

$$\widehat{\mathcal{T}}\phi := \sum_{i=1}^n \phi_i 1_{\mathcal{T}(A_i)}. \tag{3}$$

When $A_1, \dots, A_n \in \mathcal{M}_1$ are disjoint, $\mathcal{T}(A_1), \dots, \mathcal{T}(A_n) \in \mathcal{M}_2$ are also disjoint by property (ii) in Definition 2. Similar to above, we find that

$$\|\widehat{\mathcal{T}}\phi\|_2^2 = \sum_{i=1}^n \phi_i^2 \mu_2(\mathcal{T}(A_i)) = \sum_{i=1}^n \phi_i^2 \mu_{\mathcal{T}}(A_i) = \|\phi\|_{\mathcal{T}}^2, \tag{4}$$

so $\widehat{\mathcal{T}}\phi \in L^2(\mu_2)$. Hence $\widehat{\mathcal{T}}\phi$ is an elementary function in $L^2(\mu_2)$. We prove that $\widehat{\mathcal{T}}$ is linear.

Lemma 6. *The operator $\widehat{\mathcal{T}} : \mathcal{E}_{\mathcal{T}} \to L^2(\mu_2)$ defined in (3) is linear.*

Proof. Let $\phi := \sum_{i=1}^n \phi_i 1_{A_i}$ and $\psi := \sum_{j=1}^m \psi_j 1_{B_j}$ be two functions in $\mathcal{E}_{\mathcal{T}}$, where we without loss of generality can assume that $M_1 = \cup_{i=1}^n A_i = \cup_{j=1}^m B_j$. Then it holds that

$$\phi + \psi = \sum_{i=1}^{n} \sum_{j=1}^{m} (\phi_i + \psi_j) 1_{A_i \cap B_j}.$$

Since all sets of the form $A_i \cap B_j$ are disjoint, $\phi + \psi \in \mathcal{E}_{\mathcal{T}}$, so by definition of $\widehat{\mathcal{T}}$,

$$\widehat{\mathcal{T}}(\phi + \psi) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\phi_i + \psi_j) 1_{\mathcal{T}(A_i \cap B_j)}.$$

For $A, B \in \mathcal{M}_1$, we have that $\mathcal{T}(A \cap B) = \mathcal{T}(A) \cap \mathcal{T}(B)$ by Corollary 1. Then

$$\begin{aligned}
\widehat{\mathcal{T}}(\phi + \psi) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\phi_i + \psi_j) 1_{\mathcal{T}(A_i) \cap \mathcal{T}(B_j)} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_i 1_{\mathcal{T}(A_i) \cap \mathcal{T}(B_j)} + \sum_{i=1}^{n} \sum_{j=1}^{m} \psi_j 1_{\mathcal{T}(A_i) \cap \mathcal{T}(B_j)} \\
&= \sum_{i=1}^{n} \phi_i \sum_{j=1}^{m} 1_{\mathcal{T}(A_i) \cap \mathcal{T}(B_j)} + \sum_{j=1}^{m} \psi_j \sum_{i=1}^{n} 1_{\mathcal{T}(A_i) \cap \mathcal{T}(B_j)} \\
&= \sum_{i=1}^{n} \phi_i 1_{\cup_{j=1}^{m} \left(\mathcal{T}(A_i) \cap \mathcal{T}(B_j)\right)} + \sum_{j=1}^{m} \psi_j 1_{\cup_{i=1}^{n} \left(\mathcal{T}(A_i) \cap \mathcal{T}(B_j)\right)} \\
&= \sum_{i=1}^{n} \phi_i 1_{\mathcal{T}(A_i) \cap \left(\cup_{j=1}^{m} \mathcal{T}(B_j)\right)} + \sum_{j=1}^{m} \psi_j 1_{\mathcal{T}(B_j) \cap \left(\cup_{i=1}^{n} \mathcal{T}(A_i)\right)} \\
&= \sum_{i=1}^{n} \phi_i 1_{\mathcal{T}(A_i)} + \sum_{j=1}^{m} \psi_j 1_{\mathcal{T}(B_j)} \\
&= \widehat{\mathcal{T}}\phi + \widehat{\mathcal{T}}\psi,
\end{aligned}$$

where we in the second to last equality used that $\mathcal{T}(M_1) = M_2$ together with property (ii) in Definition 2. The lemma follows. □

Following similar arguments as in the proof of linearity in Lemma 6 above, we can show that $\widehat{\mathcal{T}}$ does not depend on the actual decomposition chosen in (3). We have that $\widehat{\mathcal{T}}$ can be extended to a linear operator $\widehat{\mathcal{T}} : L^2(\mu_{\mathcal{T}}) \to L^2(\mu_2)$.

Proposition 6. *There exists a unique extension of $\widehat{\mathcal{T}}$ defined in* (3) *to an isometric linear operator $\widehat{\mathcal{T}} : L^2(\mu_{\mathcal{T}}) \to L^2(\mu_2)$.*

Proof. By Prop. 6.7 in [7], $\mathcal{E}_{\mathcal{T}}$ is dense in $L^2(\mu_{\mathcal{T}})$. The result follows from Proposition 2.1.11 in [10]. □

Since for any elementary $\phi \in L^2(\mu_{\mathcal{T}})$ we find

$$\int_{M_2} \widehat{\mathcal{T}}\phi(y)\mu_2(dy) = \sum_i \phi_i \mu_2(\mathcal{T}(A_i)) = \sum_i \phi_i \mu_{\mathcal{T}}(A_i) = \int_{M_1} \phi(x)\mu_{\mathcal{T}}(dx),$$

the integration by parts formula

$$\int_{M_2} \widehat{\mathcal{T}} f(y)\mu_2(dy) = \int_{M_1} f(x)\mu_{\mathcal{T}}(dx) \tag{5}$$

holds for all $f \in L^2(\mu_{\mathcal{T}})$. We remark in passing that one may extend the operator $\widehat{\mathcal{T}}$ beyond $L^2(\mu_{\mathcal{T}})$ in the following way: if $f : M_1 \to M_2$ is a measurable function for which there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of simple functions $\phi_n : M_1 \to M_2$ such that $\phi_n \to f$ $\mu_{\mathcal{T}}$-a.e., then $\widehat{\mathcal{T}} f$ can be defined as the $\mu_{\mathcal{T}}$-a.e. limit of $\widehat{\mathcal{T}}\phi_n$. If additionally $(\phi_n)_{n\in\mathbb{N}}$ is dominated by an $\mu_{\mathcal{T}}$-integrable function, we can appeal to the dominated convergence theorem to show that (5) is valid. In this paper we will not work with this generalization.

Suppose that $L^2(\mu_{\mathcal{T}})$ is separable with an orthonormal basis (ONB) $\{e_n\}_{n\in\mathbb{N}}$. Denoting by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm of bounded linear operators on $L^2(\mu_{\mathcal{T}})$, we find that

$$\|\widehat{\mathcal{T}}\|_{\mathrm{HS}}^2 = \sum_{n=1}^{\infty} \|\widehat{\mathcal{T}}(e_n)\|_2^2 = \sum_{n=1}^{\infty} \|e_n\|_{\mathcal{T}}^2 = \infty.$$

Hence, $\widehat{\mathcal{T}}$ is not a Hilbert–Schmidt operator.

Let μ_1 be another measure on $(M_1, \mathcal{M}_1)$ such that $\mu_{\mathcal{T}} << \mu_1$ with Radon–Nikodym derivative $d\mu_{\mathcal{T}}/d\mu_1$. If $\mu_1(A) = 0$ for some $A \in \mathcal{M}_1$, then $\mu_{\mathcal{T}}(A) = 0$ by absolute continuity. Hence $\mu_2(\mathcal{T}(A)) = \mu_{\mathcal{T}}(A) = 0$, thus we can say that $\mathcal{T}$ preserves zero sets from μ_1 to μ_2. If $d\mu_{\mathcal{T}}/d\mu_1 \in L^\infty(\mu_1)$, we find that for any $f \in L^2(\mu_1)$,

$$\|f\|_{\mathcal{T}}^2 = \int_{M_1} |f(x)|^2 \frac{d\mu_{\mathcal{T}}}{d\mu_1}(x)\mu_1(dx) \le \|d\mu_{\mathcal{T}}/d\mu_1\|_{L^\infty(\mu_1)} \|f\|_{L^2(\mu_1)}^2.$$

So $L^2(\mu_1) \subseteq L^2(\mu_{\mathcal{T}})$. Moreover, for $f \in L^2(\mu_1)$,

$$\|\widehat{\mathcal{T}}(f)\|_2 = \|f\|_{\mathcal{T}} \le \|d\mu_{\mathcal{T}}/d\mu_1\|_{L^\infty(\mu_1)}^{1/2} \|f\|_{L^2(\mu_1)}.$$

Thus, we can view $\widehat{\mathcal{T}}$ as a bounded linear operator from $L^2(\mu_1)$ into $L^2(\mu_2)$. Let us consider an example.

Example 3. Let $(M_1, \mathcal{M}_1) = (M_2, \mathcal{M}_2) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, and equip this space with the Lebesgue measure (denoted *Leb*). Thus, in the above notation, $\mu_2 = Leb$. Introduce a measurable and strictly increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$. Define $\mathcal{T}(A) = f^{-1}(A) = \{x \in \mathbb{R}_+ : f(x) \in A\}$ for $A \in \mathcal{B}(\mathbb{R}_+)$. We know that $\mathcal{T}$ is a metatime, and we readily see that

$$\mathcal{T}(\mathbb{R}_+) = \{x \in \mathbb{R}_+ : f(x) \in \mathbb{R}_+\} = \mathbb{R}_+.$$

Moreover, assume that f is differentiable. Then $(f^{-1})'(y) = 1/f'(f^{-1}(y)) > 0$, and we find that

$$\begin{aligned} Leb(\mathcal{T}([0,x])) &= Leb(\{y \in \mathbb{R}_+ : 0 \le f(y) \le x\}) \\ &= Leb(\{0 \le y \le f^{-1}(x)\}) \end{aligned}$$

$$= \int_0^{f^{-1}(x)} dy$$
$$= \int_0^{x} (f^{-1})'(y)dy.$$

We conclude that $Leb_{\mathcal{T}} << Leb$ with Radon–Nikodym derivative $(f^{-1})'(y)$. If f' is bounded away from zero, then we also have $dLeb_{\mathcal{T}}/Leb \in L^{\infty}(Leb)$.

Let us now ask for a (canonical) definition of a metatime *induced* from a given linear operator $\widehat{\mathcal{T}} \in L(H_1, H_2)$, where H_1, H_2 are two Hilbert spaces. To this end, let $\mathcal{H}_i, i = 1, 2$, be the Borel σ-algebras on $H_i, i = 1, 2$, defined by the norm topologies of the respective spaces. From previous considerations (see Example 1), we can define a metatime $\mathcal{T} : \mathcal{H}_2 \to \mathcal{H}_1$ simply by $\mathcal{T}(A) := \widehat{\mathcal{T}}^{-1}(A)$ for $A \in \mathcal{H}_2$. However, in view of the above construction, we want to define a metatime going in the "same direction" as the operator. To do so, we define a metatime as the image map of $\widehat{\mathcal{T}}$, that is,

$$\mathcal{T} : \mathcal{H}_1 \to \mathcal{H}_2\,, \qquad \mathcal{H}_1 \ni A \mapsto \mathcal{T}(A) := \widehat{\mathcal{T}}(A) \subset H_2. \tag{6}$$

By $\widehat{\mathcal{T}}(A)$ we mean the image of A in H_2 with respect to the operator $\widehat{\mathcal{T}}$, that is $\widehat{\mathcal{T}}(A) = \{\widehat{\mathcal{T}}(g) \,|\, g \in A\}$. Unfortunately, it is not automatically so that $\widehat{\mathcal{T}}(A) \in \mathcal{H}_2$ for every $A \in \mathcal{H}_1$. If $\widehat{\mathcal{T}}$ is invertible, then, since the inverse is a bounded operator and thus continuous, we have that the image map of $\widehat{\mathcal{T}}$ is in $\mathcal{H}_2$.

Proposition 7. *Suppose that the image map of $\widehat{\mathcal{T}} \in L(H_1, H_2)$ maps into $\mathcal{H}_2$. If* $\ker(\widehat{\mathcal{T}}) = \{0\}$*, then $\mathcal{T}$ in (6) is a metatime.*

Proof. Let $A, B \in \mathcal{H}_1$ where $A \cap B = \emptyset$. Suppose $h \in \mathcal{T}(A) \cap \mathcal{T}(B)$. Thus, there exist $g \in A$ and $\widetilde{g} \in B$ such that $h = \widehat{\mathcal{T}}(g) = \widehat{\mathcal{T}}(\widetilde{g})$. But then $\widehat{\mathcal{T}}(g - \tilde{g}) = 0$ by linearity of $\widehat{\mathcal{T}}$, and hence $g = \widetilde{g}$ since $g - \tilde{g} \in \ker(\widehat{\mathcal{T}}) = \{0\}$ by assumption. This is a contradiction since $A \cap B = \emptyset$, and therefore $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$. Hence, $\mathcal{T}$ satisfies property (i) in Definition 1.

The following concluding lines of arguments are elementary set theory (see top of page 4 in [7]). We include the arguments for completeness. Let $A_1, A_2, \ldots \in \mathcal{H}_1$ be a countable sequence of disjoint sets. Suppose that $h \in \mathcal{T}(\cup_{i=1}^{\infty} A_i)$, which means that there is a $g \in \cup_{i=1}^{\infty} A_i$ such that $\widehat{\mathcal{T}}(g) = h$. But since the sets $A_1, A_2, \ldots$ are disjoint, $g \in A_k$ for only one $k \in \mathbb{N}$, and therefore $h \in \mathcal{T}(A_k) \subset \cup_{i=1}^{\infty} \mathcal{T}(A_i)$. Hence, $\mathcal{T}(\cup_{i=1}^{\infty} A_i) \subset \cup_{i=1}^{\infty} \mathcal{T}(A_i)$.

Suppose $h \in \cup_{i=1}^{\infty} \mathcal{T}(A_i)$, then $h \in \mathcal{T}(A_k)$ for one $k \in \mathbb{N}$ since the sets $A_1, A_2, \ldots$ are disjoint implying (from above) that $\mathcal{T}(A_i), i = 1, 2, \ldots,$ are disjoint as well. Hence, there exists a $g \in A_k$ such that $h = \widehat{\mathcal{T}}(g)$. Obviously, $g \in \cup_{i=1}^{\infty} A_i$, and thus $h \in \mathcal{T}(\cup_{i=1}^{\infty} A_i)$. This shows property (ii) of Definition 1 for $\mathcal{T}$. □

It is well known that if a linear bounded operator is surjective, it is invertible (see [10, Prop. 3.2.6]). Thus, if $\widehat{\mathcal{T}} \in L(H_1, H_2)$ is surjective, we see from above that it canonically defines a metatime. This provides us with a rich source of metatimes. For example, consider $H_1 = H_2 = \mathbb{R}^n$ equipped with the 2-norm (to have a Hilbert space). Then all quadratic matrices define linear bounded operators, and are invertible when the kernel is trivial. Hence, all invertible $n \times n$-matrices form metatimes on $\mathbb{R}^n$ by the identification in (6).

We end this section by going back to translation metatimes in Example 2.

Example 4. Let $\mathcal{T}_x$ be a translation metatime as introduced in Example 2, where we equip $(M, \mathcal{M})$ with a measure μ. If ϕ is an elementary function in $L^2(\mu_{\mathcal{T}_x})$, then we see that

$$\widehat{\mathcal{T}}_x \phi(y) = \sum_{i=1}^{n} \phi_i 1_{\mathcal{T}_x(A_i)}(y) = \phi(y - x)$$

for any $y \in M$. This holds since $y \in \mathcal{T}_x(A_i)$ is equivalent to $y - x \in A_i$. Hence, we reach that $\widehat{\mathcal{T}_x} = \widehat{\mathcal{S}}_{-x}$, the shift operator on $L^2(\mu_{\mathcal{T}_x})$.

4 Construction of a cylindrical random variable

The aim of this section is to construct a cylindrical random variable from an orthogonal random measure, and to show that it satisfies an orthogonality preserving property. We denote by $(\Omega, \mathcal{F}, P)$ a given probability space, and use the notation $L^2(P) := L^2(\Omega, \mathcal{F}, P)$. Recall the definition of a cylindrical random variable on a Hilbert space.

Definition 3. A *cylindrical random variable* $\mathbb{X}$ on a Hilbert space H is a continuous linear mapping $\mathbb{X} : H \to L^2(P)$. We say that $\mathbb{X}$ is Gaussian if $\mathbb{X}(f)$ is Gaussian for all $f \in H$.

We notice that if $\mathbb{X}$ is a cylindrical random variable on H_2 and $\widehat{\mathcal{L}} \in L(H_1, H_2)$ for two Hilbert spaces H_1 and H_2, then $\mathbb{X} \circ \widehat{\mathcal{L}}$ is a cylindrical random variable on H_1. This follows readily from the continuity and linearity of $\widehat{\mathcal{L}}$. We also recall from [12, Ch. VI(§2)] and [13, Ch. 2] the definition of an orthogonal random measure on a measurable space $(M, \mathcal{M})$.

Definition 4. An *orthogonal random measure* L on $(M, \mathcal{M})$ is a mapping $L : \mathcal{M} \to L^2(P)$ which satisfies the following:

(i) $L(A \cup B) = L(A) + L(B)$ when $A \cap B = \emptyset$.

(ii) $L(A)$ and $L(B)$ are orthogonal when $A \cap B = \emptyset$.

We say that L is Gaussian if $L(A)$ is a Gaussian random variable for all $A \in \mathcal{M}$.

We restrict our attention to random measures which have finite variance. Orthogonality means that $\mathbb{E}[L(A)L(B)] = 0$ when $A \cap B = \emptyset$. If L is standard Gaussian, that is, if $L(A)$ is a mean-zero Gaussian random variable for every $A \in \mathcal{M}$, orthogonality is equivalent to independence. Orthogonal random measures are related to Lévy bases, which are the core objects in defining ambit fields and processes (see [3]). Note that the orthogonal random measure is defined on all sets in the σ-algebra $\mathcal{M}$, unlike Lévy bases, which may be restricted to some subset of $\mathcal{M}$. Orthogonal random measures constitute a generalization of *white noise* as defined by [13, Ch. 2].

There are some simple properties of orthogonal random measures. First, metatimes act invariantly on orthogonal random measures (see [3, Thm. 14, Subsect. 5.5.2] for a similar result for Lévy bases).

Proposition 8. *Let $(M_1, \mathcal{M}_1)$ and $(M_2, \mathcal{M}_2)$ be measurable spaces, and let L be an orthogonal random measure on $(M_2, \mathcal{M}_2)$. Let $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$ a metatime. Then $L_{\mathcal{T}} := L \circ \mathcal{T}$ is an orthogonal random measure on $(M_1, \mathcal{M}_1)$.*

Proof. Since $\mathcal{T}(A) \in \mathcal{M}_2$ for every $A \in \mathcal{M}_1$, $L(\mathcal{T}(A))$ is a well-defined random variable in $L^2(P)$. If $A, B \in \mathcal{M}_1$ are disjoint, it follows by the additivity property (ii) in Definition 1 of metatimes that $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$. Moreover, by property (i) in Definition 1, $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$. Therefore, it follows by property (i) in Definition 4 that

$$L_{\mathcal{T}}(A \cup B) = L(\mathcal{T}(A \cup B)) = L(\mathcal{T}(A) \cup \mathcal{T}(B)) = L(\mathcal{T}(A)) + L(\mathcal{T}(B)).$$

Hence, $L_{\mathcal{T}}$ satisfies property (i) in Definition 4.

For property (ii) in Definition 4, assume again that $A, B \in \mathcal{M}_1$ are disjoint. As above, $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$, hence we have $L(\mathcal{T}(A)) \perp L(\mathcal{T}(B))$ since L is orthogonal. We conclude that $L_{\mathcal{T}}$ is an orthogonal random measure on $(M_1, \mathcal{M}_1)$. □

We next collect in Lemmas 7 and 8 and Proposition 9 some known results from [12, Ch. VI(§2)] and [13, Ch. 2] about orthogonal random measures which are of interest for our exposition. The first lemma provides a natural increasing property of the orthogonal random measures in $L^2(P)$ in terms of increasing sets in $\mathcal{M}$.

Lemma 7. *If $A, B \in \mathcal{M}$ and $A \subseteq B$, then $\mathbb{E}[L(A)^2] \leq \mathbb{E}[L(B)^2]$.*

From this result, we see that for a sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ with $A_{n+1} \subset A_n$, the sequence $a_n := \mathbb{E}[L(A_n)^2]$ is monotonely decreasing. Hence it has a limit. However, although $L(\emptyset) = 0$ by property (i) in Definition 4, we may have $\lim_{n\to\infty} a_n > 0$ when $A_n \downarrow \emptyset$. This leads us to the following definition taken from [13].

Definition 5. An orthogonal random measure L is L^2-*countably additive* if for any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ where $A_{n+1} \subset A_n$ and $A_n \downarrow \emptyset$,

$$\lim_{n\to\infty} \mathbb{E}[L(A_n)^2] = 0.$$

We have the following important lemma for countably additive orthogonal random measures, showing that L^2-countable additivity ensures that property (i) in Definition 4 can be extended to hold for any countable sequence of disjoint sets.

Lemma 8. *Suppose that L is an L^2-countably additive orthogonal random measure. Then for any sequence $\{A_n\}_{n=1}^{\infty} \subset \mathcal{M}$ of disjoint sets (i.e., $A_i \cap A_j = \emptyset$ for all $i \neq j$) we have*

$$L(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} L(A_n),$$

where the sum on the right-hand side converges in $L^2(P)$ and a.s.

An orthogonal random measure L on $(M, \mathcal{M})$ is said to have zero mean if for any $A \in \mathcal{M}$, $\mathbb{E}[L(A)] = 0$. The following proposition shows that the variance of L defines a measure on $(M, \mathcal{M})$ whenever L is L^2-countably additive and has zero mean.

Proposition 9. *Let L be an orthogonal random measure on $(M, \mathcal{M})$ which is L^2-countably additive and has zero mean. Define $\mu(A) := \mathbb{E}[L(A)^2]$ for $A \in \mathcal{M}$. Then μ defines a finite measure on $(M, \mathcal{M})$.*

For the remainder of this section, we assume that L is L^2-countably additive and has zero mean.

Assumption 1. *The orthogonal random measure L is L^2-countably additive and has zero mean.*

Consider the space $L^2(\mu) := L^2(M, \mathcal{M}, \mu)$, with μ as defined in Proposition 9 above. As is well known, $L^2(\mu)$ is a Hilbert space (see, e.g., [7, p. 164]). We want to construct a cylindrical random variable on $L^2(\mu)$. To this end, let $\langle \cdot, \cdot \rangle$ denote the inner product in $L^2(\mu)$, with $\| \cdot \|$ being the induced norm. We start by constructing the cylindrical random variable on the set of elementary functions in $L^2(\mu)$, and then extend it to general functions in $L^2(\mu)$ by appealing to the denseness of elementary functions.

An elementary function in $L^2(\mu)$ is a function on the form $\phi = \sum_{i=1}^n \phi_i 1_{A_i}$, where $\phi_i \in \mathbb{R}$ for $i = 1, \ldots, n$ and $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{M}$. Let $\mathcal{E}$ denote the set of elementary functions in $L^2(\mu)$. Notice that since $\phi \in L^2(\mu)$, $\mu(A_i) < \infty$ for all $i = 1, \ldots, n$. Define the map $\mathbb{L} : \mathcal{E} \to L^2(P)$ by

$$\mathbb{L}(\phi) = \sum_{i=1}^n \phi_i L(A_i), \tag{7}$$

which is known to be a linear isometry, see, e.g., [12, Ch. VI(§2)]. We have that $\mathbb{L}$ can be extended to a cylindrical random variable on $L^2(\mu)$.

Proposition 10. *Let $\mathbb{L} : \mathcal{E} \to L^2(P)$ be defined as in (7). Then there exists a unique extension of $\mathbb{L}$ to a cylindrical random variable $\mathbb{L} : L^2(\mu) \to L^2(P)$. The cylindrical random variable is isometric.*

Proof. From [7, Prop. 6.7], we know that $\mathcal{E}$ is dense in $L^2(\mu)$. The result follows from Proposition 2.1.11 in [10]. $\square$

Under the mild condition that the orthogonal random measure L has zero mean, by the proposition above, we have defined a lifting of L to a cylindrical random variable $\mathbb{L}$ defined on the space $L^2(\mu)$ of square integrable functions, where μ is induced by L. Our construction obviously gives the representation

$$\mathbb{L}(f) = \int_M f(x) L(dx), \tag{8}$$

for $f \in L^2(\mu)$, which is seen from (7). Hence, we have made a Wiener construction following the standard approach in order to construct a cylindrical random variable from an orthogonal random measure. Our construction follows the same approach as in [13], but in our case the integrands are deterministic while in [13] they are stochastic.

Since the cylindrical random variable constructed from an orthogonal random measure as above is isometric, it preserves the inner product and therefore also the orthogonality. In the case of a Gaussian cylindrical random variable, $\mathbb{L}(f)$ and $\mathbb{L}(g)$ are independent when $f \perp g$. In general, a cylindrical random variable is not necessarily preserving the orthogonality. We call cylindrical random variables with this property *orthogonality preserving*.

Definition 6. We say that a cylindrical random variable $\mathbb{X} : H \to L^2(P)$ is *orthogonality preserving* if $\mathbb{X}(f) \perp \mathbb{X}(g)$ whenever $f \perp g$ for $f, g \in H$.

Example 5. Let H be a Hilbert space of real-valued functions on some measure space $(M, \mathcal{M}, \mu)$, and assume that $\mathbb{X}$ is a cylindrical random variable on H. Let δ_x, $x \in M$, be the evaluation map on H, defined as $\delta_x f = f(x) \in \mathbb{R}$ for $f \in H$. Assume that $\delta_x \in H^*$, where H^* is the space of linear functionals on H. Let δ_x^* denote the adjoint of δ_x, i.e. $\delta_x f = \langle f, \delta_x^* 1\rangle$ with $\delta_x^* 1 \in H$. Define $X : M \to L^2(P)$ by $X(x) := \mathbb{X}(\delta_x^* 1)$. If $\mathbb{X}$ is isometric, we find that

$$\mathbb{E}[X(x)X(y)] = \mathbb{E}[\mathbb{X}(\delta_x^* 1)\mathbb{X}(\delta_y^* 1)] = \langle \delta_x^* 1, \delta_y^* 1\rangle = \delta_y \delta_x^* 1.$$

So $(X(x))_{x\in M}$ is a random field on M, with covariance structure given by $\delta_y \delta_x^* 1$. Notice that the variance of $X(x)$ is $|\delta_x^* 1|^2 = \|\delta_x\|_{\mathrm{op}}^2$.

Example 6. As another example, consider a measure space $(M, \mathcal{M}, \mu)$ where, for simplicity, M is supposed to be a metric space. Let $H = L^2(\mu)$, and consider the cylindrical random variable $\mathbb{L}$ on H constructed as a lifting of an orthogonal random measure L. In this case $\delta_x \notin H^*$. For $x \neq y$ in M, choose two disjoint open balls $B_{r_x}(x)$ and $B_{r_y}(y)$ in M with radius r_x and r_y and centers x and y, resp. Introduce two (approximation of the unit) functions ϕ_x and ϕ_y in H which have their respective supports in the balls, i.e. $\mathrm{supp}(\phi_x) \subset B_{r_x}(x)$ and $\mathrm{supp}(\phi_y) \subset B_{r_y}(y)$. Then $\phi_x \perp \phi_y$ and

$$\mathbb{E}[\mathbb{L}(\phi_x)\mathbb{L}(\phi_y)] = \langle \phi_x, \phi_y\rangle = 0.$$

Heuristically, ϕ_x is an approximation of δ_x on H. An informal calculation gives

$$f(x) = \langle f, \delta_x^* 1\rangle = \int_M f(y)\delta_x^* 1(y)\, dy.$$

Thus, $\delta_x^* 1(y) = \delta_x(y)$ for almost all $y \in M$. Hence, we can define $X : M \to L^2(P)$ by $X(x) := \mathbb{L}(\delta_x)$. Then we have that $\mathbb{E}[X(x)X(y)] = \delta_y(x)$.

To this end, we have shown how to lift a metatime $\mathcal{T}$ to a linear operator $\widehat{\mathcal{T}}$ (see the previous section), and how to lift an orthogonal random measure L to a cylindrical random variable $\mathbb{L}$. Our final concern in this section is to study the combination of a cylindrical random measure with a linear operator.

Let $(M_1, \mathcal{M}_1)$ and $(M_2, \mathcal{M}_2)$ be measurable spaces, and let L be an orthogonal random measure on $(M_2, \mathcal{M}_2)$. Let μ_2 be the measure on $(M_2, \mathcal{M}_2)$ induced by L, i.e. $\mu_2(A) = \mathbb{E}[L(A)^2]$ for $A \in \mathcal{M}_2$. For a metatime $\mathcal{T} : \mathcal{M}_1 \to \mathcal{M}_2$, $L_{\mathcal{T}} := L \circ \mathcal{T}$ is an orthogonal random measure on $(M_1, \mathcal{M}_1)$ by Proposition 8. We can therefore lift $L_{\mathcal{T}}$ to a cylindrical random variable $\mathbb{L}_{\mathcal{T}}$ on $L^2(\mu_1)$, where $\mu_1(A) := \mathbb{E}[L_{\mathcal{T}}(A)^2]$. Consider the measure $\mu_{\mathcal{T}} := \mu_2(\mathcal{T}\cdot)$ on $(M_1, \mathcal{M}_1)$. We have that

$$\mu_1(A) = \mathbb{E}[L_{\mathcal{T}}(A)^2] = \mathbb{E}[L(\mathcal{T}(A))^2] = \mu_2(\mathcal{T}(A)) = \mu_{\mathcal{T}}(A).$$

Hence $\mathbb{L}_{\mathcal{T}}$ is a cylindrical random variable on $L^2(\mu_{\mathcal{T}})$.

On the other hand, we can lift the metatime $\mathcal{T}$ to a bounded linear operator $\widehat{\mathcal{T}}$: $L^2(\mu_{\mathcal{T}}) \to L^2(\mu_2)$. For a cylindrical random variable $\mathbb{L}$ on $L^2(\mu_2)$, $\mathbb{L} \circ \widehat{\mathcal{T}}$ is a cylindrical random variable on $L^2(\mu_{\mathcal{T}})$ by the continuity of $\widehat{\mathcal{T}}$. For an elementary function $\phi = \sum_{i=1}^n \phi_i 1_{A_i}$ in $L^2(\mu_{\mathcal{T}})$ it holds that

$$\mathbb{L}(\widehat{\mathcal{T}}(\phi)) = \mathbb{L}(\sum_{i=1}^n \phi_i 1_{\mathcal{T}(A_i)}) = \sum_{i=1}^n \phi_i L(\mathcal{T}(A_i)) = \sum_{i=1}^n \phi_i L_{\mathcal{T}}(A_i) = \mathbb{L}_{\mathcal{T}}(\phi).$$

Hence it follows that $\mathbb{L} \circ \widehat{\mathcal{T}} = \mathbb{L}_{\mathcal{T}}$. This equality can be expressed as

$$\int_{M_1} f(x) L_{\mathcal{T}}(dx) = \int_{M_2} (\widehat{\mathcal{T}} f)(y) L(dy) \tag{9}$$

for $f \in L^2(\mu_{\mathcal{T}})$. This is, of course, simply a change-of-variables formula for the metatime $\mathcal{T}$ (recall the analogous formula (5)).

5 Stochastic processes from cylindrical random variables

In this section we look at some applications of changing the argument in a cylindrical random variable to define stochastic processes. To this end, we consider $\mathbb{R}_+ := [0, \infty)$ and some Hilbert space H, and equip both spaces with their respective Borel σ-algebras. We let $\langle \cdot, \cdot \rangle$ denote the inner product in H and $\| \cdot \|$ denote the induced norm on H. We define *trawl processes*, which are constructed by inserting an H-valued function of time as the argument of a cylindrical random variable.

Definition 7. Let $\mathbb{X} : H \to L^2(P)$ be a cylindrical random variable and let $f : \mathbb{R}_+ \to H$ be a measurable function. Define $\mathbb{X}_f : \mathbb{R}_+ \to L^2(P)$ by $\mathbb{X}_f(t) := \mathbb{X}(f(t))$. Then $\mathbb{X}_f$ is a *trawl process*.

As the cylindrical random variable $\mathbb{X}$ is continuous from H into $L^2(P)$, we find that the trawl process is a measurable mapping from $\mathbb{R}_+$ into $L^2(P)$. A trawl process $\mathbb{X}_f$ is thus a stochastic process with finite second moment. Moreover, if $\mathbb{E}[\mathbb{X}(h)^2] = \|h\|^2$ for all $h \in H$, then

$$\mathbb{E}[\mathbb{X}_f(t)\mathbb{X}_f(s)] = \langle f(t), f(s) \rangle$$

defines the covariance structure of the trawl process.

As the next example shows, our Definition 7 can be particularized to coincide with the classical definition of trawl processes by [5].

Example 7. Let us consider an example of a classical trawl process. Let $\mathbb{L}$ be a cylindrical random variable induced by an orthogonal random measure L with mean zero. In this case, $H = L^2(\mu)$ where μ is induced by L. Let $A : \mathbb{R}_+ \to \mathcal{M}$, and assume that $\mu(A(t)) < \infty$ for all $t \geq 0$. Define $f : \mathbb{R}_+ \to H$ by $f(t) := 1_{A(t)}$ for $t \geq 0$. Suppose furthermore that the family of sets $\{A(t)\}_{t \geq 0}$ is such that $\mathbb{R}_+ \ni t \to f(t) \in H$ is measurable. Define the trawl process $\mathbb{L}_f(t) := \mathbb{L}(f(t))$ for $t \geq 0$. By the construction of $\mathbb{L}$, we find that

$$\mathbb{L}_f(t) = \mathbb{L}(1_{A(t)}) = L(A(t)),$$

and the covariance structure becomes

$$\mathbb{E}[\mathbb{L}_f(t)\mathbb{L}_f(s)] = \mu(A(t) \cap A(s)).$$

Hence, $t \to \mathbb{L}_f(t)$ coincides with the trawl process introduced in [5] (see also [3, Ch. 8]).

Hence, Definition 7 provides us with a generalization of classical trawl processes to cylindrical random variables on a Hilbert space.

Example 8. An example of a trawl process beyond the classical one could be the following. Let U be a separable Hilbert space and define $H = L_{\text{HS}}(U)$, the set of bounded linear operators on H which are Hilbert–Schmidt. Equipped with the Hilbert–Schmidt norm, $L_{\text{HS}}(U)$ is again a separable Hilbert space. Let $\mathbb{X}$ be a cylindrical random variable on H and consider a measurable map $\mathbb{R}_+ \ni t \mapsto \widehat{\mathcal{S}}(t) \in H$. Then $\mathbb{R}_+ \ni t \mapsto \mathbb{X}(\widehat{\mathcal{S}}(t))$ is a trawl process.

Suppose now that H has a partial order $\geq$ and let $\mathbb{X}$ be a cylindrical random variable on H. Let $f : \mathbb{R}_+ \to H$ be a measurable function, and assume that $t \mapsto f(t)$ is monotonely nondecreasing, that is, $f(t) \geq f(s)$ for $t \geq s$. Define the trawl process $\mathbb{X}_f$ as in Definition 7. For $t \geq 0$, define the σ-algebra $\mathcal{F}_t$ generated by $\mathbb{X}_f(s)$ for $s \leq t$. The generating sets of $\mathcal{F}_t$ are

$$\{\omega \in \Omega \,|\, \mathbb{X}_f(s_1) \in C_1, \dots, \mathbb{X}_f(s_n) \in C_n, 0 \leq s_1 < s_2 < \cdots < s_n \leq t\}$$

for any $n \in \mathbb{N}$ and $C_1, \dots, C_n$ Borel sets on $\mathbb{R}$.

Lemma 9. *$\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration and $t \mapsto \mathbb{X}_f(t)$ is $\mathcal{F}_t$-adapted.*

Proof. It is clear that any set in the generator of $\mathcal{F}_s$ is an element of $\mathcal{F}_t$ for $s \leq t$ (in fact, it is in the generator of $\mathcal{F}_t$). Hence, $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s \leq t$. This shows that $\{\mathcal{F}_t\}_{t \geq 0}$ is a filtration. Notice that for a given $t \geq 0$, we have that $\mathbb{X}_f(t) \in L^2(P)$. Therefore it is in particular a random variable (using a representation in the equivalence class), and so $\mathbb{X}_f(t)^{-1}(C) \in \mathcal{F}$ for any Borel set C on $\mathbb{R}$. But $\mathbb{X}_f(t)^{-1}(C)$ is a particular set in the generator of $\mathcal{F}_t$, and hence $\mathbb{X}_f(t)$ is $\mathcal{F}_t$-measurable. Adaptedness follows. □

We can create a trawl process with independent increments using the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

Proposition 11. *Assume that $\mathbb{X}$ is mean-zero Gaussian and orthogonality preserving variable. Assume also that $f(t) - f(s) \perp f(u)$ for all $u \leq s \leq t$. Then $\mathbb{X}_f$ has independent increments with respect to the filtration $\mathcal{F}_t$, and in particular it is an $\mathcal{F}_t$-martingale.*

Proof. Let $0 \leq s < t$. From the linearity of cylindrical random variables we have that

$$\mathbb{X}_f(t) - \mathbb{X}_f(s) = \mathbb{X}(f(t)) - \mathbb{X}(f(s)) = \mathbb{X}(f(t) - f(s)).$$

Since $f(t) - f(s) \perp f(s)$, $\mathbb{X}(f(t) - f(s))$ is orthogonal to $\mathbb{L}(f(u))$ for all $u \leq s$. Therefore $\mathbb{X}(f(t) - f(s))$ is independent of the generating sets of $\mathcal{F}_s$ since $\mathbb{X}$ is Gaussian (with mean zero). This shows the independent increment claim.

As $\mathbb{X}$ is cylindrical, it holds that $\mathbb{X}(f(t)) \in L^1(P)$ for all $t \geq 0$. We find from the independent increment property shown above that

$$\mathbb{E}\left[\mathbb{X}_f(t) - \mathbb{X}_f(s) \,|\, \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{X}_f(t) - \mathbb{X}_f(s)\right].$$

The zero mean assumption on $\mathbb{X}$ proves the martingale property. □

Note that if L is a Gaussian orthogonal random measure on $(M, \mathcal{M})$, then the cylindrical random variable $\mathbb{L}$ constructed in the last section is Gaussian as well. This can be seen from the definition of $\mathbb{L}$ on simple functions ϕ in $L^2(\mu)$, where $\mathbb{L}(\phi) = \sum \phi_i L(A_1)$ is a sum of independent normally distributed random variables, and hence normal. Taking limits in $L^2(\mu)$ preserves normality. Moreover, $\mathbb{L}$ is also orthogonality preserving since it is isometric, and it has mean zero whenever $\mathbb{E}[L(A)] = 0$ for all $A \in \mathcal{M}$.

From the above, we see that we need $t \mapsto f(t)$ to be increasing with respect to some order $\geq$ in H to define a filtration for the trawl. Additionally, we need that $t \mapsto f(t)$ has "orthogonal increments" in H to have an independent increment process for Gaussian cylindrical random variables.

We next consider an extension of trawl processes. First recall the definition of a cylindrical process.

Definition 8. A *cylindrical process* in a Hilbert space H is a family $\{\mathbb{X}(t)\}_{t\geq 0}$ of cylindrical random variables in H.

Definition 9. Let $\mathbb{X} : H \to L^2(P)$ be a cylindrical random variable on H, and let $\widehat{\mathcal{S}} : \mathbb{R}_+ \to L(H)$ be a measurable map. Define the *cylindrical trawl process* $\mathbb{X}_{\widehat{\mathcal{S}}} : \mathbb{R}_+ \times H \to L^2(P)$ by $\mathbb{X}_{\widehat{\mathcal{S}}}(t, h) = \mathbb{X}(\widehat{\mathcal{S}}(t)h)$.

We see that the cylindrical trawl process is a cylindrical process, since for a fixed $t \geq 0$, the map $h \mapsto \mathbb{X}(\widehat{\mathcal{S}}(t)h)$ is linear and continuous by the linearity and continuity of both $\mathbb{X}$ and $\widehat{\mathcal{S}}(t)$. Let us calculate the covariance operator of a cylindrical trawl process when $\mathbb{X}$ is isometric (i.e. $\mathbb{E}[\mathbb{X}(h)^2] = \|h\|^2$ for all $h \in H$). For $h, g \in H$ and $s, t \in \mathbb{R}_+$, we find from the polarization identity and linearity of $\mathbb{X}$ that

$$\begin{aligned}
4\mathbb{E}[\mathbb{X}(\widehat{\mathcal{S}}(t)h)\mathbb{X}(\widehat{\mathcal{S}}(s)g)] &= \mathbb{E}[\mathbb{X}(\widehat{\mathcal{S}}(t)h + \widehat{\mathcal{S}}(s)g))^2] - \mathbb{E}[\mathbb{X}(\widehat{\mathcal{S}}(t)h - \widehat{\mathcal{S}}(s)g))^2] \\
&= \|\widehat{\mathcal{S}}(t)h + \widehat{\mathcal{S}}(s)g\|^2 - \|\widehat{\mathcal{S}}(t)h - \widehat{\mathcal{S}}(s)g\|^2 \\
&= 4\langle \widehat{\mathcal{S}}(s)^*\widehat{\mathcal{S}}(t)h, g\rangle.
\end{aligned}$$

Hence, the covariance operator is

$$\widehat{\mathcal{Q}}_{s,t} := \widehat{\mathcal{S}}(s)^*\widehat{\mathcal{S}}(t). \tag{10}$$

We notice that $\widehat{\mathcal{Q}}^*_{s,t} = \widehat{\mathcal{Q}}_{t,s}$. Furthermore, $\widehat{\mathcal{Q}}_{t,t}$ is symmetric and positive definite.

Consider now a separable Hilbert space H.

Proposition 12. *Suppose that $\widehat{\mathcal{S}}(t) \in L_{HS}(H)$ for each $t \geq 0$. Then $\mathbb{X}_{\widehat{\mathcal{S}}}(t, \cdot) \in H^*$ a.s., that is, there exists an H-valued square-integrable stochastic process $\{X_{\widehat{\mathcal{S}}}(t)\}_{t\geq 0}$ such that $\mathbb{X}_{\widehat{\mathcal{S}}}(t, h) = \langle X_{\widehat{\mathcal{S}}}(t), h\rangle$ a.s., for any $h \in H$.*

Proof. Fix $t \geq 0$ and let $\{e_i\}_{i \in \mathbb{N}}$ be an orthonormal basis (ONB) of H. Define

$$X_{\widehat{\mathcal{S}}}(t) := \sum_{i=1}^{\infty} \mathbb{X}_{\widehat{\mathcal{S}}}(t, e_i) e_i.$$

We show that $X_{\widehat{\mathcal{S}}}(t)$ is an element of H. It holds by Parseval's identity and monotone convergence that

$$\begin{aligned}\mathbb{E}[\|X_{\widehat{\mathcal{S}}}(t)\|^2] &= \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{X}_{\widehat{\mathcal{S}}}(t, e_i)^2] = \sum_{i=1}^{\infty} \mathbb{E}[\mathbb{X}(\widehat{\mathcal{S}}(t) e_i)^2] = \sum_{i=1}^{\infty} \|\widehat{\mathcal{S}}(t) e_i\|^2 \\ &= \|\widehat{\mathcal{S}}(t)\|_{\mathrm{HS}}^2 < \infty.\end{aligned}$$

Thus, $X_{\widehat{\mathcal{S}}}(t) \in H$ a.s.

Let $h \in H$, and $h^n := \sum_{i=1}^n \langle h, e_i \rangle e_i$. Then

$$\mathbb{X}_{\widehat{\mathcal{S}}}(t, h) - \langle X_{\widehat{\mathcal{S}}}(t), h \rangle = \mathbb{X}(\widehat{\mathcal{S}}(t) h) - \langle X_{\widehat{\mathcal{S}}}(t), h^n \rangle + \langle X_{\widehat{\mathcal{S}}}(t), h^n - h \rangle.$$

Since by definition,

$$\langle X_{\widehat{\mathcal{S}}}(t), h^n \rangle = \sum_{i=1}^{n} \mathbb{X}_{\widehat{\mathcal{S}}}(t, e_i) \langle h, e_i \rangle = \mathbb{X}_{\widehat{\mathcal{S}}}(t, h^n),$$

it holds that

$$\mathbb{X}_{\widehat{\mathcal{S}}}(t, h) - \langle X_{\widehat{\mathcal{S}}}(t), h \rangle = \mathbb{X}_{\widehat{\mathcal{S}}}(t, h - h^n) + \langle X_{\widehat{\mathcal{S}}}(t), h^n - h \rangle.$$

By an elementary inequality followed by the Cauchy–Schwarz inequality, we find that

$$\begin{aligned}\mathbb{E}[(\mathbb{X}_{\widehat{\mathcal{S}}}(t, h) - \langle X_{\widehat{\mathcal{S}}}(t), h \rangle)^2] &\leq 2\mathbb{E}[\mathbb{X}_{\widehat{\mathcal{S}}}(t, h - h^n)^2] + 2\mathbb{E}[\langle X_{\widehat{\mathcal{S}}}(t), h^n - h \rangle^2] \\ &\leq 2\|\widehat{\mathcal{S}}(t)(h - h^n)\|^2 + 2\mathbb{E}[\|X_{\widehat{\mathcal{S}}}(t)\|^2] \|h - h^n\|^2 \\ &\leq 2\|\widehat{\mathcal{S}}(t)(h - h^n)\|^2 + 2\|\widehat{\mathcal{S}}(t)\|_{\mathrm{HS}}^2 \|h - h^n\|^2.\end{aligned}$$

But, for any $h \in H$, we find by the triangle inequality and the Cauchy–Schwarz inequality for sequences that

$$\begin{aligned}\|\widehat{\mathcal{S}}(t) h\|^2 &= \| \sum_{i=1}^{\infty} \langle h, e_i \rangle \widehat{\mathcal{S}}(t) e_i \|^2 \\ &\leq \left(\sum_{i=1}^{\infty} |\langle h, e_i \rangle| \|\widehat{\mathcal{S}}(t) e_i\| \right)^2 \\ &\leq \left(\sum_{i=1}^{\infty} \langle h, e_i \rangle^2 \right) \left(\sum_{i=1}^{\infty} \|\widehat{\mathcal{S}}(t) e_i\|^2 \right) \\ &= \|h\|^2 \|\widehat{\mathcal{S}}(t)\|_{\mathrm{HS}}^2.\end{aligned}$$

Thus,

$$\mathbb{E}[(\mathbb{X}_{\widehat{\mathcal{S}}}(t,h) - \langle X_{\widehat{\mathcal{S}}}(t), h\rangle)^2] \leq 4\|\widehat{\mathcal{S}}(t)\|_{\mathrm{HS}}^2 \|h - h^n\|^2.$$

Since $h^n \to h$ in H, it follows that $\mathbb{X}_{\widehat{\mathcal{S}}}(t,h) = \langle X_{\widehat{\mathcal{S}}}(t), h\rangle$ a.s.

If $\{\ell_j\}_{j\in\mathbb{N}}$ is another ONB of H, we define $Y_{\widehat{\mathcal{S}}}(t) := \sum_{j=1}^{\infty} \mathbb{X}_{\widehat{\mathcal{S}}}(t,\cdot)\ell_j$. From the arguments above we have that for any $h \in H$,

$$\langle X_{\widehat{\mathcal{S}}}(t) - Y_{\widehat{\mathcal{S}}}(t), h\rangle = \mathbb{X}_{\widehat{\mathcal{S}}}(t,h) - \mathbb{X}_{\widehat{\mathcal{S}}}(t,h) = 0.$$

Thus, $X_{\widehat{\mathcal{S}}}(t) = Y_{\widehat{\mathcal{S}}}(t)$, and the definition of $X_{\widehat{\mathcal{S}}}(t)$ is independent of the choice of ONB. □

A natural class of time-parametric operators is a C_0-semigroup. Suppose that $\{\widehat{\mathcal{S}}(t)\}_{t\geq 0}$ is a C_0-semigroup on H with generator A being an unbounded operator on H with a densely defined domain denoted $\mathcal{D}(A)$. The semigroup property $\widehat{\mathcal{S}}(t+s) = \widehat{\mathcal{S}}(t)\widehat{\mathcal{S}}(s)$ for $t, s \geq 0$ may be viewed as an extension of the exponential function to infinite dimensions, and as such a cylindrical trawl process can be interpreted as a cylindrical Ornstein–Uhlenbeck process. The covariance operator will be $\widehat{\mathcal{Q}}_{s,t} = \widehat{\mathcal{S}}(s)^*\widehat{\mathcal{S}}(t-s)\widehat{\mathcal{S}}(s)$, assuming $s \leq t$. Also, since the semigroup is the identity operator at time zero, the initial state of the cylindrical trawl process is $\mathbb{X}_{\widehat{\mathcal{S}}}(0,\cdot) = \mathbb{X}(\cdot)$. We also remark that a semigroup family of operators is not in the set of Hilbert–Schmidt operators, as the identity operator is not Hilbert–Schmidt. As the next proposition shows, the cylindrical trawl process has differentiable paths on a dense subset of H.

Proposition 13. *Suppose that $\{\widehat{\mathcal{S}}(t)\}_{t\geq 0}$ is a C_0-semigroup on H, with generator A being unbounded and densely defined. Then, for every $h \in \mathcal{D}(A)$, $t \mapsto \mathbb{X}_{\widehat{\mathcal{S}}}(t,h)$ is differentiable, with derivative given by*

$$\frac{d}{dt}\mathbb{X}_{\widehat{\mathcal{S}}}(t,h) = \mathbb{X}_{\widehat{\mathcal{S}}}(t, Ah).$$

Proof. Let $t, u > 0$ and observe that by the linearity and semigroup property,

$$\frac{1}{u}\left(\mathbb{X}_{\widehat{\mathcal{S}}}(t+u,h) - \mathbb{X}_{\widehat{\mathcal{S}}}(t,h)\right) = \mathbb{X}\left(\widehat{\mathcal{S}}(t)\frac{1}{u}(\widehat{\mathcal{S}}(u)h - h)\right).$$

But as $h \in \mathcal{D}(A)$, $\frac{1}{u}(\widehat{\mathcal{S}}(u)h - h) \to Ah$ in H as $u \downarrow 0$. The result follows from the continuity of $\mathbb{X}$. □

Example 9. Let us return to the examples with translation metatimes, Examples 2 and 4. We recall that for the translation metatime $\mathcal{T}_x$, the associated linear operator is $\widehat{\mathcal{T}}_x = \widehat{\mathcal{S}}_{-x}$, the shift operator. Considering the time-dependent translation metatime on $M = \mathbb{R}^{d+1}$ appearing in the context of trawl processes as discussed in Example 2, $x(t) = (\mathbf{0}, t)$, $t \geq 0$, we see that $x(t+s) = x(t) + x(s)$ for $t, s \geq 0$. By the semigroup property of the shift operator, we see that the time-dependent linear operator associated with $\mathcal{T}_{x(t)}$ is a semigroup since

$$\widehat{\mathcal{S}}_{-x(t+s)} = \widehat{\mathcal{S}}_{-x(t)-x(s)} = \widehat{\mathcal{S}}_{-x(t)}\widehat{\mathcal{S}}_{-x(s)}$$

and as $x(0) = (\mathbf{0}, 0)$, $\widehat{\mathcal{S}}_{-x(0)} = \mathrm{Id}$, the identity operator. Furthermore, notice that if we allow negative times, the semigroup becomes a group. We recall that the operator $\widehat{\mathcal{S}}_{-x}$ in general is defined on the Hilbert space $L^2(\mu_{\mathcal{T}_x})$, so when we vary t we also vary $x(t)$ and thus the space where $\widehat{\mathcal{S}}_{-x(t)}$ is defined. However, if we suppose that $\mu_{\mathcal{T}_x} << \nu$ for some measure ν, uniformly in $x \in M$, with an L^∞-Radon–Nikodym derivative (recalling the discussion in Section 3), we can define the translation semigroup $\widehat{\mathcal{S}}_{-x}$ on $L^2(\nu)$ for all $x \in M$. Notice that the Lebesgue measure on $\mathbb{R}^{d+1}$ is translation invariant, and therefore $Leb_{\mathcal{T}_x}(A) = Leb(A + \{x\}) = Leb(A)$. Hence, $Leb_{\mathcal{T}_x}$ is absolutely continuous with respect to Leb, with Radon–Nikodym derivative being the constant 1. In conclusion, $\widehat{\mathcal{S}}_{-x(t)}$ defines a semigroup on $L^2(\mathbb{R}^{d+1})$. Moreover, by Prop. 8.5 in [7], the translation operator is continuous in $L^2(\mathbb{R}^{d+1})$-norm, and therefore $\widehat{\mathcal{S}}_{-x(t)}$ defines a C_0-semigroup.

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References

[1] Barndorff-Nielsen, O.E.: Lévy bases and extended subordination. Thiele Research Report, University of Aarhus, Denmark 2010-12 (2010)

[2] Barndorff-Nielsen, O.E.: Stationary infinitely divisible processes. Braz. J. Probab. Stat. **25**, 294–322 (2011). MR2832888. https://doi.org/10.1214/11-BJPS140

[3] Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Ambit Stochastics. Springer, Cham (2018). 2018. MR3839270. https://doi.org/10.1007/978-3-319-94129-5

[4] Barndorff-Nielsen, O.E., Pedersen, J.: Meta-times and extended subordination. Theory Probab. Appl. **56**(2), 319–327 (2012). MR3136481. https://doi.org/10.4213/tvp4384

[5] Barndorff-Nielsen, O.E., Lunde, A., Shephard, N., Veraart, A.: Integer-valued trawl processes: a class of stationary infinitely divisible processes. Scand. J. Stat. **41**(3), 693–724 (2014). MR3249424. https://doi.org/10.1111/sjos.12056

[6] Barndorff-Nielsen, O.E., Shiryaev, A.: Change of Time and Change of Measure. World Scientific, Singapore (2015). MR3363697. https://doi.org/10.1142/9609

[7] Folland, G.B.: Real Analysis. Wiley-Interscience, New York (1984). MR0767633

[8] Metivier, M., Pellaumail, J.: Stochastic Integration. Academic Press, New York (1980). MR0578177

[9] Molchanov, I.: Theory of Random Sets. Springer, London (2005). MR2132405

[10] Pedersen, G.K.: Analysis Now. Springer, New York (1989). MR0971256. https://doi.org/10.1007/978-1-4612-1007-8

[11] Rajput, B.S., Rosinski, J.: Spectral representations of infinitely divisible processes. Probab. Theory Relat. Fields **82**(3), 451–487 (1989). 1989. MR1001524. https://doi.org/10.1007/BF00339998

[12] Shiryaev, A.N.: Probability. Springer, New York (1996). MR1368405. https://doi.org/10.1007/978-1-4757-2539-1

[13] Walsh, J.: An introduction to stochastic partial differential equations. In: Carmona, R., Kesten, H., Walsh, J. (eds.) Ecole d'Ete de Probabilites de Saint-Flour XIV. Lecture Notes in Mathematics, vol. 1180. Springer, New York (1984). 1986. MR0876085. https://doi.org/10.1007/BFb0074920