Modern Stochastics: Theory and Applications 7 (4) (2020) 379–394
https://doi.org/10.15559/20-VMSTA164

On shortfall risk minimization for game options

Yan Dolinsky

Hebrew University, Jerusalem

yan.dolinsky@mail.huji.ac.il (Y. Dolinsky)

Received: 13 July 2020, Revised: 21 September 2020, Accepted: 22 October 2020,
Published online: 29 October 2020

Abstract In this paper we study the existence of an optimal hedging strategy for the shortfall risk measure in the game options setup. We consider the continuous time Black–Scholes (BS) model. Our first result says that in the case where the game contingent claim (GCC) can be exercised only on a finite set of times, there exists an optimal strategy. Our second and main result is an example which demonstrates that for the case where the GCC can be stopped on the whole time interval, optimal portfolio strategies need not always exist.

Keywords Complete market, game options, shortfall risk, stochastic optimal control
2010 MSC 91G10, 91E20

1 Introduction

A game contingent claim (GCC) or game option, which was introduced in [8], is defined as a contract between the seller and the buyer of the option such that both have the right to exercise it at any time up to a maturity date (horizon) T. If the buyer exercises the contract at time t then he receives the payment Y_t, but if the seller exercises (cancels) the contract before the buyer then the latter receives X_t. The difference $\Delta_t = X_t - Y_t$ is the penalty which the seller pays to the buyer for the contract cancellation. In short, if the seller will exercise at a stopping time $\sigma \le T$ and the buyer at a stopping time $\tau \le T$ then the former pays to the latter the amount $H(\sigma, \tau)$ where

$$H(\sigma, \tau) := X_\sigma \mathbb{I}_{\sigma<\tau} + Y_\tau \mathbb{I}_{\tau\le\sigma} \tag{1}$$

and we set $\mathbb{I}_Q = 1$ if an event Q occurs and $\mathbb{I}_Q = 0$ if not.

A hedge (for the seller) against a GCC is defined here as a pair (π, σ) which consists of a self–financing strategy π and a stopping time σ which is the cancellation time for the seller. A hedge is called perfect if no matter what exercise time the buyer chooses, the seller can cover his liability to the buyer (with probability 1). The option price is defined as the minimal initial capital which is required for a perfect hedge. Recall (see [8]) that pricing a GCC in a complete market leads to the value of a zero sum optimal stopping (Dynkin's) game under the unique martingale measure. For additional information about pricing of game options see, for instance [5–7, 10–12].

In real market conditions an investor (seller) may not be willing for various reasons to tie in a hedging portfolio the full initial capital required for a perfect hedge. In this case the seller is ready to accept a risk that his portfolio value at an exercise time may be less than his obligation to pay and he will need additional funds to fulfill the contract.

We consider the shortfall risk measure which is given by (see [2])

$$R(\pi, \sigma) := \sup_{\tau} \mathbb{E}_{\mathbb{P}} \left[\left(H(\sigma, \tau) - V^{\pi}_{\sigma \wedge \tau} \right)^{+} \right]$$

where $\{V^{\pi}_t\}_{t=0}^{T}$ is the wealth process of the portfolio strategy π and $\mathbb{E}_{\mathbb{P}}$ denotes the expectation with respect to the market measure. The supremum is taken over all exercise times of the buyer and corresponds to the case where the investor has no information on the buyer exercise strategy. The only assumption is that the buyer exercise strategy is a stopping time with respect to a given filtration.

A natural question to ask, is whether for a given initial capital there exists a hedging strategy which minimizes the shortfall risk (an optimal hedge). For American options the existence of an optimal hedging strategy is proved by applying the Komlós lemma and relies heavily on the fact that the shortfall risk measure is a convex functional of the wealth process (see [14, 16]). For the game options setup, the shortfall risk measure, as a functional of the wealth process is given by

$$R(\pi) := \inf_{\sigma} \sup_{\tau} \mathbb{E}_{\mathbb{P}} \left[\left(H(\sigma, \tau) - V^{\pi}_{\sigma \wedge \tau} \right)^{+} \right]. \tag{2}$$

This functional is not necessarily convex (because of the inf) and so the Komlós lemma can not be applied here.

In this paper we treat the simplest complete, continuous time model, namely the Black–Scholes (BS) model. Our first result (Theorem 1) which is proved in the next section says that for the case where the option can be exercised only on a finite set of times, there exists an optimal hedging strategy. The proof is based on the dynamical programming approach and the randomization technique developed in [17, 18]. Up to date there are several existence results for risk minimization in the game options setup (see [2, 3] and Section 5.2 in [9]). The above papers treat essentially discrete time trading and due to *admissability* conditions the trading strategies are compact. In the current setup trading is done continuously, and so it requires a new method of proof.

In Section 3 we provide the second result of the paper (Theorem 2). This is an example which demonstrates that for the case where the GCC can be stopped on the whole time interval, optimal portfolio strategies need not always exist. We combine the machinery developed in [13] with additional ideas which allow us to treat the shortfall risk measure for game options. Formally, we show that the inf in (2) which ruins the convexity leads to non existence of optimal hedging strategies.

2 Existence result

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a standard one–dimensional Brownian motion $\{W_t\}_{t=0}^{\infty}$, and the filtration $\mathcal{F}_t = \sigma\{W_s | s \leq t\}$ completed by the null sets. We consider a simple BS financial market with time horizon $T < \infty$, which consists of a riskless savings account bearing zero interest (for simplicity) and of a risky asset S, whose value at time t is given by

$$S_t = S_0 \exp\left(\kappa W_t + (\vartheta - \kappa^2/2)t\right), \quad t \in [0, T]$$

where $S_0, \kappa > 0$ and $\vartheta \in \mathbb{R}$ are constants.

Define the exponential martingale

$$Z_t := \exp\left(-\frac{\vartheta}{\kappa} W_t - \frac{\vartheta^2}{2\kappa^2} t\right), \quad t \in [0, T]. \tag{3}$$

From the Girsanov theorem it follows that the probability measure $\mathbb{Q}$ which is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} | \mathcal{F}_t := Z_t, \quad t \in [0, T] \tag{4}$$

is the unique martingale measure for the risky asset S.

Next, let $\mathbb{T} := \{0 = T_0 < T_1 < \cdots < T_n = T\}$ be a finite set of a deterministic times. Consider a game option that can be exercised on the set $\mathbb{T}$. Denote by $\mathcal{T}_{\mathbb{T}}$ the set of all stopping times with values in $\mathbb{T}$. For any $k = 0, 1, \ldots, n$ the payoffs at time T_k are path–independent and given by $Y_{T_k} = f_k(S_{T_k})$ and $X_{T_k} = g_k(S_{T_k})$ where $f_k, g_k : (0, \infty) \to \mathbb{R}$ are measurable functions and $0 \leq f_k \leq g_k$. The payoff function H is given by (1). We will assume the following integrability condition

$$\mathbb{E}_{\mathbb{P}}[X_{T_k}] < \infty, \quad k = 0, 1, \ldots, n. \tag{5}$$

A portfolio strategy with an initial capital $x \geq 0$ is a pair $\pi = (x, \gamma)$ such that $\gamma = \{\gamma_t\}_{t=0}^{T}$ is a predictable S–integrable process and the corresponding wealth process

$$V_t^{\pi} := x + \int_0^t \gamma_u dS_u, \quad t \in [0, T]$$

satisfies the *admissibility* condition $V_t^{\pi} \geq 0$ a.s. for all t.

Let us recall some elementary properties that will be used in the sequel (for details see Chapters IV-V in [19]). The continuity of S implies that the wealth process $\{V_t^{\pi}\}_{t=0}^{T}$ is continuous as well. Moreover, since $\{S_t\}_{t=0}^{T}$ is a $\mathbb{Q}$–martingale then the wealth process $\{V_t^{\pi}\}_{t=0}^{T}$ is a $\mathbb{Q}$–local martingale, and so from the *admissibility* condition we get that $\{V_t^{\pi}\}_{t=0}^{T}$ is a $\mathbb{Q}$–super martingale. On the other hand, due to the martingale representation theorem, for any nonnegative $\mathbb{Q}$–martingale $\{M_t\}_{t=0}^T$ there exists a portfolio strategy π such that $V_t^\pi = M_t$ for all t a.s.

For any $x \geq 0$ denote by $\mathcal{A}(x)$ the set of all portfolio strategies with an initial capital x. A hedging strategy with an initial capital x is a pair $(\pi, \sigma) \in \mathcal{A}(x) \times \mathcal{T}_{\mathbb{T}}$.

The shortfall risk measure is given by

$$R_{\mathbb{T}}(\pi, \sigma) := \sup_{\tau \in \mathcal{T}_{\mathbb{T}}} \mathbb{E}_{\mathbb{P}} \left[\left(H(\sigma, \tau) - V^\pi_{\sigma \wedge \tau} \right)^+ \right], \quad (\pi, \sigma) \in \mathcal{A}(x) \times \mathcal{T}_{\mathbb{T}},$$
$$R_{\mathbb{T}}(x) := \inf_{(\pi,\sigma) \in \mathcal{A}(x) \times \mathcal{T}_{\mathbb{T}}} R_{\mathbb{T}}(\pi, \sigma).$$

Now, we are ready to formulate our first result.

Theorem 1. *For any $x \geq 0$ there exists a hedging strategy $(\hat{\pi}, \hat{\sigma}) \in \mathcal{A}(x) \times \mathcal{T}_{\mathbb{T}}$ such that*

$$R_{\mathbb{T}}(\hat{\pi}, \hat{\sigma}) = R_{\mathbb{T}}(x).$$

Remark 1. We emphasis that in contrast to previous work on game options (see [2, 3] and Section 5.2 in [9]) the trading in our setup is done continuously. Namely, the investor trades the risky asset continuously, but the GCC can be exercised only on a finite set of deterministic times. This can be viewed as a game version of the Bermudan options.

2.1 Proof of Theorem 1

We start with some preparations. Let $U : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be a measurable function such that for any $y > 0$, $U(\cdot, y)$ is a bounded, nondecreasing and continuous function. Let $U_c : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be the concave envelop of U with respect to the first variable. Namely, for any $y > 0$ the function $U_c(\cdot, y)$ is the minimal concave function which satisfies $U_c(\cdot, y) \geq U(\cdot, y)$. Clearly, U_c is continuous in the first variable. Thus, for any $y > 0$ the set $\{x : U(x, y) < U_c(x, y)\}$ is open and so can be written as a countable union of disjoint intervals

$$\{x : U(x, y) < U_c(x, y)\} = \bigcup_{n \in \mathbb{N}} (a_n(y), b_n(y)). \tag{6}$$

From Lemma 2.8 in [17] it follows that $U_c(\cdot, y)$ is affine on each of the intervals $(a_n(y), b_n(y))$. Since U, U_c are continuous in the first variable, then the functions $a_n, b_n : (0, \infty) \to \mathbb{R}_+$, $n \in \mathbb{N}$ are determined by the countable collection of functions $U(q, \cdot), U_c(q, \cdot) : (0, \infty) \to \mathbb{R}$ for nonnegative rational q, and so they are measurable.

For any $0 \leq t_1 < t_2 \leq T$ and a $\mathcal{F}_{t_1}$ measurable random variable $\Theta_1 \geq 0$ denote by $\mathcal{H}_{t_1,t_2}(\Theta_1)$ the set of all random variables $\Theta_2 \geq 0$ which are $\mathcal{F}_{t_2}$ measurable and satisfy $\mathbb{E}_{\mathbb{Q}}(\Theta_2 | \mathcal{F}_{t_1}) \leq \Theta_1$.

The following auxiliary result is an extension of Theorem 5.1 in [17].

Lemma 1. *Let $0 \leq t_1 < t_2 \leq T$ and let $\Theta_1 \geq 0$ be a $\mathcal{F}_{t_1}$ measurable random variable. For a function U as above, assume that there exists a function $G : \mathbb{R} \to \mathbb{R}$ such that $|U(x, y)| \leq G(y)$ for all x, y and $\mathbb{E}_{\mathbb{P}}[G(S_{t_2})] < \infty$. Then there exists a random variable $\Theta \in \mathcal{H}_{t_1,t_2}(\Theta_1)$ such that*

$$\mathbb{E}_{\mathbb{P}} \left[U(\Theta, S_{t_2}) | \mathcal{F}_{t_1} \right]$$

$$= ess\ \sup_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[U(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$= ess\ \sup_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[U_c(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right].$$

Proof. Since $U_c \geq U$, it is sufficient to show that there exists $\Theta \in \mathcal{H}_{t_1,t_2}(\Theta_1)$ such that

$$\mathbb{E}_{\mathbb{P}}\left[U(\Theta, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$= ess\ \sup_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[U_c(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right].$$

Choose a sequence $\Theta^{(n)} \in \mathcal{H}_{t_1,t_2}(\Theta_1)$, $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[U_c(\Theta^{(n)}, S_{t_2})|\mathcal{F}_{t_1}\right] = ess \sup_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[U_c(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right]. \tag{7}$$

From Lemma A1.1 in [4] we obtain a sequence $\Lambda^{(m)} \in conv(\Theta^{(m)}, \Theta^{(m+1)}, \ldots)$, $m \in \mathbb{N}$ converging $\mathbb{P}$ a.s. to a random variable Λ. The Fatou lemma implies that $\Lambda \in \mathcal{H}_{t_1,t_2}(\Theta_1)$.

By applying the dominated convergence theorem, the inequality $|U_c(\cdot, S_{t_2})| \leq G(S_{t_2})$ and the fact that U_c is concave and continuous in the first variable we obtain

$$\mathbb{E}_{\mathbb{P}}\left[U_c(\Lambda, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$= \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[U_c(\Lambda^{(n)}, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$\geq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[U_c(\Lambda^{(n)}, S_{t_2})|\mathcal{F}_{t_1}\right].$$

This together with (7) gives

$$\mathbb{E}_{\mathbb{P}}\left[U_c(\Lambda, S_{t_2})|\mathcal{F}_{t_1}\right] = ess \sup_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[U_c(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right]. \tag{8}$$

Next, introduce the normal random variable

$$\Gamma := (W_{\frac{t_1+t_2}{2}} - W_{t_1}) - \frac{1}{2}(W_{t_2} - W_{t_1}).$$

Observe that $\mathbb{E}_{\mathbb{P}}[\Gamma W_{t_2}] = 0$ and so we conclude that Γ is independent of the $\sigma-$algebra generated by $W_t, t \in [0, t_1] \cup \{t_2\}$. From Theorem 1 in [20] it follows that there exists a measurable function $\Phi : C[0, t_1] \times \mathbb{R}^2 \to \mathbb{R}$ such that we have the following equality of the joint laws

$$\left(\left(W_{[0,t_1]}, W_{t_2}, \Lambda\right); \mathbb{P}\right) = \left(\left(W_{[0,t_1]}, W_{t_2}, \Phi\left(W_{[0,t_1]}, W_{t_2}, \Gamma\right)\right); \mathbb{P}\right).$$

In particular $\Phi\left(W_{[0,t_1]}, W_{t_2}, \Gamma\right) \in \mathcal{H}_{t_1,t_2}(\Theta_1)$ and

$$\mathbb{E}_{\mathbb{P}}\left[U_c\left(\Lambda, S_{t_2}\right)|\mathcal{F}_{t_1}\right] = \mathbb{E}_{\mathbb{P}}\left[U_c\left(\Phi\left(W_{[0,t_1]}, W_{t_2}, \Gamma\right), S_{t_2}\right)|\mathcal{F}_{t_1}\right].$$

Thus, without loss of generality we assume that $\Lambda = \Phi\left(W_{[0,t_1]}, W_{t_2}, \Gamma\right)$.

We arrive to the final step of the proof. Introduce the normal random variable

$$\hat{\Gamma} := (W_{\frac{t_1+2t_2}{3}} - W_{t_1}) - \frac{2}{3}(W_{t_2} - W_{t_1}) - \frac{2}{3}\Gamma.$$

Observe that $\mathbb{E}_{\mathbb{P}}[\hat{\Gamma} W_{t_2}] = \mathbb{E}_{\mathbb{P}}[\hat{\Gamma}\Gamma] = 0$. Thus, $\hat{\Gamma}$ is independent of the σ–algebra generated by $W_t, t \in [0, t_1] \cup \{t_2\}$ and Γ. Let F^{-1} be the inverse function of the cumulative distribution function $F(\cdot) := \mathbb{P}(\hat{\Gamma} \leq \cdot)$. Recall (6) and define the random variable

$$\Theta := \Lambda \mathbb{I}_{\Lambda \notin \bigcup_{n\in\mathbb{N}}(a_n(S_{t_2}), b_n(S_{t_2}))}$$
$$+ \sum_{n\in\mathbb{N}} b_n(S_{t_2}) \mathbb{I}_{\Lambda \in (a_n(S_{t_2}), b_n(S_{t_2}))} \mathbb{I}_{\hat{\Gamma} < F^{-1}\left(\frac{\Lambda - a_n(S_{t_2})}{b_n(S_{t_2}) - a_n(S_{t_2})}\right)}$$
$$+ \sum_{n\in\mathbb{N}} a_n(S_{t_2}) \mathbb{I}_{\Lambda \in (a_n(S_{t_2}), b_n(S_{t_2}))} \mathbb{I}_{\hat{\Gamma} > F^{-1}\left(\frac{\Lambda - a_n(S_{t_2})}{b_n(S_{t_2}) - a_n(S_{t_2})}\right)}.$$

Let $\mathcal{G}$ be the σ–algebra generated by $W_t, t \in [0, t_1] \cup \{t_2\}$ and Γ. From the Bayes theorem, the tower property for conditional expectation and (4) we get

$$\mathbb{E}_{\mathbb{Q}}\left(\Theta|\mathcal{F}_{t_1}\right) = \mathbb{E}_{\mathbb{P}}\left(\frac{\Theta Z_{t_2}}{Z_{t_1}}|\mathcal{F}_{t_1}\right)$$
$$= \mathbb{E}_{\mathbb{P}}\left(\mathbb{E}_{\mathbb{P}}\left(\frac{\Theta Z_{t_2}}{Z_{t_1}}|\mathcal{G}\right)|\mathcal{F}_{t_1}\right) = \mathbb{E}_{\mathbb{P}}\left(\frac{\Lambda Z_{t_2}}{Z_{t_1}}|\mathcal{F}_{t_1}\right) = \mathbb{E}_{\mathbb{Q}}\left(\Lambda|\mathcal{F}_{t_1}\right).$$

Thus $\Theta \in \mathcal{H}_{t_1,t_2}(\Theta_1)$. Finally, let us notice that $U(\Theta, S_{t_2}) = U_c(\Theta, S_{t_2})$, and so, from the tower property of conditional expectation and the fact that $U_c(\cdot, y)$ is affine on each of the intervals $(a_n(y), b_n(y))$ we obtain

$$\mathbb{E}_{\mathbb{P}}\left[U(\Theta, S_{t_2})|\mathcal{F}_{t_1}\right] = \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(U(\Theta, S_{t_2})|\mathcal{G}\right)|\mathcal{F}_{t_1}\right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left(U_c(\Theta, S_{t_2})|\mathcal{G}\right)|\mathcal{F}_{t_1}\right] = \mathbb{E}_{\mathbb{P}}\left[U_c(\Lambda, S_{t_2})|\mathcal{F}_{t_1}\right].$$

This together with (8) completes the proof. □

We arrive at the following Corollary.

Corollary 1. *Let $B : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be a measurable function such that for any $y > 0$, $B(\cdot, y)$ is a bounded, nonincreasing and continuous function. Let $B^c : [0, \infty) \times (0, \infty) \to \mathbb{R}$ be the convex envelop of B with respect to the first variable.*

(i). Let $0 \leq t_1 < t_2 \leq T$ and let $\Theta_1 \geq 0$ be a $\mathcal{F}_{t_1}$ measurable random variable. Assume that there exists a function $G : \mathbb{R} \to \mathbb{R}$ such that $|B(x, y)| \leq G(y)$ for all x, y and $\mathbb{E}_{\mathbb{P}}[G(S_{t_2})] < \infty$. Then there exists a random variable $\Theta \in \mathcal{H}_{t_1,t_2}(\Theta_1)$ such that

$$\mathbb{E}_{\mathbb{P}}\left[B(\Theta, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$= ess\inf_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[B(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right]$$
$$= ess\inf_{\Theta_2 \in \mathcal{H}_{t_1,t_2}(\Theta_1)} \mathbb{E}_{\mathbb{P}}\left[B^c(\Theta_2, S_{t_2})|\mathcal{F}_{t_1}\right].$$

(ii). Let $t_1 = 0$. The function $b : [0, \infty) \to \mathbb{R}$ which is defined by

$$b(x) := \inf_{\Theta \in \mathcal{H}_{0,t_2}(x)} \mathbb{E}_{\mathbb{P}}\left[B(\Theta, S_{t_2})\right] = \inf_{\Theta \in \mathcal{H}_{0,t_2}(x)} \mathbb{E}_{\mathbb{P}}\left[B^c(\Theta, S_{t_2})\right], \quad x \geq 0$$

is convex and continuous.

Proof. (i). The result follows immediately by applying Lemma 1 for $U := -B$.

(ii). The convexity of b follows from the convexity of B^c in the first variable and the fact that for any $x_1, x_2 \geq 0$ and $\lambda \in (0, 1)$,

$$\lambda \mathcal{A}(x_1) + (1-\lambda)\mathcal{A}(x_2) \subset \mathcal{A}(\lambda x_1 + (1-\lambda)x_2).$$

In particular b is continuous in $(0, \infty)$. It remans to prove continuity at $x = 0$. Since B is nonincreasing in the first variable, then b is nonincreasing as well. Thus, it is sufficient to show that $b(0) \leq \lim_{n\to\infty} b(1/n)$. To that end, choose $\Theta^{(n)} \in \mathcal{H}_{0,t_2}(1/n)$, $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[B^c(\Theta^{(n)}, S_{t_2})\right] = \lim_{n\to\infty} b(1/n).$$

From Lemma A1.1 in [4] we obtain a sequence $\Lambda^{(m)} \in conv(\Theta^{(m)}, \Theta^{(m+1)}, \ldots)$, $m \in \mathbb{N}$ converging $\mathbb{P}$ a.s. to a random variable Λ. The Fatou lemma implies that $\Lambda = 0$, and so by applying the dominated convergence theorem together with convexity and continuity of B^c in the first variable we get,

$$b(0) = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[B^c(\Lambda^{(n)}, S_{t_2})\right] \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}\left[B^c(\Theta^{(n)}, S_{t_2})\right] = \lim_{n\to\infty} b(1/n)$$

and continuity follows. □

Now we are ready to prove Theorem 1.

Proof. Let $x \geq 0$. For any $\pi \in \mathcal{A}(x)$ we define $R_{\mathbb{T}}(\pi)$ as in (2) where the infimum and the supremum are taken over the set $\mathcal{T}_{\mathbb{T}}$.

Moreover, define the random variables Ψ_k^π, $k = 0, 1, \ldots, n$ by

$$\Psi_n^\pi := \left(Y_T - V_T^\pi\right)^+$$

and for $k = 0, 1, \ldots, n-1$ by the recursive relations

$$\Psi_k^\pi := \min\left(\left(X_{T_k} - V_{T_k}^\pi\right)^+, \max\left(\left(Y_{T_k} - V_{T_k}^\pi\right)^+, \mathbb{E}_{\mathbb{P}}(\Psi_{k+1}^\pi | \mathcal{F}_{T_k})\right)\right). \tag{9}$$

In view of (5) the random variables Ψ_k^π, $k = 0, 1, \ldots, n$ are well defined. From the standard theory of zero–sum Dynkin games (see [15]) it follows that

$$\Psi_0^\pi = R_{\mathbb{T}}(\pi).$$

Moreover, for the stopping time

$$\sigma := T \wedge \min\left\{t \in \mathbb{T} : \Psi_t^\pi = \left(X_t - V_t^\pi\right)^+\right\}$$

we have $R_{\mathbb{T}}(\pi) = R_{\mathbb{T}}(\pi, \sigma)$.

Thus, in order to conclude the proof we need to show that there exists $\hat{\pi} \in \mathcal{A}(x)$ such that

$$\Psi_0^{\hat{\pi}} = \inf_{\pi \in \mathcal{A}(x)} \Psi_0^\pi. \tag{10}$$

We apply dynamical programming. Introduce the functions $B_k : [0,\infty)\times(0,\infty)\to\mathbb{R}$, $k=0,1,\dots,n$ by

$$B_n(z,y) := (f_n(y)-z)^+,$$

and for $k=0,1,\dots,n-1$ by the recursive relations

$$B_k(z,y) = \min\Big((g_k(y)-z)^+, \max\Big((f_k(y)-z)^+, \\ \inf_{\Theta_{k+1}\in\mathcal{H}_{0,T_{k+1}-T_k}(z)} \mathbb{E}_{\mathbb{P}}\left[B_{k+1}(\Theta_{k+1}, yS_{T_{k+1}-T_k})\right]\Big)\Big).$$

Let us argue by backward induction that for any k, $B_k(z,y)$ is measurable, and for any y the function $B_k(\cdot,y)$ is continuous and nonincreasing. For $k=n$ this is clear. Assume that the statement holds for $k+1$, let us prove it for k. From Corollary 1(ii) it follows that for any y the function $B_k(\cdot,y)$ is continuous and nonincreasing. For any $z>0$ the measurability of the function $B_k(z,\cdot)$ follows from the fact that the set $\mathcal{H}_{0,T_{k+1}-T_k}(z)$ is separable (with respect to convergence in probability). Since B_k is continuous in the first variable we conclude joint measurability and complete the argument.

Next, from Corollary 1(i) it follows that we can construct a sequence of random variables D_0, D_1, ..., D_n such that $D_0=x$ and for any $k=1,\dots,n$ $D_k\in\mathcal{H}_{T_{k-1},T_k}(D_{k-1})$ satisfies

$$\mathbb{E}_{\mathbb{P}}\left[B_k(D_k,S_{T_k})|\mathcal{F}_{T_{k-1}}\right] = ess \inf_{\Theta_k\in\mathcal{H}_{T_{k-1},T_k}(D_{k-1})} \mathbb{E}_{\mathbb{P}}\left[B_k(\Theta_k,S_{T_k})|\mathcal{F}_{T_{k-1}}\right]. \quad (11)$$

Since B_k, $k=0,1,\dots,n$ are nonincreasing in the first variable then without loss of generality we assume that $\mathbb{E}_{\mathbb{Q}}[D_k|\mathcal{F}_{T_{k-1}}]=D_{k-1}$ for all k.

Finally, the completeness of the BS model implies that there exists $\hat{\pi}\in\mathcal{A}(x)$ such that $V^{\hat{\pi}}_{T_k}=D_k$ for all $k=0,1,\dots,n$. Observe that $\frac{S_{T_k}}{S_{T_{k-1}}}$ is independent of $\mathcal{F}_{T_{k-1}}$ and has the same distribution as $S_{T_k-T_{k-1}}$. Thus, from (9) and (11) we obtain (by backward induction)

$$B_k(V^{\hat{\pi}}_{T_k},S_{T_k}) = \Psi^{\hat{\pi}}_k \ \text{ a.s. } \forall k=0,1,\dots,n. \quad (12)$$

On the other hand, for an arbitrary $\pi\in\mathcal{A}(x)$ we have $V^{\pi}_{T_k}\in\mathcal{H}_{T_{k-1},T_k}(V^{\pi}_{T_{k-1}})$, $k=1,\dots,n$. Hence, similar arguments as before (12) yield

$$B_k(V^{\pi}_{T_k},S_{T_k}) \le \Psi^{\pi}_k \ \text{ a.s. } \forall k=0,1,\dots,n. \quad (13)$$

By combining (12)–(13) for $k=0$ gives that for any $\pi\in\mathcal{A}(x)$

$$\Psi^{\hat{\pi}}_0 = B_0(x,S_0) \le \Psi^{\pi}_0$$

and (10) follows. □

Remark 2. We observe that the proof of Theorem 1 and Lemma 1 can be adjusted to the case where the volatility and the drift are deterministic functions of time. However, in order to make the presentation more friendly we assume constant parameters.

3 Example where no optimal strategy exists

In this section we consider a game option which can be exercised at any time in the interval [0, 1]. The payoffs are given by

$$X_t = (1 + \sin(\pi t)) \max(Z_t, 1/2), \quad t \in [0, 1]$$
$$Y_1 = X_1,$$
$$Y_t = 0, \quad \text{for } t < 1$$

where Z_t was defined in (3). Notice that $\mathbb{E}_{\mathbb{P}}[\sup_{0\le t\le 1} X_t] < \infty$.

Denote by $\mathcal{T}$ the set of all stopping times with values in the interval [0, 1]. Obviously, the equalities $Y_{[0,1)} \equiv 0$ and $Y_1 = X_1$ imply that (the buyer of the game option will not stop before $t = 1$) the shortfall risk measure is given by

$$R(\pi, \sigma) = \mathbb{E}_{\mathbb{P}}\left[\left(X_\sigma - V_\sigma^\pi\right)^+\right].$$

As in (2), for a portfolio strategy π we have

$$R(\pi) := \inf_{\sigma \in \mathcal{T}} R(\pi, \sigma) = \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[\left(X_\sigma - V_\sigma^\pi\right)^+\right].$$

Similarly to Section 2, for an initial capital x we define

$$R(x) := \inf_{(\pi,\sigma)\in\mathcal{A}(x)\times\mathcal{T}} R(\pi, \sigma) = \inf_{(\pi,\sigma)\in\mathcal{A}(x)\times\mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[\left(X_\sigma - V_\sigma^\pi\right)^+\right].$$

For any π the process $\{(X_t - V_t^\pi)^+\}_{t=0}^1$ is continuous, and so from the general theory of optimal stopping (see Section 6 in [9]) it follows that there exists $\sigma = \sigma(\pi)$ such that $R(\pi, \sigma) = R(\pi)$. Namely, the existence of an optimal hedging strategy is equivalent to the existence of an optimal portfolio strategy. We say that $\pi \in \mathcal{A}(x)$ is an optimal portfolio strategy if $R(\pi) = R(x)$.

We arrive at the main result.

Theorem 2. *Assume that the drift term $\vartheta \neq 0$ and let*

$$\nu := \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[Z_1 \mathbb{I}_{Z_1<1/2}\right], \tag{14}$$

observe that $\vartheta \neq 0$ implies that $\nu > 0$. Then for any initial capital $x \in (0, \nu)$ there is no optimal strategy.

Remark 3. If $\vartheta = 0$ then $\mathbb{P} = \mathbb{Q}$. In this specific case (see Theorem 7.1 in [2]) there exists an optimal hedging strategy.

Remark 4. Let us notice that for a given stopping time $\sigma \in \mathcal{T}$ the functional

$$\pi \to R(\pi, \sigma) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[\left(H(\sigma, \tau) - V_{\sigma\wedge\tau}^\pi\right)^+\right]$$

is convex. Thus, by following the same arguments as in [14] (which are based on the Komlós lemma) one can prove that (for a given σ) the infimum in the expression $\inf_{\pi\in\mathcal{A}(x)} R(\pi, \sigma)$ is attained. Hence, Theorem 2 implies that for any $x \in (0, \nu)$ we have the following

$$\inf_{\pi\in\mathcal{A}(x)} R(\pi, \sigma) = \min_{\pi\in\mathcal{A}(x)} R(\pi, \sigma) > R(x) \ \forall \sigma \in \mathcal{T}_T.$$

Before we prove Theorem 2 we will need some auxiliary results. We start with the following lemma.

Lemma 2. *The function* $R : [0, \infty) \to [0, \infty)$ *is convex and continuous. Namely, the shortfall risk measure* R *is convex and continuous as a function of the initial capital.*

Proof. The proof will be done by approximating $R(\cdot)$. For any $n \in \mathbb{N}$ let $\mathcal{T}_n$ be the set of all stopping times with values in the set $\{1/n, 2/n, \ldots, 1\}$ (0 is not included). Set,

$$R_n(\pi) := \inf_{\sigma \in \mathcal{T}_n} R(\pi, \sigma)$$
$$R_n(x) := \inf_{\sigma \in \mathcal{T}_n} \inf_{\pi \in \mathcal{A}(x)} R(\pi, \sigma).$$

We argue that R_n converge uniformly to R. First, we have the obvious observation $R_n(\cdot) \geq R(\cdot)$. Next, let $x \geq 0$ and $(\pi, \sigma) \in \mathcal{A}(x) \times \mathcal{T}$. Define $\sigma_n \in \mathcal{T}_n$ by

$$\sigma_n := \frac{1}{n} \min\{k \in \mathbb{N} : \ k/n \geq \sigma\}.$$

Clearly, $\sigma_n \geq \sigma$. Thus, there exists a portfolio $\pi_n \in \mathcal{A}(x)$ such that $V^{\pi_n}_{\sigma_n} = V^{\pi}_{\sigma}$. From the inequality $\sigma_n - \sigma \leq 1/n$ we obtain

$$R(\pi_n, \sigma_n) - R(\pi, \sigma) \leq \mathbb{E}_{\mathbb{P}} \left[|X_\sigma - X_{\sigma_n}| \right] \leq \mathbb{E}_{\mathbb{P}} \left[\sup_{|t-s| \leq 1/n} |X_t - X_s| \right].$$

Since $(\pi, \sigma) \in \mathcal{A}(x) \times \mathcal{T}$ was arbitrary we conclude that

$$0 \leq R_n(x) - R(x) \leq \mathbb{E}_{\mathbb{P}} \left[\sup_{|t-s| \leq 1/n} |X_t - X_s| \right].$$

From the dominated convergence theorem

$$\lim_{n \to \infty} \mathbb{E}_{\mathbb{P}} \left[\sup_{|t-s| \leq 1/n} |X_t - X_s| \right] = 0$$

and uniform convergence follows.

It remains to argue that for any n the function $R_n : [0, \infty) \to [0, \infty)$ is convex and continuous. Fix $n \in \mathbb{N}$. For any $k = 1, \ldots, n$ let $g_k : (0, \infty) \to (0, \infty)$ be such that $X_{k/n} = g_k(S_{k/n})$. Introduce the functions $\hat{B}_k : [0, \infty) \times (0, \infty) \to \mathbb{R}$, $k = 0, 1, \ldots, n$ by

$$\hat{B}_n(z, y) := (g_n(y) - z)^+,$$

for $k = 1, \ldots, n-1$ by the recursive relations

$$\hat{B}_k(z, y) = \min \left((g_k(y) - z)^+, \inf_{\Theta_{k+1} \in \mathcal{H}_{0,1/n}(z)} \mathbb{E}_{\mathbb{P}} \left[\hat{B}_{k+1}(\Theta_{k+1}, y S_{1/n}) \right] \right),$$

and for $k = 0$

$$\hat{B}_0(z, y) = \inf_{\Theta_1 \in \mathcal{H}_{0,1/n}(z)} \mathbb{E}_{\mathbb{P}} \left[\hat{B}_1(\Theta_1, y S_{1/n}) \right]. \tag{15}$$

Observe that $R_n(\cdot)$ is "almost" as $R_{\mathbb{T}}(\cdot)$ defined in Section 2 for the set $\mathbb{T} := \{0, 1/n, 2/n, \ldots, 1\}$, the only difference is that for $R_n(x)$ stopping at zero is not allowed. This is why in (15) we do not take minimum with $(g_0(y)-z)^+$. Using similar arguments as in the proof of Theorem 1 we obtain that $R_n(x) = \hat{B}_0(x, S_0)$. Finally, from Corollary 1(ii) we get that for any y, $\hat{B}_0(\cdot, y)$ is convex and continuous. This completes the proof. $\square$

Next, we observe that for any stopping time $\sigma \in \mathcal{T}$ and $\lambda > 0$

$$\inf_{\Upsilon \geq 0} \left[(X_\sigma - \Upsilon)^+ + \lambda Z_\sigma \Upsilon\right] = X_\sigma \min(1, \lambda Z_\sigma). \tag{16}$$

This brings us to introducing the function

$$F(\lambda) = \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}} \left[X_\sigma \min(1, \lambda Z_\sigma)\right], \quad \lambda > 0. \tag{17}$$

Obviously $F : (0, \infty) \to [0, \infty)$ is concave and nondecreasing. Inspired by Corollary 8.3 in [13] we prove the following.

Lemma 3. *(i). For any $x \geq 0$ and $\lambda > 0$,*

$$R(x) \geq F(\lambda) - \lambda x.$$

(ii). Let $\lambda > 0$ be such that F is differentiable at λ. Then for $x = F'(\lambda)$ we have the equality

$$R(x) = F(\lambda) - \lambda x.$$

Proof. (i). Let $x \geq 0$ and $\lambda > 0$. Choose arbitrary $(\pi, \sigma) \in \mathcal{A}(x) \in \mathcal{T}$. Then, from the super–martingale property of an *admissible* portfolio we have

$$x = V_0^\pi \geq \mathbb{E}_{\mathbb{Q}}[V_\sigma^\pi] = \mathbb{E}_{\mathbb{P}}[Z_\sigma V_\sigma^\pi]. \tag{18}$$

This together with (16) gives

$$R(\pi, \sigma) + \lambda x \geq \mathbb{E}_{\mathbb{P}} \left[(X_\sigma - V_\sigma^\pi)^+ + \lambda Z_\sigma V_\sigma^\pi\right] \geq F(\lambda).$$

Since $(\pi, \sigma) \in \mathcal{A}(x) \in \mathcal{T}$ was arbitrary we complete the proof.

(ii). In view of (i), it is sufficient to show that $R(x) \leq F(\lambda) - \lambda x$. Let $\sigma_\lambda \in \mathcal{T}$ be an optimal stopping time in (17), i.e.

$$F(\lambda) = \mathbb{E}_{\mathbb{P}} \left[X_{\sigma_\lambda} \min(1, \lambda Z_{\sigma_\lambda})\right]. \tag{19}$$

Such stopping time exists because the process $\{X_t \min(1, \lambda Z_t)\}_{t=0}^1$ is continuous. Set $\Upsilon_\lambda = X_{\sigma_\lambda} \mathbb{I}_{Z_{\sigma_\lambda} < 1/\lambda}$. From (16) it follows that for any $\tilde{\lambda} > 0$

$$F(\tilde{\lambda}) \leq \mathbb{E}_{\mathbb{P}} \left[(X_{\sigma_\lambda} - \Upsilon_\lambda)^+ + \tilde{\lambda} Z_{\sigma_\lambda} \Upsilon_\lambda\right].$$

On the other hand from (19)

$$F(\lambda) = \mathbb{E}_{\mathbb{P}} \left[(X_{\sigma_\lambda} - \Upsilon_\lambda)^+ + \lambda Z_{\sigma_\lambda} \Upsilon_\lambda\right].$$

Thus,

$$\frac{F(\tilde{\lambda})-F(\lambda)}{\tilde{\lambda}-\lambda} \leq \mathbb{E}_{\mathbb{P}}\left[Z_{\sigma_\lambda}\Upsilon_\lambda\right], \quad \text{for } \tilde{\lambda} > \lambda$$
$$\text{and } \frac{F(\tilde{\lambda})-F(\lambda)}{\tilde{\lambda}-\lambda} \geq \mathbb{E}_{\mathbb{P}}\left[Z_{\sigma_\lambda}\Upsilon_\lambda\right] \quad \text{for } \tilde{\lambda} < \lambda.$$

From the fact that $F'(\lambda) = x$ we conclude that

$$x = \mathbb{E}_{\mathbb{P}}\left[Z_{\sigma_\lambda}\Upsilon_\lambda\right] = \mathbb{E}_{\mathbb{Q}}\left[\Upsilon_\lambda\right].$$

The completeness of the BS model implies that there exists $\pi \in \mathcal{A}(x)$ such that $V^{\pi}_{\sigma_\lambda} = \Upsilon_\lambda$. From (19) we get

$$\begin{aligned} R(x) + \lambda x \leq R(\pi, \sigma_\lambda) + \lambda x &= \mathbb{E}_{\mathbb{P}}\left[(X_{\sigma_\lambda} - \Upsilon_\lambda)^+ + \lambda Z_{\sigma_\lambda}\Upsilon_\lambda\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[X_{\sigma_\lambda}\min(1, \lambda Z_{\sigma_\lambda})\right] = F(\lambda) \end{aligned}$$

as required. □

While Lemmas 2–3 are quite general, the following lemma uses the explicit structure of the payoff process $\{X_t\}_{t=0}^1$.

Lemma 4. *(i). For any $\lambda \geq 2$, $F(\lambda) = 1$.*

(ii). The derivative of F from the left (exists because F is concave) satisfies $F'_{-}(2) \geq \nu$ where ν is given by (14).

Proof. (i). Let $\lambda \geq 2$. Obviously, $\mathbb{E}_{\mathbb{P}}[Z_\sigma] = 1$ for all $\sigma \in \mathcal{T}$. Hence, from the simple formula $\max(z, 1/2)\min(1, 2z) \equiv z$ we obtain

$$F(\lambda) \geq F(2) = \inf_{\sigma \in \mathcal{T}} \mathbb{E}_{\mathbb{P}}\left[Z_\sigma(1 + \sin(\pi\sigma))\right] \geq 1.$$

On the other hand, taking $\sigma \equiv 0$ in (17), we get $F(\lambda) \leq 1$ and so $F \equiv 1$ on the interval $[2, \infty)$.

(ii). Choose $\lambda < 2$. Clearly, (we take $\sigma \equiv 1$ in (17))

$$\begin{aligned} F(\lambda) &\leq \mathbb{E}_{\mathbb{P}}\left[\max(Z_1, 1/2)\min(1, \lambda Z_1)\right] \\ &\leq \mathbb{E}_{\mathbb{P}}\left[Z_1\mathbb{I}_{Z_1 > 1/2} + \tfrac{\lambda}{2} Z_1\mathbb{I}_{Z_1 < 1/2}\right] = 1 - \tfrac{2-\lambda}{2}\mathbb{E}_{\mathbb{P}}\left[Z_1\mathbb{I}_{Z_1 < 1/2}\right]. \end{aligned}$$

This together with the equality $F(2) = 1$ gives $F'_{-}(2) \geq \nu$. □

Now, we have all the ingredients for the proof of Theorem 2.

Proof. From Lemma 3(i) and Lemma 4(i) it follows that for any x

$$R(x) \geq F(2) - 2x = 1 - 2x.$$

Let us prove that

$$R(x) = 1 - 2x, \quad \forall x \leq F'_{-}(2). \tag{20}$$

Since R is convex (Lemma 2) then it is sufficient to show that $R(0) \leq 1$ and $R(F'_{-}(2)) \leq 1 - 2F'_{-}(2)$.

The first inequality is trivial, $R(0) \le X_0 = 1$. Let us show the second inequality. The concavity of F implies that there exists a sequence $\lambda_n \uparrow 2$ such that for any n the derivative $F'(\lambda_n)$ exists. Hence, from the continuity of R (Lemma 2), the concavity of F and Lemma 3(ii) we obtain

$$R(F'_-(2)) = \lim_{n\to\infty} R(F'(\lambda_n)) = \lim_{n\to\infty} [F(\lambda_n) - \lambda_n F'(\lambda_n)] = 1 - 2F'_-(2)$$

and (20) follows.

Next, let $x \in (0, \nu)$. Assume by contradiction that there exists a hedging strategy $(\pi, \sigma) \in \mathcal{A}(x) \times \mathcal{T}$ such that $R(\pi, \sigma) = R(x)$. From Lemma 4(ii) and (20) we obtain

$$R(\pi, \sigma) = 1 - 2x. \tag{21}$$

Observe that if σ takes on values (with positive probability) in the interval (0, 1) then

$$\mathbb{E}_{\mathbb{P}}[X_\sigma \min(1, 2Z_\sigma)] = \mathbb{E}_{\mathbb{P}}[Z_\sigma(1 + \sin(\pi\sigma))] > \mathbb{E}_{\mathbb{P}}[Z_\sigma] = 1.$$

Thus, from (16) and (18)

$$R(\pi, \sigma) + 2x \ge \mathbb{E}_{\mathbb{P}}\left[(X_\sigma - V^\pi_\sigma)^+ + 2Z_\sigma V^\pi_\sigma\right] > 1$$

which is a contradiction to (21). On the other hand if $\sigma \equiv 0$ then

$$R(\pi, \sigma) = X_0 - x = 1 - x,$$

also a contradiction to (21).

We conclude that the only remaining possibility is $\sigma \equiv 1$. Let us show that there is a contradiction in this case as well. Introduce the event

$$A := \{\max(Z_1, V^\pi_1) < 1/2\}.$$

Observe that on the event A we have

$$(X_1 - V^\pi_1)^+ + 2Z_1 V^\pi_1 = (1/2 - Z_1)(1 - 2V^\pi_1) + Z_1 > Z_1 = X_1 \min(1, 2Z_1). \tag{22}$$

From (18) and the fact that $x < \nu$ it follows that

$$\mathbb{E}_{\mathbb{P}}[Z_1 V^\pi_1] < \nu = \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[Z_1 \mathbb{I}_{Z_1 < 1/2}\right].$$

This together with the inequality $V^\pi_1 \ge 0$ gives $\mathbb{P}(A) > 0$. Thus, by combining (16), (18) and (22) we obtain

$$\begin{aligned} R(\pi, \sigma) + 2x &= \mathbb{E}_{\mathbb{P}}\left[(X_1 - V^\pi_1)^+ + 2Z_1 V^\pi_1\right] \\ &> \mathbb{E}_{\mathbb{P}}[X_1 \min(1, 2Z_1)] = \mathbb{E}_{\mathbb{P}}[Z_1] = 1 \end{aligned}$$

which is a contradiction to (21). □

We end this section with the following two remarks.

Remark 5. The message of Theorem 2 is that the inf in (2) which ruins the convexity of the shortfall risk functional $R(\pi)$ can lead to non existence of an optimal strategy. Observe that in the above constructed example, the payoff process X is continuous and the payoff process Y has a positive jump in the maturity date.

One can ask, what if we require that both of the payoff processes X and Y will be continuous, is there a counter example in this case as well?

The answer is yes. Let us apply Theorem 2 in order to construct a counter example with continuous payoffs.

Consider a simple BS financial market with time horizon $T = 2$ which consists of a riskless savings account bearing zero interest and of a risky asset S, whose value at time t is given by

$$S_t = S_0 \exp\left(\kappa W_t + (\vartheta - \kappa^2/2)t\right), \quad t \in [0, 1]$$
$$S_t = S_1 \exp\left(\kappa (W_t - W_1) - \kappa^2 (t-1)/2\right), \quad t \in (1, 2]$$

where, as before, $S_0, \kappa > 0$ and $\vartheta \neq 0$ are constants. Namely, this is a BS model which has a drift jump in $t = 1$. Obviously this market is complete and the unique martingale measure is given by $\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t := Z_{t\wedge 1}$ where Z_t is given by (3). Consider a game option with the continuous payoffs

$$\hat{X}_t = (1 + \sin(\pi t)) \max(Z_t, 1/2), \quad t \in [0, 1]$$
$$\hat{X}_t = \hat{X}_1, \quad t \in (1, 2],$$
$$\hat{Y}_t = 0, \quad t \in [0, 1]$$
$$\hat{Y}_t = (t-1)\hat{X}_1, \quad t \in (1, 2].$$

Denote by $\hat{R}$ the corresponding shortfall risk. We argue that for an initial capital $0 < x < \nu := \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[Z_1 \mathbb{I}_{Z_1 < 1/2}\right]$ there is no optimal hedging strategy.

Indeed, let π be an *admissible* portfolio strategy and σ be a stopping time with values in the interval $[0, 2]$. From the super–martingale property of the portfolio value and the fact that Z is a constant random variable after $t = 1$ we obtain

$$V^\pi_{\sigma\wedge 1} \geq \mathbb{E}_{\mathbb{P}}[V^\pi_\sigma | \mathcal{F}_1].$$

This together with the Jensen inequality and the fact that $\hat{X}$ is a constant random variable after $t = 1$ gives

$$\mathbb{E}_{\mathbb{P}}\left[(\hat{X}_{\sigma\wedge 1} - V^\pi_{\sigma\wedge 1})^+\right] \leq \mathbb{E}_{\mathbb{P}}\left[\mathbb{E}_{\mathbb{P}}\left[(\hat{X}_\sigma - V^\pi_\sigma)^+ | \mathcal{F}_1\right]\right] = \mathbb{E}_{\mathbb{P}}\left[(\hat{X}_\sigma - V^\pi_\sigma)^+\right]. \quad (23)$$

From (23), and the relations $\hat{Y}_{[0,1]} \equiv 0$, $\hat{Y}_2 = \hat{X}_2$ we obtain

$$\hat{R}(\pi, \sigma \wedge 1) = \mathbb{E}_{\mathbb{P}}\left[(\hat{X}_{\sigma\wedge 1} - V^\pi_{\sigma\wedge 1})^+\right] \leq \mathbb{E}_{\mathbb{P}}\left[(\hat{X}_\sigma - V^\pi_\sigma)^+\right] \leq \hat{R}(\pi, \sigma).$$

Namely, we can restrict the investor to stopping times in the interval $[0, 1]$, but this is exactly the setup that was studied in Theorem 2. From Theorem 2 we conclude that there is no optimal hedging strategy for $x \in (0, \nu)$.

Remark 6. From Theorem 1 it follows that for any n there exists a hedging strategy $(\pi_n, \sigma_n) \in \mathcal{A}(x) \times \mathcal{T}_n$ such that

$$R_n(x) = \mathbb{E}_{\mathbb{P}}\left[\left(X_{\sigma_n} - V^{\pi_n}_{\sigma_n}\right)^+\right]$$

where R_n was defined in the beginning of the proof of Lemma 2.

Theorem 2 implies that we should not expect that these optimal hedging strategies $(\pi_n, \sigma_n) \in \mathcal{A}(x) \times \mathcal{T}_n$, $n \in \mathbb{N}$ will converge in a strong sense when n goes to infinity. Indeed, if these hedging strategies would converge in a strong sense, then we can argue that the limit is an optimal hedging strategy for the continuous time problem. This is a contradiction to Theorem 2 (at least for $x \in (0, \nu)$).

By applying the weak convergence theory we can show that $(\pi_n, \sigma_n) \in \mathcal{A}(x) \times \mathcal{T}_n$, $n \in \mathbb{N}$ has a cluster point (with respect to convergence in law). However, in view of Theorem 2 we conclude that the representation of the cluster point will require an enlargement of the probability space, i.e. an additional randomization.

An interesting question which is left for the future, is whether by allowing the investor to randomize from the start (in the spirit of [1]) will provide an existence of an optimal hedging strategy.

Acknowledgments

I would like to thank Yuri Kifer for introducing me to the problems which are treated in this paper and also for related fruitful discussions. I also would like to thank Walter Schachermayer for sharing some ideas a while ago which turned out to be helpful for proving Theorem 1.

Funding

Supported in part by the GIF Grant 1489-304.6/2019 and the ISF grant 160/17.

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