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The risk model with stochastic premiums and a multi-layer dividend strategy

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Abstract The paper deals with a generalization of the risk model with stochastic premiums where dividends are paid according to a multi-layer dividend strategy. First of all, we derive piecewise integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments until ruin. In addition, we concentrate on the detailed investigation of the model in the case of exponentially distributed claim and premium sizes and find explicit formulas for the ruin probability as well as for the expected discounted dividend payments. Lastly, numerical illustrations for some multi-layer dividend strategies are presented.

Keywords Risk model with stochastic premiums, multi-layer dividend strategy, Gerber–Shiu function, expected discounted dividend payments, ruin probability, piecewise integro-differential equation

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1 Introduction

The ruin measures such as the ruin probability, the surplus prior to ruin and the deficit at ruin have attracted great interest of researchers recently (see, e.g., [2, 28, 33] and references therein). Gerber and Shiu [16] introduced the expected discounted penalty function for the classical risk model, which enabled to study those risk measures together by combining them into one function. After that, the so-called Gerber–Shiu function has been investigated by many authors in more general risk models (see, e.g., [6–8, 10, 11, 17, 18, 36, 39, 45, 47]).

In particular, a lot of attention has been paid to the study of risk models where shareholders receive dividends from their insurance company. De Finetti [14], who first considered dividend strategies in insurance, dealt with a binomial model. For the classical risk model and its different generalizations, different dividend strategies have been studied in a number of papers (see, e.g., [9, 12, 13, 20, 22–24, 26, 27, 35, 37, 40]). In addition, the monograph by Schmidli [34] is devoted to optimal dividend problems in insurance risk models.

Applying multi-layer dividend strategies enables to change the dividend payment intensity depending on the current surplus. Albrecher and Hartinger [1] consider the modification of the classical risk model where both the premium intensity and the dividend payment intensity are assumed to be step functions depending on the current surplus level. The authors derive algorithmic schemes for the determination of explicit expressions for the Gerber–Shiu function and the expected discounted dividend payments. A similar risk model is considered by Lin and Sendova [25], who derive a piecewise integro-differential equation for the Gerber–Shiu function and provide a recursive approach to obtain general solutions to that equation and its generalizations. Developing a recursive algorithm to calculate the moments of the expected discounted dividend payments for a class of risk models with Markovian claim arrivals, Badescu and Landriault [3] generalize some of the results obtained in [1] (see also [4] for some results related to the class of Markovian risk models with a multi-layer dividend strategy).

The absolute ruin problem in the classical risk model with constant interest force and a multi-layer dividend strategy is investigated in [43], where a piecewise integro-differential equation for the discounted penalty function is derived, some explicit expressions are given when claims are exponentially distributed and an asymptotic formula for the absolute ruin probability is obtained for heavy-tailed claim sizes. The dual model of the compound Poisson risk model with a multi-layer dividend strategy under stochastic interest is considered in [44]. Results related to perturbed compound Poisson risk models under multi-layer dividend strategies can be found in [31, 42]. In addition, different classes of more general renewal risk models are investigated in [15, 19, 40, 41], and some recent papers deal with risk models that incorporate various dependence structures (see, e.g., [21, 38, 46, 48]).

The present paper generalizes the risk model with stochastic premiums introduced and investigated in [5] (see also [28]). In that risk model, both claims and premiums are modeled as compound Poisson processes, whereas premiums arrive with constant intensity and are not random in the classical compound Poisson risk model (see also [29, 30] for a generalization of the classical risk model where an insurance company gets additional funds whenever a claim arrives). In [5], claim sizes and inter-claim times are assumed to be mutually independent, and the same assumption is made concerning premium arrivals. In contrast to [5], the recent paper [32] deals with the risk model with stochastic premiums where the dependence structures between claim sizes and inter-claim times as well as premium sizes and inter-premium times are modeled by the Farlie–Gumbel–Morgenstern copulas, and dividends are paid according to a threshold dividend strategy. The Gerber–Shiu function, a special case of which is the ruin probability, and the expected discounted dividend payments until ruin are studied in [32]. In the present paper, we develop those results and make the assumption that dividends are paid according to a multi-layer dividend strategy and all random variables and processes are mutually independent.

The rest of the paper is organized as follows. In Section 2, we give a description of the risk model with stochastic premiums and a multi-layer dividend strategy. In Sections 3 and 4, we derive piecewise integro-differential equations for the Gerber–Shiu function and the expected discounted dividend payments until ruin. Next, in Section 5, we deal with exponentially distributed claim and premium sizes and obtain explicit formulas for the ruin probability and the expected discounted dividend payments. Finally, Section 6 provides some numerical illustrations.

2 Description of the model

Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space satisfying the usual conditions, and let all the stochastic objects we use below be defined on it.

In the risk model with stochastic premiums introduced in [5] (see also [28]), claim sizes form a sequence $(Y_i)_{i\ge1}$ of non-negative independent and identically distributed (i.i.d.) random variables (r.v.'s) with cumulative distribution function (c.d.f.) $F_Y(y) = \mathbb{P}[Y_i \le y]$, and the number of claims on the time interval $[0, t]$ is a Poisson process $(N_t)_{t\ge0}$ with constant intensity $\lambda > 0$. In addition, premium sizes form a sequence $(\bar{Y}_i)_{i\ge1}$ of non-negative i.i.d. r.v.'s with c.d.f. $\bar{F}_{\bar{Y}}(y) = \mathbb{P}[\bar{Y}_i \le y]$, and the number of premiums on the time interval $[0, t]$ is a Poisson process $(\bar{N}_t)_{t\ge0}$ with constant intensity $\bar{\lambda} > 0$. Thus, the total claims and premiums on $[0, t]$ equal $\sum_{i=1}^{N_t} Y_i$ and $\sum_{i=1}^{\bar{N}_t} \bar{Y}_i$, respectively.

It is worth pointing out that, here and subsequently, a sum is always set to 0 if the upper summation index is less than the lower one. In particular, we have $\sum_{i=1}^{0} Y_i = 0$ if $N_t = 0$, and $\sum_{i=1}^{0} \bar{Y}_i = 0$ if $\bar{N}_t = 0$. In what follows, we also assume that the r.v.'s $(Y_i)_{i\ge1}$ and $(\bar{Y}_i)_{i\ge1}$ have finite expectations $\mu > 0$ and $\bar{\mu} > 0$, respectively. Furthermore, we suppose that $(Y_i)_{i\ge1}$, $(\bar{Y}_i)_{i\ge1}$, $(N_t)_{t\ge0}$ and $(\bar{N}_t)_{t\ge0}$ are mutually independent.

Next, we denote a non-negative initial surplus of the insurance company by x, and let $X_t(x)$ be its surplus at time t provided that the initial surplus is x. Then the surplus process $(X_t(x))_{t\ge0}$ is defined by the equality

$$X_t(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i, \quad t \ge 0. \tag{1}$$

In contrast to the risk model considered in [5], we make the additional assumption that the insurance company pays dividends to its shareholders according to a k-layer dividend strategy with $k \ge 2$. Let $\mathbf{b} = (b_1, \dots, b_{k-1})$ be a $(k-1)$-dimensional vector with real-valued components such that $0 < b_1 < \cdots < b_{k-1} < \infty$. Besides that, we set $b_0 = 0$ and $b_k = \infty$. Let $(X_t^{\mathbf{b}}(x))_{t\ge0}$ denote the modified surplus process under the k-layer dividend strategy $\mathbf{b}$, which implies that dividends are paid continuously at a rate $d_j > 0$ whenever $b_{j-1} \le X_t^{\mathbf{b}}(x) < b_j$, i.e. the process $(X_t^{\mathbf{b}}(x))_{t\ge0}$ is in the jth layer at time t, where $1 \le j \le k$. Then

$$X_t^{\mathbf{b}}(x) = x + \sum_{i=1}^{\bar{N}_t} \bar{Y}_i - \sum_{i=1}^{N_t} Y_i - \int_0^t \sum_{j=1}^{k} d_j \mathbb{1}\big(b_{j-1} \le X_s^{\mathbf{b}}(x) < b_j\big)\, \mathrm{d}s, \quad t \ge 0, \quad (2)$$

where $\mathbb{1}(\cdot)$ is the indicator function.

From now on, we suppose that the net profit condition holds, which in this case means that

$$\bar{\lambda}\bar{\mu} > \lambda\mu + \max_{1\le j\le k} \{d_j\}. \quad (3)$$

Let $(D_t)_{t\ge0}$ denote the dividend distributing process. For the k-layer dividend strategy described above, we have

$$\mathrm{d}D_t = d_j\, \mathrm{d}s \quad \text{if } b_{j-1} \le X_t^{\mathbf{b}}(x) < b_j, \quad 1 \le j \le k.$$

Next, let $\tau_{\mathbf{b}}(x) = \inf\{t \ge 0\colon X_t^{\mathbf{b}}(x) < 0\}$ be the ruin time for the risk process $(X_t^{\mathbf{b}}(x))_{t\ge0}$ defined by (2). In what follows, we omit the dependence on x and write $\tau_{\mathbf{b}}$ instead of $\tau_{\mathbf{b}}(x)$ when no confusion can arise.

For $\delta_0 \ge 0$, the Gerber–Shiu function is defined by

$$m(x, \mathbf{b}) = \mathbb{E}\big[e^{-\delta_0\tau_{\mathbf{b}}}\, w\big(X_{\tau_{\mathbf{b}}-}^{\mathbf{b}}(x), \big|X_{\tau_{\mathbf{b}}}^{\mathbf{b}}(x)\big|\big)\, \mathbb{1}(\tau_{\mathbf{b}} < \infty) \mid X_0^{\mathbf{b}}(x) = x\big], \quad x \ge 0,$$

where $w(\cdot, \cdot)$ is a bounded non-negative measurable function, $X_{\tau_{\mathbf{b}}-}^{\mathbf{b}}(x)$ is the surplus immediately before ruin and $|X_{\tau_{\mathbf{b}}}^{\mathbf{b}}(x)|$ is a deficit at ruin. Note that if $w(\cdot, \cdot) \equiv 1$ and $\delta_0 = 0$, then $m(x, \mathbf{b})$ becomes the infinite-horizon ruin probability

$$\psi(x, \mathbf{b}) = \mathbb{E}\big[\mathbb{1}(\tau_{\mathbf{b}} < \infty) \mid X_0^{\mathbf{b}}(x) = x\big].$$

For $\delta > 0$, the expected discounted dividend payments until ruin are defined by

$$v(x, \mathbf{b}) = \mathbb{E}\bigg[\int_0^{\tau_{\mathbf{b}}} e^{-\delta t}\, \mathrm{d}D_t \mid X_0^{\mathbf{b}}(x) = x\bigg], \quad x \ge 0.$$

For simplicity of notation, we also write $m(x)$, $\psi(x)$ and $v(x)$ instead of $m(x, \mathbf{b})$, $\psi(x, \mathbf{b})$ and $v(x, \mathbf{b})$, respectively. For all $1 \le j \le k$ and $b_{j-1} \le x \le b_j$, we also set $m_j(x) = m(x, \mathbf{b})$, $\psi_j(x) = \psi(x, \mathbf{b})$ and $v_j(x) = v(x, \mathbf{b})$. Thus, the functions $m_j(x)$, $\psi_j(x)$ and $v_j(x)$ are defined on $[b_{j-1}, b_j]$, and we have $m_j(b_j) = m_{j+1}(b_j)$, $\psi_j(b_j) = \psi_{j+1}(b_j)$ and $v_j(b_j) = v_{j+1}(b_j)$ for all $1 \le j \le k - 1$.

Remark 1. Note that although we consider the interval $[b_{k-1}, \infty)$ instead of $[b_{j-1}, b_j]$ if $j = k$, for the sake of convenience and compactness, here and subsequently, we do write $[b_{j-1}, b_j]$ for all $1 \le j \le k$. In addition, in what follows, the derivatives of all functions at the ends of the closed intervals $[b_{j-1}, b_j]$ are assumed to be one-sided.

3 Piecewise integro-differential equation for the Gerber–Shiu function

Theorem 1. *Let the surplus process $(X_t^b(x))_{t\ge0}$ be defined by (2) under the above assumptions, and let $F_Y(y)$ and $w(u_1, u_2)$ be continuous on $\mathbb{R}_+$ and $\mathbb{R}_+^2$, respectively.*

Then the function $m(x)$ is differentiable on the intervals $[b_{j-1}, b_j]$ for all $1 \le j \le k$ and satisfies the piecewise integro-differential equation

$$d_j m'(x) + (\lambda + \bar{\lambda} + \delta_0) m(x) = \lambda \int_0^x m(x-y)\,\mathrm{d}F_Y(y) + \lambda \int_x^\infty w(x, y-x)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^\infty m(x+y)\,\mathrm{d}F_{\bar{Y}}(y), \quad x \in [b_{j-1}, b_j]. \tag{4}$$

Proof. We now fix any j such that $1 \le j \le k$ and deal with the case $x \in [b_{j-1}, b_j]$. For all $x \in [b_{j-1}, b_j]$, we define the following functions:

$$a_1(x) = (x - b_{j-1})/d_j + (b_{j-1} - b_{j-2})/d_{j-1} + \cdots + (b_2 - b_1)/d_2 + (b_1 - b_0)/d_1,$$
$$a_2(x) = (x - b_{j-1})/d_j + (b_{j-1} - b_{j-2})/d_{j-1} + \cdots + (b_2 - b_1)/d_2,$$
$$\cdots$$
$$a_{j-1}(x) = (x - b_{j-1})/d_j + (b_{j-1} - b_{j-2})/d_{j-1},$$
$$a_j(x) = (x - b_{j-1})/d_j.$$

From these equalities we conclude that for all $x \in [b_{j-1}, b_j]$, the process $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ up to its first jump is in the jth layer if $t \in [0, a_j(x)]$ and in the ith layer if $t \in [a_{i+1}(x), a_i(x)]$, where $1 \le i \le j-1$. Thus, for any $x \in [b_{j-1}, b_j]$, the sequence $a_j(x), a_{j-1}(x), \ldots, a_1(x)$ defines the times when $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ passes through the values $b_{j-1}, b_{j-2}, \ldots, b_0$ provided that it has no jumps until those times.

It is easily seen that the time of the first jump of $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ is exponentially distributed with mean $1/(\lambda + \bar{\lambda})$. Considering the time and the size of the first jump of that process and applying the law of total probability we obtain

$$m(x) = I_j(x) + I_{j-1}(x) + \cdots + I_1(x) + I_0(x), \quad x \in [b_{j-1}, b_j], \tag{5}$$

where

$$I_j(x) = \int_0^{a_j(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\lambda \int_0^{x - d_j t} e^{-\delta_0 t}\, m(x - d_j t - y)\,\mathrm{d}F_Y(y) + \lambda \int_{x - d_j t}^\infty e^{-\delta_0 t}\, w(x - d_j t,\ y - x + d_j t)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^\infty e^{-\delta_0 t}\, m(x - d_j t + y)\,\mathrm{d}F_{\bar{Y}}(y) \bigg) \mathrm{d}t,$$

$$I_{j-1}(x) = \int_{a_j(x)}^{a_{j-1}(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\lambda \int_0^{b_{j-1} - d_{j-1}(t - a_j(x))} e^{-\delta_0 t} \times m\big(b_{j-1} - d_{j-1}\big(t - a_j(x)\big) - y\big)\,\mathrm{d}F_Y(y) + \lambda \int_{b_{j-1} - d_{j-1}(t - a_j(x))}^\infty e^{-\delta_0 t}\, w\big(b_{j-1} - d_{j-1}\big(t - a_j(x)\big), y - b_{j-1} + d_{j-1}\big(t - a_j(x)\big)\big)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^\infty e^{-\delta_0 t}\, m\big(b_{j-1} - d_{j-1}\big(t - a_j(x)\big) + y\big)\,\mathrm{d}F_{\bar{Y}}(y) \bigg) \mathrm{d}t,$$

$$\dots$$

$$I_1(x) = \int_{a_2(x)}^{a_1(x)} e^{-(\lambda+\bar{\lambda})t} \biggl(\lambda \int_0^{b_1-d_1(t-a_2(x))} e^{-\delta_0 t} \times m\bigl(b_1 - d_1\bigl(t-a_2(x)\bigr) - y\bigr)\,\mathrm{d}F_Y(y) + \lambda \int_{b_1-d_1(t-a_2(x))}^{\infty} e^{-\delta_0 t}\, w\bigl(b_1 - d_1\bigl(t-a_2(x)\bigr), y - b_1 + d_1\bigl(t-a_2(x)\bigr)\bigr)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^{\infty} e^{-\delta_0 t}\, m\bigl(b_1 - d_1\bigl(t-a_2(x)\bigr) + y\bigr)\,\mathrm{d}F_{\bar{Y}}(y) \biggr)\mathrm{d}t,$$

$$I_0(x) = e^{-(\lambda+\bar{\lambda}+\delta_0)\,a_1(x)}\, w(0,0).$$

Note that the term $I_i(x)$, $1 \le i \le j$, in (5) corresponds to the case where $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ is in the ith layer when its first jump occurs, and the term $I_0(x)$ corresponds to the case where there are no jumps of $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ up to the time $a_1(x)$.

Changing the variable $x - d_j t = s$ in the outer integral in the expression for $I_j(x)$ yields

$$I_j(x) = \frac{1}{d_j} e^{-(\lambda+\bar{\lambda}+\delta_0)x/d_j} \int_{b_{j-1}}^{x} e^{(\lambda+\bar{\lambda}+\delta_0)s/d_j} \biggl(\lambda \int_0^s m(s-y)\,\mathrm{d}F_Y(y) + \lambda \int_s^{\infty} w(s, y-s)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^{\infty} m(s+y)\,\mathrm{d}F_{\bar{Y}}(y) \biggr)\mathrm{d}s. \tag{6}$$

Changing the variable $b_{j-1} - d_{j-1}(t - a_j(x)) = s$ in the outer integral in the expression for $I_{j-1}(x)$ gives

$$I_{j-1}(x) = \frac{1}{d_{j-1}} e^{-(\lambda+\bar{\lambda}+\delta_0)(a_j(x)+b_{j-1}/d_{j-1})} \times \int_{b_{j-2}}^{b_{j-1}} e^{(\lambda+\bar{\lambda}+\delta_0)s/d_{j-1}} \biggl(\lambda \int_0^s m(s-y)\,\mathrm{d}F_Y(y) + \lambda \int_s^{\infty} w(s, y-s)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^{\infty} m(s+y)\,\mathrm{d}F_{\bar{Y}}(y) \biggr)\mathrm{d}s. \tag{7}$$

In the same manner we change variables in all the outer integrals on the right-hand side of (5). Finally, changing the variable $b_1 - d_1(t - a_2(x)) = s$ in the outer integral in the expression for $I_1(x)$ yields

$$I_1(x) = \frac{1}{d_1} e^{-(\lambda+\bar{\lambda}+\delta_0)(a_2(x)+b_1/d_1)} \int_{b_0}^{b_1} e^{(\lambda+\bar{\lambda}+\delta_0)s/d_1} \times \biggl(\lambda \int_0^s m(s-y)\,\mathrm{d}F_Y(y) + \lambda \int_s^{\infty} w(s, y-s)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^{\infty} m(s+y)\,\mathrm{d}F_{\bar{Y}}(y) \biggr)\mathrm{d}s. \tag{8}$$

Thus, from the above and equality (5) we deduce that $m(x)$ is continuous on $[b_{j-1}, b_j]$, and hence, on $\mathbb{R}_+$. Therefore, (6) implies that $I_j(x)$ is differentiable on $[b_{j-1}, b_j]$, and for all $x \in [b_{j-1}, b_j]$, we get

$$I_j'(x) = -\frac{\lambda + \bar{\lambda} + \delta_0}{d_j} I_j(x) + \frac{1}{d_j}\bigg(\lambda \int_0^x m(x-y)\,\mathrm{d}F_Y(y) \\ + \lambda \int_x^\infty w(x, y-x)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^\infty m(x+y)\,\mathrm{d}F_{\bar{Y}}(y)\bigg).$$

Moreover, it is easily seen, e.g. from (7) and (8), that all the functions $I_{j-1}(x), \dots,$ $I_1(x)$ and $I_0(x)$ are differentiable on $[b_{j-1}, b_j]$, and for all $x \in [b_{j-1}, b_j]$, we have

$$I_{j-1}'(x) = -\frac{\lambda + \bar{\lambda} + \delta_0}{d_j} I_{j-1}(x), \quad \dots, \quad I_1'(x) = -\frac{\lambda + \bar{\lambda} + \delta_0}{d_j} I_1(x),$$

$$I_0'(x) = -\frac{\lambda + \bar{\lambda} + \delta_0}{d_j} I_0(x).$$

From (5) it follows that $m(x)$ is also differentiable on $[b_{j-1}, b_j]$. Differentiating (5) and taking into account expressions for $I_j'(x)$, $I_{j-1}'(x), \dots, I_1'(x)$, $I_0'(x)$ we obtain

$$m'(x) = -\frac{\lambda + \bar{\lambda} + \delta_0}{d_j} m(x) + \frac{1}{d_j}\bigg(\lambda \int_0^x m(x-y)\,\mathrm{d}F_Y(y) \\ + \lambda \int_x^\infty w(x, y-x)\,\mathrm{d}F_Y(y) + \bar{\lambda} \int_0^\infty m(x+y)\,\mathrm{d}F_{\bar{Y}}(y)\bigg), \quad x \in [b_{j-1}, b_j],$$

from which equation (4) follows immediately. □

Remark 2. To solve equation (4), we use the following boundary conditions. The first $k-1$ conditions are easily obtained from the equality $m_j(b_j) = m_{j+1}(b_j)$ or, equivalently, $\lim_{x \uparrow b_j} m(x) = \lim_{x \downarrow b_j} m(x)$ for all $1 \le j \le k-1$. In addition, for the ruin probability, using standard considerations (see, e.g., [28, 30, 33]) we can show that $\lim_{x \to \infty} \psi(x) = 0$ provided that the net profit condition holds. Finally, it is evident that $\psi(0) = 1$. Although equation (4) is not solvable analytically in the general case, we can find explicit expressions for the corresponding ruin probability in the case where claim and premium sizes are exponentially distributed (see Section 5). The uniqueness of the required solution to (4) should be justified in each case.

Remark 3. In the assertion of Theorem 1, we require the continuity of $F_Y(y)$. Note that if $F_Y(y)$ has positive points of discontinuity, then $m(x)$ may be not differentiable at some interior points of the intervals $[b_{j-1}, b_j]$, $1 \le j \le k$ (for details, see [28, 30]). Moreover, it is easily seen from (4) that $m(x)$ is not differentiable at $x = b_j$, $1 \le j \le k-1$, since its one-sided derivatives do not coincide at those points.

4 Piecewise integro-differential equation for the expected discounted dividend payments until ruin

Theorem 2. *Let the surplus process* $(X_t^b(x))_{t\ge 0}$ *be defined by* (2) *under the above assumptions, and let* $F_Y(y)$ *be continuous on* $\mathbb{R}_+$. *Then the function* $v(x)$ *is differentiable on the intervals* $[b_{j-1}, b_j]$ *for all* $1 \le j \le k$ *and satisfies the piecewise integro-differential equation*

$$d_j v'(x) + (\lambda + \bar\lambda + \delta) v(x) = \lambda \int_0^x v(x-y)\,\mathrm{d}F_Y(y) + \bar\lambda \int_0^\infty v(x+y)\,\mathrm{d}F_{\bar Y}(y) + d_j, \quad x \in [b_{j-1}, b_j]. \tag{9}$$

Proof. We now fix any j such that $1 \le j \le k$ and deal with the case $x \in [b_{j-1}, b_j]$. As in the proof of Theorem 1, considering the time and the size of the first jump of $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ and applying the law of total probability we have

$$v(x) = I_{1,j}(x) + I_{2,j}(x) + I_{1,j-1}(x) + I_{2,j-1}(x) + \cdots + I_{1,1}(x) + I_{2,1}(x) + I_{1,0}(x), \quad x \in [b_{j-1}, b_j], \tag{10}$$

where

$$I_{1,j}(x) = \int_0^{a_j(x)} (\lambda+\bar\lambda) e^{-(\lambda+\bar\lambda)t} \left(\int_0^t d_j e^{-\delta s}\,\mathrm{d}s \right) \mathrm{d}t,$$

$$I_{2,j}(x) = \int_0^{a_j(x)} e^{-(\lambda+\bar\lambda)t} \bigg(\lambda \int_0^{x-d_j t} e^{-\delta t}\, v(x - d_j t - y)\,\mathrm{d}F_Y(y) + \bar\lambda \int_0^\infty e^{-\delta t}\, v(x - d_j t + y)\,\mathrm{d}F_{\bar Y}(y) \bigg) \mathrm{d}t,$$

$$I_{1,j-1}(x) = \int_{a_j(x)}^{a_{j-1}(x)} (\lambda+\bar\lambda) e^{-(\lambda+\bar\lambda)t} \left(\int_0^{a_j(x)} d_j e^{-\delta s}\,\mathrm{d}s + \int_{a_j(x)}^t d_{j-1} e^{-\delta s}\,\mathrm{d}s \right) \mathrm{d}t,$$

$$I_{2,j-1}(x) = \int_{a_j(x)}^{a_{j-1}(x)} e^{-(\lambda+\bar\lambda)t} \bigg(\lambda \int_0^{b_{j-1} - d_{j-1}(t - a_j(x))} e^{-\delta t} \times v\big(b_{j-1} - d_{j-1}\big(t - a_j(x)\big) - y\big)\,\mathrm{d}F_Y(y) + \bar\lambda \int_0^\infty e^{-\delta t}\, v\big(b_{j-1} - d_{j-1}\big(t - a_j(x)\big) + y\big)\,\mathrm{d}F_{\bar Y}(y) \bigg) \mathrm{d}t,$$

$$\cdots$$

$$I_{1,1}(x) = \int_{a_2(x)}^{a_1(x)} (\lambda+\bar\lambda) e^{-(\lambda+\bar\lambda)t} \bigg(\int_0^{a_j(x)} d_j e^{-\delta s}\,\mathrm{d}s + \int_{a_j(x)}^{a_{j-1}(x)} d_{j-1} e^{-\delta s}\,\mathrm{d}s + \cdots + \int_{a_3(x)}^{a_2(x)} d_2 e^{-\delta s}\,\mathrm{d}s + \int_{a_2(x)}^t d_1 e^{-\delta s}\,\mathrm{d}s \bigg) \mathrm{d}t,$$

$$I_{2,1}(x) = \int_{a_2(x)}^{a_1(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\lambda \int_0^{b_1 - d_1(t - a_2(x))} e^{-\delta t} \times v\big(b_1 - d_1\big(t - a_2(x)\big) - y\big) \, \mathrm{d}F_Y(y) + \bar{\lambda} \int_0^{\infty} e^{-\delta t} \, v\big(b_1 - d_1\big(t - a_2(x)\big) + y\big) \, \mathrm{d}F_{\bar{Y}}(y) \bigg) \mathrm{d}t,$$

$$I_{1,0}(x) = \int_{a_1(x)}^{\infty} (\lambda + \bar{\lambda}) e^{-(\lambda+\bar{\lambda})t} \bigg(\int_0^{a_j(x)} d_j e^{-\delta s} \, \mathrm{d}s + \int_{a_j(x)}^{a_{j-1}(x)} d_{j-1} e^{-\delta s} \, \mathrm{d}s + \cdots + \int_{a_3(x)}^{a_2(x)} d_2 e^{-\delta s} \, \mathrm{d}s + \int_{a_2(x)}^{a_1(x)} d_1 e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t,$$

and the functions $a_1(x), a_2(x), \ldots, a_j(x)$ are defined in the proof of Theorem 1.

Note that the terms $I_{1,i}(x)$ and $I_{2,i}(x)$, $1 \le i \le j$, in (10) correspond to the case where $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ is in the ith layer when its first jump occurs, and the term $I_{1,0}(x)$ corresponds to the case where there are no jumps of $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ up to the time $a_1(x)$. The terms $I_{1,i}(x)$, $0 \le i \le j$, are equal to the discounted dividend payments until the first jump of $(X_t^{\mathbf{b}}(x))_{t \ge 0}$ provided that the process is in the ith layer, whereas the terms $I_{2,i}(x)$, $1 \le i \le j$, are equal to the corresponding expected discounted dividend payments after that time.

Next, we set

$$I_{1,*}(x) = I_{1,j}(x) + I_{1,j-1}(x) + \cdots + I_{1,1}(x) + I_{1,0}(x), \quad x \in [b_{j-1}, b_j]. \tag{11}$$

Thus, $I_{1,*}(x)$ describes the expected discounted dividend payments until the first jump of $(X_t^{\mathbf{b}}(x))_{t \ge 0}$.

Rearranging terms in the expression for $I_{1,*}(x)$ gives

$$\begin{aligned} I_{1,*}(x) = (\lambda + \bar{\lambda}) \bigg(& \int_0^{a_j(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_0^t d_j e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t \\ & + \int_{a_j(x)}^{a_{j-1}(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_{a_j(x)}^t d_{j-1} e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t + \cdots \\ & + \int_{a_2(x)}^{a_1(x)} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_{a_2(x)}^t d_1 e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t \\ & + \int_{a_j(x)}^{\infty} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_0^{a_j(x)} d_j e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t \\ & + \int_{a_{j-1}(x)}^{\infty} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_{a_j(x)}^{a_{j-1}(x)} d_{j-1} e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t + \cdots \\ & + \int_{a_1(x)}^{\infty} e^{-(\lambda+\bar{\lambda})t} \bigg(\int_{a_2(x)}^{a_1(x)} d_1 e^{-\delta s} \, \mathrm{d}s \bigg) \mathrm{d}t \bigg). \end{aligned} \tag{12}$$

Taking all the integrals on the right-hand side of (12) and simplifying the resulting expression we get

$$I_{1,*}(x) = \frac{d_j}{\lambda + \bar{\lambda} + \delta} \big(1 - e^{-(\lambda+\bar{\lambda}+\delta)\, a_j(x)}\big)$$

$$+ \frac{d_{j-1}}{\lambda+\bar{\lambda}+\delta}\left(e^{-(\lambda+\bar{\lambda}+\delta)\,a_j(x)} - e^{-(\lambda+\bar{\lambda}+\delta)\,a_{j-1}(x)}\right) + \cdots$$
$$+ \frac{d_1}{\lambda+\bar{\lambda}+\delta}\left(e^{-(\lambda+\bar{\lambda}+\delta)\,a_2(x)} - e^{-(\lambda+\bar{\lambda}+\delta)\,a_1(x)}\right). \tag{13}$$

Changing the variable $x - d_j t = s$ in the outer integral in the expression for $I_{2,j}(x)$ gives

$$I_{2,j}(x) = \frac{1}{d_j}\,e^{-(\lambda+\bar{\lambda}+\delta)x/d_j}\int_{b_{j-1}}^{x} e^{(\lambda+\bar{\lambda}+\delta)s/d_j} \times \left(\lambda\int_0^s v(s-y)\,\mathrm{d}F_Y(y) + \bar{\lambda}\int_0^\infty v(s+y)\,\mathrm{d}F_{\bar{Y}}(y)\right)\mathrm{d}s. \tag{14}$$

Likewise, changing the variable $b_{j-1} - d_{j-1}(t - a_j(x)) = s$ in the outer integral in the expression for $I_{2,j-1}(x)$ yields

$$I_{2,j-1}(x) = \frac{1}{d_{j-1}}\,e^{-(\lambda+\bar{\lambda}+\delta)(a_j(x)+b_{j-1}/d_{j-1})}\int_{b_{j-2}}^{b_{j-1}} e^{(\lambda+\bar{\lambda}+\delta)s/d_{j-1}} \times \left(\lambda\int_0^s v(s-y)\,\mathrm{d}F_Y(y) + \bar{\lambda}\int_0^\infty v(s+y)\,\mathrm{d}F_{\bar{Y}}(y)\right)\mathrm{d}s. \tag{15}$$

Next, in the same manner we change variables in all those outer integrals on the right-hand side of (10) that are not included in the sum (11). Eventually, changing the variable $b_1 - d_1(t - a_2(x)) = s$ in the outer integral in the expression for $I_{2,1}(x)$ we obtain

$$I_{2,1}(x) = \frac{1}{d_1}\,e^{-(\lambda+\bar{\lambda}+\delta)(a_2(x)+b_1/d_1)}\int_{b_0}^{b_1} e^{(\lambda+\bar{\lambda}+\delta)s/d_1} \times \left(\lambda\int_0^s v(s-y)\,\mathrm{d}F_Y(y) + \bar{\lambda}\int_0^\infty v(s+y)\,\mathrm{d}F_{\bar{Y}}(y)\right)\mathrm{d}s. \tag{16}$$

From (13) it follows immediately that $I'_{1,*}(x)$ is differentiable on $[b_{j-1}, b_j]$, and for all $x \in [b_{j-1}, b_j]$, we get

$$I'_{1,*}(x) = -\frac{\lambda+\bar{\lambda}+\delta}{d_j}\left(I_{1,*}(x) - \frac{d_j}{\lambda+\bar{\lambda}+\delta}\right).$$

Next, from the above and equality (10) we conclude that $v(x)$ is continuous on $[b_{j-1}, b_j]$, and hence, on $\mathbb{R}_+$. Hence, (14) implies that $I_{2,j}(x)$ is differentiable on $[b_{j-1}, b_j]$, and for all $x \in [b_{j-1}, b_j]$, we have

$$I'_{2,j}(x) = -\frac{\lambda+\bar{\lambda}+\delta}{d_j}\,I_{2,j}(x) + \frac{1}{d_j}\left(\lambda\int_0^x v(x-y)\,\mathrm{d}F_Y(y) + \bar{\lambda}\int_0^\infty v(x+y)\,\mathrm{d}F_{\bar{Y}}(y)\right).$$

Furthermore, it follows immediately, e.g. from (15) and (16), that all the functions $I_{2,j-1}(x), \dots, I_{2,1}(x)$ are differentiable on $[b_{j-1}, b_j]$, and for all $x \in [b_{j-1}, b_j]$, we obtain

$$I'_{2,j-1}(x) = -\frac{\lambda + \bar{\lambda} + \delta}{d_j} I_{2,j-1}(x), \quad \dots, \quad I'_{2,1}(x) = -\frac{\lambda + \bar{\lambda} + \delta}{d_j} I_{2,1}(x).$$

By (10), we conclude that $v(x)$ is also differentiable on $[b_{j-1}, b_j]$. Differentiating (10) and taking into account expressions for $I'_{1,*}(x)$, $I'_{2,j}(x)$, $I'_{2,j-1}(x), \dots, I'_{2,1}(x)$ we get

$$\begin{aligned} v'(x) = -\frac{\lambda + \bar{\lambda} + \delta}{d_j} v(x) + 1 + \frac{1}{d_j}\bigg(\lambda \int_0^x v(x-y)\,\mathrm{d}F_Y(y) \\ + \bar{\lambda} \int_0^\infty v(x+y)\,\mathrm{d}F_{\bar{Y}}(y)\bigg), \quad x \in [b_{j-1}, b_j], \end{aligned}$$

which immediately yields equation (9). □

Remark 4. To solve equation (9), we obtain the first $k-1$ boundary conditions from the equality $v_j(b_j) = v_{j+1}(b_j)$ or, equivalently, $\lim_{x \uparrow b_j} v(x) = \lim_{x \downarrow b_j} v(x)$ for all $1 \le j \le k-1$. Moreover, if the net profit condition holds, applying arguments similar to those in [34, p. 70] we can show that $\lim_{x \to \infty} v(x) = d_k/\delta$. Lastly, it is easily seen that $v(0) = 0$. The uniqueness of the required solution to (9) should be justified in each case. Explicit expressions for $v(x)$ in the case where claim and premium sizes are exponentially distributed are given in Section 5.

Remark 5. If $F_Y(y)$ has positive points of discontinuity, then $v(x)$ may be not differentiable at some interior points of the intervals $[b_{j-1}, b_j]$, $1 \le j \le k$. Furthermore, from (9) we deduce that $v(x)$ is not differentiable at $x = b_j$, $1 \le j \le k-1$.

5 Exponentially distributed claim and premium sizes

In this section, we concentrate on the case where claim and premium sizes are exponentially distributed, i.e.

$$f_Y(y) = \frac{1}{\mu} e^{-y/\mu} \quad \text{and} \quad f_{\bar{Y}}(y) = \frac{1}{\bar{\mu}} e^{-y/\bar{\mu}}, \quad y \ge 0. \tag{17}$$

5.1 *Explicit formulas for the ruin probability*

Let now $w(\cdot,\cdot) \equiv 1$ and $\delta_0 = 0$. Taking into account (17), equation (4) for the ruin probability can be written as

$$\begin{aligned} &d_j \psi'(x) + (\lambda + \bar{\lambda})\psi(x) \\ &\quad = \frac{\lambda}{\mu} e^{-x/\mu} \int_0^x \psi(u) e^{u/\mu}\,\mathrm{d}u + \lambda e^{-x/\mu} + \frac{\bar{\lambda}}{\bar{\mu}} e^{x/\bar{\mu}} \int_x^\infty \psi(u) e^{-u/\bar{\mu}}\,\mathrm{d}u \end{aligned} \tag{18}$$

for all $x \in [b_{j-1}, b_j]$ and $1 \le j \le k$.

We now reduce piecewise integro-differential equation (18) to a piecewise linear differential equation with constant coefficients.

Lemma 1. *Let the surplus process* $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ *be defined by* (2) *under the above assumptions, and let claim and premium sizes be exponentially distributed with means* μ *and* $\bar{\mu}$*, respectively. Then* $\psi(x)$ *is a solution to the piecewise differential equation*

$$d_j\mu\bar{\mu}\psi'''(x) + \big(d_j(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda})\big)\psi''(x) + (\bar{\lambda}\bar{\mu} - \lambda\mu - d_j)\psi'(x) = 0 \quad (19)$$

for all $x \in [b_{j-1}, b_j]$ *and* $1 \le j \le k$*.*

Proof. It is easily seen that the right-hand side of (18) is differentiable on $[b_{j-1}, b_j]$. Therefore, $\psi(x)$ is twice differentiable on $[b_{j-1}, b_j]$. Differentiating (18) gives

$$\begin{aligned} d_j\psi''(x) + (\lambda+\bar{\lambda})\psi'(x) &= -\frac{1}{\mu}\left(\frac{\lambda}{\mu}e^{-x/\mu}\int_0^x \psi(u)e^{u/\mu}\,\mathrm{d}u + \lambda e^{-x/\mu}\right) \\ &+ \frac{\bar{\lambda}}{\bar{\mu}^2}e^{x/\bar{\mu}}\int_x^\infty \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u + \left(\frac{\lambda}{\mu} - \frac{\bar{\lambda}}{\bar{\mu}}\right)\psi(x), \quad x \in [b_{j-1}, b_j]. \end{aligned} \quad (20)$$

Multiplying (20) by μ and adding (18) we get

$$\begin{aligned} &d_j\mu\psi''(x) + \big(d_j + \mu(\lambda+\bar{\lambda})\big)\psi'(x) + \bar{\lambda}\left(1+\frac{\mu}{\bar{\mu}}\right)\psi(x) \\ &= \frac{\bar{\lambda}}{\bar{\mu}}\left(1+\frac{\mu}{\bar{\mu}}\right)e^{x/\bar{\mu}}\int_x^\infty \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u, \quad x \in [b_{j-1}, b_j]. \end{aligned} \quad (21)$$

From (21) it follows that $\psi(x)$ has the third derivative on $x \in [b_{j-1}, b_j]$. Differentiating (21) yields

$$\begin{aligned} &d_j\mu\psi'''(x) + \big(d_j + \mu(\lambda+\bar{\lambda})\big)\psi''(x) + \bar{\lambda}\left(1+\frac{\mu}{\bar{\mu}}\right)\psi'(x) \\ &= \frac{\bar{\lambda}}{\bar{\mu}^2}\left(1+\frac{\mu}{\bar{\mu}}\right)e^{x/\bar{\mu}}\int_x^\infty \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u - \frac{\bar{\lambda}}{\bar{\mu}}\left(1+\frac{\mu}{\bar{\mu}}\right)\psi(x), \quad x \in [b_{j-1}, b_j]. \end{aligned} \quad (22)$$

Finally, multiplying (22) by $(-\bar{\mu})$ and adding (21) we obtain (19). □

For $1 \le j \le k$, we now define the following constants, which are used in the assertion of Theorem 3 below:

$$\mathrm{D}_j = \big(d_j(\mu+\bar{\mu}) + \mu\bar{\mu}(\lambda-\bar{\lambda})\big)^2 + 4\lambda\bar{\lambda}\mu^2\bar{\mu}^2,$$

$$z_{1,j} = \frac{-(d_j(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda})) + \sqrt{\mathrm{D}_j}}{2d_j\mu\bar{\mu}}$$

and

$$z_{2,j} = \frac{-(d_j(\bar{\mu}-\mu) + \mu\bar{\mu}(\lambda+\bar{\lambda})) - \sqrt{\mathrm{D}_j}}{2d_j\mu\bar{\mu}}.$$

Theorem 3. *Let the surplus process* $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ *be defined by* (2) *under the above assumptions, and let claim and premium sizes be exponentially distributed with means*

μ and $\bar{\mu}$, respectively. If the net profit condition (3) holds, then we have

$$\psi_j(x) = C_{1,j}e^{z_{1,j}x} + C_{2,j}e^{z_{2,j}x} + C_{3,j} \tag{23}$$

for all $x \in [b_{j-1}, b_j]$ and $1 \le j \le k$, where $C_{3,k} = 0$ and all the other constants $C_{1,j}$, $C_{2,j}$ and $C_{3,j}$ are determined from the system of linear equations (24)–(27):

$$\begin{aligned}
&\lambda e^{-b_{j-1}/\mu} \sum_{l=1}^{j-1} \Bigg(\sum_{i=1}^{2} \frac{C_{i,l}}{\mu z_{i,l} + 1} \big(e^{(z_{i,l}+1/\mu)b_l} - e^{(z_{i,l}+1/\mu)b_{l-1}}\big) \\
&\quad + \big(e^{b_l/\mu} - e^{b_{l-1}/\mu}\big)C_{3,l}\Bigg) + \sum_{i=1}^{2} \Bigg(\frac{\bar{\lambda} e^{b_{j-1}/\bar{\mu}}}{\bar{\mu} z_{i,j} - 1} \big(e^{(z_{i,j}-1/\bar{\mu})b_j} - e^{(z_{i,j}-1/\bar{\mu})b_{j-1}}\big) \\
&\quad - (d_j z_{i,j} + \lambda + \bar{\lambda})e^{z_{i,j}b_{j-1}}\Bigg) C_{i,j} - \big(\bar{\lambda} e^{(b_{j-1}-b_j)/\bar{\mu}} + \lambda\big) C_{3,j} \\
&\quad + \bar{\lambda} e^{b_{j-1}/\bar{\mu}} \sum_{l=j+1}^{k} \Bigg(\sum_{i=1}^{2} \frac{C_{i,l}}{\bar{\mu} z_{i,l} - 1} \big(e^{(z_{i,l}-1/\bar{\mu})b_l} - e^{(z_{i,l}-1/\bar{\mu})b_{l-1}}\big) \\
&\quad - \big(e^{-b_l/\bar{\mu}} - e^{-b_{l-1}/\bar{\mu}}\big)C_{3,l}\Bigg) = -\lambda e^{-b_{j-1}/\mu}, \quad 1 \le j \le k,
\end{aligned} \tag{24}$$

$$C_{1,1} + C_{2,1} + C_{3,1} = 1, \tag{25}$$

$$d_j \sum_{i=1}^{2} z_{i,j} e^{z_{i,j}b_j} C_{i,j} - d_{j+1} \sum_{i=1}^{2} z_{i,j+1} e^{z_{i,j+1}b_j} C_{i,j+1} = 0, \quad 1 \le j \le k-1, \tag{26}$$

and

$$\sum_{i=1}^{2} e^{z_{i,j}b_j} C_{i,j} + C_{3,j} - \sum_{i=1}^{2} e^{z_{i,j+1}b_j} C_{i,j+1} - C_{3,j+1} = 0, \quad 1 \le j \le k-1, \tag{27}$$

provided that its determinant is not equal to 0.

Proof. Taking into account the notation introduced in Section 2 and applying Lemma 1 we conclude that the function $\psi_j(x)$ is a solution to (19) on $x \in [b_{j-1}, b_j]$ for each $1 \le j \le k$. The corresponding characteristic equation has the form

$$d_j \mu\bar{\mu} z^3 + \big(d_j(\bar{\mu} - \mu) + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)z^2 + (\bar{\lambda}\bar{\mu} - \lambda\mu - d_j)z = 0 \tag{28}$$

for all $1 \le j \le k$. We first show that the equation

$$d_j \mu\bar{\mu} z^2 + \big(d_j(\bar{\mu} - \mu) + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)z + (\bar{\lambda}\bar{\mu} - \lambda\mu - d_j) = 0 \tag{29}$$

has two negative roots. Indeed, its discriminant is equal to

$$\begin{aligned}
&\big(d_j(\bar{\mu} - \mu) + \mu\bar{\mu}(\lambda + \bar{\lambda})\big)^2 - 4d_j\mu\bar{\mu}(\bar{\lambda}\bar{\mu} - \lambda\mu - d_j) \\
&\quad = d_j^2(\bar{\mu} - \mu)^2 + \mu^2\bar{\mu}^2(\lambda + \bar{\lambda})^2 + 2d_j\mu\bar{\mu}(\lambda + \bar{\lambda})(\bar{\mu} - \mu) \\
&\qquad + 4d_j\mu\bar{\mu}(d_j + \lambda\mu - \bar{\lambda}\bar{\mu})
\end{aligned}$$

$$= d_j^2(\mu + \bar{\mu})^2 + \mu^2\bar{\mu}^2(\lambda + \bar{\lambda})^2 + 2d_j\mu\bar{\mu}(\lambda - \bar{\lambda})(\mu + \bar{\mu})$$
$$= \big(d_j(\mu + \bar{\mu}) + \mu\bar{\mu}(\lambda - \bar{\lambda})\big)^2 + 4\lambda\bar{\lambda}\mu^2\bar{\mu}^2.$$

Hence, it is positive and coincides with the constant D_j defined above. Consequently, $z_{1,j}$ and $z_{2,j}$ defined before the assertion of the theorem are two real roots of equation (29). By the net profit condition (3), we have

$$\bar{\lambda}\bar{\mu} - \lambda\mu - d_j > 0$$

and

$$d_j(\bar{\mu} - \mu) + \mu\bar{\mu}(\lambda + \bar{\lambda}) = \mu(\bar{\lambda}\bar{\mu} - \lambda\mu - d_j) + \lambda\mu^2 + \lambda\mu\bar{\mu} + d_j\bar{\mu} > 0$$

for all $1 \le j \le k$, which implies that $z_{1,j} < 0$ and $z_{2,j} < 0$.

Therefore, $z_{1,j} < 0$, $z_{2,j} < 0$ and $z_{3,j} = 0$ are roots of equation (28), and we get (23) with some constants $C_{1,j}$, $C_{2,j}$ and $C_{3,j}$. Moreover, since condition (3) holds, using standard considerations (see, e.g., [28, 30, 33]) we can easily show that $\lim_{x\to\infty} \psi(x) = 0$, which yields $C_{3,k} = 0$.

To determine all the other constants $C_{1,j}$, $C_{2,j}$ and $C_{3,j}$, we use the following boundary conditions. The first k conditions are obtained by letting $x = b_{j-1}$ in (18) for $1 \le j \le k$:

$$d_j\psi'(b_{j-1}) + (\lambda + \bar{\lambda})\psi(b_{j-1}) = \frac{\lambda}{\mu}\, e^{-b_{j-1}/\mu} \int_0^{b_{j-1}} \psi(u)e^{u/\mu}\,\mathrm{d}u$$
$$+ \lambda e^{-b_{j-1}/\mu} + \frac{\bar{\lambda}}{\bar{\mu}}\, e^{b_{j-1}/\bar{\mu}} \int_{b_{j-1}}^{\infty} \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u. \tag{30}$$

One more condition is obtained from the equality $\psi(0) = 1$. Finally, the last $2(k-1)$ conditions are derived from the equalities $d_j\psi_j'(b_j) = d_{j+1}\psi_{j+1}'(b_j)$ and $\psi_j(b_j) = \psi_{j+1}(b_j)$ for $1 \le j \le k-1$. Note that the first equality easily follows from (18).

Taking into account (23), for all $1 \le j \le k$, we get:

$$\psi_j'(x) = C_{1,j}z_{1,j}e^{z_{1,j}x} + C_{2,j}z_{2,j}e^{z_{2,j}x}, \quad x \in [b_{j-1}, b_j], \tag{31}$$

$$\int_0^{b_{j-1}} \psi(u)e^{u/\mu}\,\mathrm{d}u = \sum_{l=1}^{j-1} \int_{b_{l-1}}^{b_l} \psi_l(u)e^{u/\mu}\,\mathrm{d}u = \sum_{l=1}^{j-1}\Bigg(\sum_{i=1}^{2} \frac{C_{i,l}}{z_{i,l} + 1/\mu}$$
$$\times \left(e^{(z_{i,l}+1/\mu)b_l} - e^{(z_{i,l}+1/\mu)b_{l-1}}\right) + C_{3,l}\mu\left(e^{b_l/\mu} - e^{b_{l-1}/\mu}\right)\Bigg) \tag{32}$$

and

$$\int_{b_{j-1}}^{\infty} \psi(u)e^{-u/\bar{\mu}}\,\mathrm{d}u = \sum_{l=j}^{k} \int_{b_{l-1}}^{b_l} \psi_l(u)e^{-u/\bar{\mu}}\,\mathrm{d}u = \sum_{l=j}^{k}\Bigg(\sum_{i=1}^{2} \frac{C_{i,l}}{z_{i,l} - 1/\bar{\mu}}$$
$$\times \left(e^{(z_{i,l}-1/\bar{\mu})b_l} - e^{(z_{i,l}-1/\bar{\mu})b_{l-1}}\right) - C_{3,l}\bar{\mu}\left(e^{-b_l/\bar{\mu}} - e^{-b_{l-1}/\bar{\mu}}\right)\Bigg). \tag{33}$$

Substituting (23), (31), (32) and (33) into (30) and doing some simplifications yield (24). Next, from the equality $\psi(0) = 1$ we get (25). Lastly, substituting (31) into $d_j\psi'_j(b_j) = d_{j+1}\psi'_{j+1}(b_j)$ and (23) into $\psi_j(b_j) = \psi_{j+1}(b_j)$ for $1 \le j \le k-1$ immediately yields (26) and (27), respectively.

Thus, we get the system of $3k-1$ linear equations (24)–(27) to determine $3k-1$ unknown constants. That system has a unique solution provided that its determinant is not equal to 0. Hence, piecewise differential equation (19) has a unique solution satisfying certain conditions and that solution is given by (23). Since we have derived (19) from (18) without any additional assumptions concerning the differentiability of $\psi(x)$, we conclude that the functions $\psi_j(x)$, $1 \le j \le k$, given by (23) are unique solutions to (18) on the intervals $[b_{j-1}, b_j]$ satisfying certain conditions. This guarantees that the functions $\psi_j(x)$ we have found coincide with the ruin probability on $[b_{j-1}, b_j]$, which completes the proof. □

Remark 6. In particular, if $k = 2$, then $C_{3,2} = 0$ and the constants $C_{1,1}$, $C_{2,1}$, $C_{3,1}$, $C_{1,2}$ and $C_{2,2}$ are determined from the system of linear equations (34)–(38):

$$\sum_{i=1}^{2}\left(\frac{\bar{\lambda}}{\bar{\mu}z_{i,1}-1}\left(1-e^{(z_{i,1}-1/\bar{\mu})b_1}\right)+d_1 z_{i,1}+\lambda+\bar{\lambda}\right)C_{i,1}$$
$$+\left(\bar{\lambda}e^{-b_1/\bar{\mu}}+\lambda\right)C_{3,1}+\sum_{i=1}^{2}\frac{\bar{\lambda}e^{(z_{i,2}-1/\bar{\mu})b_1}}{\bar{\mu}z_{i,2}-1}C_{i,2}=\lambda, \tag{34}$$

$$\sum_{i=1}^{2}\frac{\lambda e^{-b_1/\mu}}{\mu z_{i,1}+1}\left(1-e^{(z_{i,1}+1/\mu)b_1}\right)C_{i,1}+\lambda\left(e^{-b_1/\mu}-1\right)C_{3,1}$$
$$+\sum_{i=1}^{2}\left(\frac{\bar{\lambda}e^{z_{i,2}b_1}}{\bar{\mu}z_{i,2}-1}+(d_2 z_{i,2}+\lambda+\bar{\lambda})e^{z_{i,2}b_1}\right)C_{i,2}=\lambda e^{-b_1/\mu}, \tag{35}$$

$$C_{1,1}+C_{2,1}+C_{3,1}=1, \tag{36}$$

$$d_1\sum_{i=1}^{2}z_{i,1}e^{z_{i,1}b_1}C_{i,1}-d_2\sum_{i=1}^{2}z_{i,2}e^{z_{i,2}b_1}C_{i,2}=0 \tag{37}$$

and

$$\sum_{i=1}^{2}e^{z_{i,1}b_1}C_{i,1}+C_{3,1}-\sum_{i=1}^{2}e^{z_{i,2}b_1}C_{i,2}=0 \tag{38}$$

provided that its determinant is not equal to 0.

The proposition below enables us to check whether the system of equations (34)–(38) has a unique solution. Let

$$\Delta=\frac{1}{\bar{\lambda}(\mu+\bar{\mu})^2}$$
$$\times\left(d_1\bar{\mu}(z_{1,1}-z_{2,1})\left(e^{b_1/\bar{\mu}}-e^{-b_1/\mu}\right)\left((\bar{\lambda}\bar{\mu}-\lambda\mu-d_1)e^{b_1/\bar{\mu}}-(\bar{\lambda}\bar{\mu}-\lambda\mu-d_2)\right)\right.$$
$$\left.-d_1\bar{\mu}e^{b_1/\bar{\mu}}(z_{1,1}-z_{2,1})(\bar{\lambda}\bar{\mu}-\lambda\mu-d_1)\left(e^{b_1/\bar{\mu}}-e^{-b_1/\mu}\right)\right.$$

$$+ d_1\mu(1-1/\bar{\mu})\big(e^{z_{1,1}b_1} - e^{z_{2,1}b_1}\big)\big((\bar{\lambda}\bar{\mu} - \lambda\mu - d_1)e^{b_1/\bar{\mu}} - (\bar{\lambda}\bar{\mu} - \lambda\mu - d_2)\big)$$
$$+ (d_2 - d_1)\big(e^{z_{1,1}b_1} - e^{z_{2,1}b_1}\big)(\bar{\lambda}\bar{\mu} - \lambda\mu - d_1)\big(e^{b_1/\bar{\mu}} - e^{-b_1/\mu}\big)$$
$$+ d_1^2\mu e^{b_1/\bar{\mu}}(\bar{\mu}-1)\big(z_{2,1}e^{z_{1,1}b_1} - z_{1,1}e^{z_{2,1}b_1}\big)$$
$$+ d_1\bar{\mu}(d_2 - d_1)\big(e^{b_1/\bar{\mu}} - e^{-b_1/\mu}\big)\big(z_{2,1}e^{z_{1,1}b_1} - z_{1,1}e^{z_{2,1}b_1}\big)\big).$$

Proposition 1. *The system of linear equations* (34)–(38) *has a unique solution if and only if* $\Delta \neq 0$.

Proof. From (36) we have $C_{3,1} = 1 - C_{1,1} - C_{2,1}$. Substituting that into (34), (35) and (38) yields

$$\sum_{i=1}^{2}\left(\frac{\bar{\lambda}}{\bar{\mu}z_{i,1}-1}\left(1 - e^{(z_{i,1}-1/\bar{\mu})b_1}\right) + d_1 z_{i,1} + \bar{\lambda} - \bar{\lambda}e^{-b_1/\bar{\mu}}\right)C_{i,1} + \sum_{i=1}^{2}\frac{\bar{\lambda}e^{(z_{i,2}-1/\bar{\mu})b_1}}{\bar{\mu}z_{i,2}-1}C_{i,2} = -\bar{\lambda}e^{-b_1/\bar{\mu}}, \tag{39}$$

$$\sum_{i=1}^{2}\left(\frac{\lambda e^{-b_1/\mu}}{\mu z_{i,1}+1}\left(1 - e^{(z_{i,1}+1/\mu)b_1}\right) + \lambda\big(1 - e^{-b_1/\mu}\big)\right)C_{i,1} + \sum_{i=1}^{2}\left(\frac{\bar{\lambda}e^{z_{i,2}b_1}}{\bar{\mu}z_{i,2}-1} + (d_2 z_{i,2} + \lambda + \bar{\lambda})e^{z_{i,2}b_1}\right)C_{i,2} = \lambda \tag{40}$$

and

$$\sum_{i=1}^{2}\big(e^{z_{i,1}b_1} - 1\big)C_{i,1} - \sum_{i=1}^{2} e^{z_{i,2}b_1}C_{i,2} = -1. \tag{41}$$

Thus, the system of equations (34)–(38) has a unique solution if and only if the system of equations (39), (40), (37) and (41) does.

Multiplying (41) by $(-d_2 z_{1,2})$, adding (37) and rearranging the terms we obtain

$$e^{z_{2,2}b_1}C_{2,2} = \frac{d_2 z_{1,2} - \sum_{i=1}^{2}(d_1 z_{i,1}e^{z_{i,1}b_1} + d_2 z_{1,2}(1 - e^{z_{i,1}b_1}))C_{i,1}}{d_2(z_{1,2} - z_{2,2})}. \tag{42}$$

Similarly, multiplying (41) by $(-d_2 z_{2,2})$, adding (37) and rearranging the terms we obtain

$$e^{z_{1,2}b_1}C_{1,2} = \frac{d_2 z_{2,2} - \sum_{i=1}^{2}(d_1 z_{i,1}e^{z_{i,1}b_1} + d_2 z_{2,2}(1 - e^{z_{i,1}b_1}))C_{i,1}}{d_2(z_{2,2} - z_{1,2})}. \tag{43}$$

Substituting (42) and (43) into (39) and doing some simplifications yield

$$\sum_{i=1}^{2}\left(\frac{\bar{\mu}z_{i,1}e^{b_1/\bar{\mu}} - e^{z_{i,1}b_1}}{\bar{\mu}z_{i,1}-1} + d_1 z_{i,1}e^{b_1/\bar{\mu}} - 1 + \frac{-d_1\bar{\mu}z_{i,1}e^{z_{i,1}b_1} + d_2(1 - \bar{\mu}z_{1,2} - \bar{\mu}z_{2,2})(1 - e^{z_{i,1}b_1})}{d_2(\bar{\mu}z_{1,2}-1)(\bar{\mu}z_{2,2}-1)}\right)C_{i,1}$$

$$= -\frac{\bar{\mu}^2 z_{1,2} z_{2,2}}{(\bar{\mu} z_{1,2} - 1)(\bar{\mu} z_{2,2} - 1)}. \tag{44}$$

By Vieta's theorem applied to (29) for $j = 2$, we have

$$z_{1,2} z_{2,2} = \frac{\bar{\lambda}\bar{\mu} - \lambda\mu - d_2}{d_2 \mu \bar{\mu}},$$

$$1 - \bar{\mu} z_{1,2} - \bar{\mu} z_{2,2} = \frac{d_2 \bar{\mu} + \mu\bar{\mu}(\lambda + \bar{\lambda})}{d_2 \mu}$$

and

$$(\bar{\mu} z_{1,2} - 1)(\bar{\mu} z_{2,2} - 1) = \frac{\bar{\lambda}\bar{\mu}(\mu + \bar{\mu})}{d_2 \mu}.$$

Substituting these equalities into (44) gives

$$\begin{aligned} &\sum_{i=1}^{2} \bigg(\frac{\bar{\mu} z_{i,1} e^{b_1/\bar{\mu}} - e^{z_{i,1} b_1}}{\bar{\mu} z_{i,1} - 1} + d_1 z_{i,1} e^{b_1/\bar{\mu}} - 1 \\ &\quad + \frac{-d_1 \mu z_{i,1} e^{z_{i,1} b_1} + (d_2 + \mu(\lambda + \bar{\lambda}))(1 - e^{z_{i,1} b_1})}{\bar{\lambda}(\mu + \bar{\mu})} \bigg) C_{i,1} \\ &= \frac{d_2 + \lambda\mu - \bar{\lambda}\bar{\mu}}{\bar{\lambda}(\mu + \bar{\mu})}. \end{aligned} \tag{45}$$

Next, multiplying (41) by $(\lambda + \bar{\lambda})$ and adding (37) and (40) we get

$$\begin{aligned} &\sum_{i=1}^{2} \bigg(\frac{\lambda e^{-b_1/\mu}}{\mu z_{i,1} + 1} \big(1 - e^{(z_{i,1} + 1/\mu) b_1}\big) + \lambda\big(1 - e^{-b_1/\mu}\big) + d_1 z_{i,1} e^{z_{i,1} b_1} \\ &\quad + (\lambda + \bar{\lambda})\big(e^{z_{i,1} b_1} - 1\big) \bigg) C_{i,1} + \sum_{i=1}^{2} \frac{\bar{\lambda} e^{z_{i,2} b_1}}{\bar{\mu} z_{i,2} - 1} C_{i,2} = -\bar{\lambda}. \end{aligned} \tag{46}$$

Multiplying (39) by $(-e^{b_1/\bar{\mu}})$ and adding (46) we obtain

$$\begin{aligned} &\sum_{i=1}^{2} \bigg(\frac{-\lambda\mu z_{i,1} e^{-b_1/\mu} - \lambda e^{z_{i,1} b_1}}{\mu z_{i,1} + 1} + \frac{-\bar{\lambda} e^{b_1/\bar{\mu}} + \bar{\lambda} e^{z_{i,1} b_1}}{\bar{\mu} z_{i,1} - 1} \\ &\quad - (d_1 z_{i,1} + \bar{\lambda}) e^{b_1/\bar{\mu}} + (d_1 z_{i,1} + \lambda + \bar{\lambda}) e^{z_{i,1} b_1} \bigg) C_{i,1} = \lambda - \bar{\lambda}. \end{aligned} \tag{47}$$

Thus, if the system of equations (45) and (47) has a unique solution, then $C_{1,2}$ and $C_{2,2}$ can be found from (43) and (42), respectively. Consequently, the system of equations (39), (40), (37) and (41) has a unique solution if and only if the system of equations (45) and (47) does.

A standard computation shows that the determinant of the system of equations (45) and (47) is equal to Δ defined above. In particular, here we use Vieta's theorem applied to (29) for $j = 1$. Therefore, the system of equations (34)–(38) has a unique solution if and only if $\Delta \neq 0$. □

5.2 Explicit formulas for the expected discounted dividend payments until ruin

By (17), equation (9) for the expected discounted dividend payments can be written as

$$\begin{aligned} &d_j v'(x) + (\lambda + \bar\lambda + \delta) v(x) \\ &\quad = \frac{\lambda}{\mu} e^{-x/\mu} \int_0^x v(u) e^{u/\mu}\,\mathrm{d}u + \frac{\bar\lambda}{\bar\mu} e^{x/\bar\mu} \int_x^\infty v(u) e^{-u/\bar\mu}\,\mathrm{d}u + d_j \end{aligned} \tag{48}$$

for all $x \in [b_{j-1}, b_j]$ and $1 \le j \le k$.

The piecewise integro-differential equation (48) can also be reduced to a piecewise linear differential equation with constant coefficients.

Lemma 2. *Let the surplus process $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ be defined by (2) under the above assumptions, and let claim and premium sizes be exponentially distributed with means μ and $\bar\mu$, respectively. Then $v(x)$ is a solution to the piecewise differential equation*

$$\begin{aligned} &d_j \mu\bar\mu v'''(x) + \big(d_j(\bar\mu - \mu) + \mu\bar\mu(\lambda + \bar\lambda + \delta)\big) v''(x) \\ &\quad + \big(\bar\mu(\bar\lambda + \delta) - \mu(\lambda + \delta) - d_j\big) v'(x) - \delta v(x) = -d_j \end{aligned} \tag{49}$$

for all $x \in [b_{j-1}, b_j]$ and $1 \le j \le k$.

The proof of the lemma is similar to the proof of Lemma 1.

For $1 \le j \le k$, let

$$\begin{aligned} \tilde{\mathrm{D}}_j = &-18\delta d_j \mu\bar\mu \big(d_j(\bar\mu - \mu) + \mu\bar\mu(\lambda + \bar\lambda + \delta)\big)\big(\bar\mu(\bar\lambda + \delta) - \mu(\lambda + \delta) - d_j\big) \\ &+ 4\delta\big(d_j(\bar\mu - \mu) + \mu\bar\mu(\lambda + \bar\lambda + \delta)\big)^3 \\ &+ \big(d_j(\bar\mu - \mu) + \mu\bar\mu(\lambda + \bar\lambda + \delta)\big)^2 \big(\bar\mu(\bar\lambda + \delta) - \mu(\lambda + \delta) - d_j\big)^2 \\ &- 4 d_j \mu\bar\mu \big(\bar\mu(\bar\lambda + \delta) - \mu(\lambda + \delta) - d_j\big)^3 - 27(\delta d_j \mu\bar\mu)^2. \end{aligned}$$

Theorem 4. *Let the surplus process $(X_t^{\mathbf{b}}(x))_{t\ge 0}$ be defined by (2) under the above assumptions, and let claim and premium sizes be exponentially distributed with means μ and $\bar\mu$, respectively. If the net profit condition (3) holds and $\min_{1\le j\le k} \tilde{\mathrm{D}}_j > 0$, then we have*

$$v_j(x) = \tilde C_{1,j} e^{\tilde z_{1,j} x} + \tilde C_{2,j} e^{\tilde z_{2,j} x} + \tilde C_{3,j} e^{\tilde z_{3,j} x} + d_j/\delta \tag{50}$$

for all $x \in [b_{j-1}, b_j]$ and $1 \le j \le k$, where $\tilde z_{1,j}$, $\tilde z_{2,j}$ and $\tilde z_{3,j}$ are distinct real roots of the cubic equation

$$d_j \mu\bar\mu z^3 + \big(d_j(\bar\mu - \mu) + \mu\bar\mu(\lambda + \bar\lambda + \delta)\big) z^2 + \big(\bar\mu(\bar\lambda + \delta) - \mu(\lambda + \delta) - d_j\big) z - \delta = 0, \tag{51}$$

$\tilde C_{3,k} = 0$ and all the other constants $\tilde C_{1,j}$, $\tilde C_{2,j}$ and $\tilde C_{3,j}$ are determined from the system of linear equations (52)–(55):

$$\begin{aligned} &\lambda e^{-b_{j-1}/\mu} \sum_{l=1}^{j-1} \bigg(\sum_{i=1}^{3} \frac{\tilde C_{i,l}}{\mu \tilde z_{i,l} + 1} \big(e^{(\tilde z_{i,l} + 1/\mu) b_l} - e^{(\tilde z_{i,l} + 1/\mu) b_{l-1}} \big) \bigg) \\ &\quad + \sum_{i=1}^{3} \bigg(\frac{\bar\lambda e^{b_{j-1}/\bar\mu}}{\bar\mu \tilde z_{i,j} - 1} \big(e^{(\tilde z_{i,j} - 1/\bar\mu) b_j} - e^{(\tilde z_{i,j} - 1/\bar\mu) b_{j-1}} \big) \end{aligned}$$

$$
\begin{aligned}
&- (d_j\tilde{z}_{i,j} + \lambda + \bar{\lambda} + \delta)e^{\tilde{z}_{i,j}b_{j-1}}\Big)\tilde{C}_{i,j} \\
&+ \bar{\lambda}e^{b_{j-1}/\bar{\mu}} \sum_{l=j+1}^{k} \Bigg(\sum_{i=1}^{3} \frac{\tilde{C}_{i,l}}{\bar{\mu}\tilde{z}_{i,l} - 1}\left(e^{(\tilde{z}_{i,l}-1/\bar{\mu})b_l} - e^{(\tilde{z}_{i,l}-1/\bar{\mu})b_{l-1}}\right)\Bigg) \\
&= \frac{d_j(\lambda+\bar{\lambda})}{\delta} - \frac{\lambda e^{-b_{j-1}/\mu}}{\delta}\sum_{l=1}^{j-1} d_l\big(e^{b_l/\mu} - e^{b_{l-1}/\mu}\big) \\
&\quad + \frac{\bar{\lambda}e^{b_{j-1}/\bar{\mu}}}{\delta}\sum_{l=j}^{k} d_l\big(e^{-b_l/\bar{\mu}} - e^{-b_{l-1}/\bar{\mu}}\big), \quad 1 \le j \le k, \qquad (52)
\end{aligned}
$$

$$
\tilde{C}_{1,1} + \tilde{C}_{2,1} + \tilde{C}_{3,1} = -d_1/\delta, \qquad (53)
$$

$$
\begin{aligned}
&d_j \sum_{i=1}^{3} \tilde{z}_{i,j}e^{\tilde{z}_{i,j}b_j}\tilde{C}_{i,j} - d_{j+1}\sum_{i=1}^{3} \tilde{z}_{i,j+1}e^{\tilde{z}_{i,j+1}b_j}\tilde{C}_{i,j+1} \\
&\quad = d_j - d_{j+1}, \quad 1 \le j \le k-1, \qquad (54)
\end{aligned}
$$

and

$$
\sum_{i=1}^{3} e^{\tilde{z}_{i,j}b_j}\tilde{C}_{i,j} - \sum_{i=1}^{3} e^{\tilde{z}_{i,j+1}b_j}\tilde{C}_{i,j+1} = \frac{d_{j+1} - d_j}{\delta}, \quad 1 \le j \le k-1, \qquad (55)
$$

provided that its determinant is not equal to 0.

Proof. By Lemma 2 and the notation introduced in Section 2, we deduce that the function $v_j(x)$ is a solution to (49) on $x \in [b_{j-1}, b_j]$ for each $1 \le j \le k$. In addition, it is easily seen that (51) is the corresponding characteristic equation and its discriminant coincides with the constant $\tilde{\mathrm{D}}_j$ introduced above. The assumption $\min_{1\le j\le k}\tilde{\mathrm{D}}_j > 0$ guarantees that cubic equation (51) has three distinct real roots $\tilde{z}_{1,j}$, $\tilde{z}_{2,j}$ and $\tilde{z}_{3,j}$. Hence, the general solution to (49) is given by (50) with some constants $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ and $\tilde{C}_{3,j}$.

By Vieta's theorem, we conclude that (51) has either two or no negative roots for each $1 \le j \le k$. Since the net profit condition (3) holds, applying arguments similar to those in [34, p. 70] shows that $\lim_{x\to\infty} v_k(x) = d_k/\delta$. Consequently, if equation (51) for $j = k$ had no negative roots, the function $v_k(x)$ would be constant, which is impossible. Therefore, equation (51) for $j = k$ has two negative roots. We denote those negative roots by $\tilde{z}_{1,k}$ and $\tilde{z}_{2,k}$, and let $\tilde{z}_{3,k}$ be the third root. Since $\tilde{z}_{3,k} > 0$, we conclude that $\tilde{C}_{3,k} = 0$.

To determine all the other constants $\tilde{C}_{1,j}$, $\tilde{C}_{2,j}$ and $\tilde{C}_{3,j}$, we need $3k-1$ boundary conditions. The first k conditions are obtained by letting $x = b_{j-1}$ in (48) for $1 \le j \le k$:

$$
\begin{aligned}
&d_j v'(b_{j-1}) + (\lambda + \bar{\lambda} + \delta)v(b_{j-1}) \\
&\quad = \frac{\lambda}{\mu}e^{-b_{j-1}/\mu}\int_0^{b_{j-1}} v(u)e^{u/\mu}\,\mathrm{d}u + \frac{\bar{\lambda}}{\bar{\mu}}e^{b_{j-1}/\bar{\mu}}\int_{b_{j-1}}^{\infty} v(u)e^{-u/\bar{\mu}}\,\mathrm{d}u + d_j. \qquad (56)
\end{aligned}
$$

One more condition is obtained from the equality $v(0) = 0$. The last $2(k-1)$ conditions are derived from the equalities $d_j v_j'(b_j) - d_j = d_{j+1} v_{j+1}'(b_j) - d_{j+1}$, which easily follow from (48), and $v_j(b_j) = v_{j+1}(b_j)$ for $1 \le j \le k-1$.

Substituting (50) into (56) as well as into the equalities $v(0) = 0$, $d_j v_j'(b_j) - d_j = d_{j+1} v_{j+1}'(b_j) - d_{j+1}$ and $v_j(b_j) = v_{j+1}(b_j)$ for $1 \le j \le k-1$ and doing some simplifications yield the system of linear equations (52)–(55), which has a unique solution provided that its determinant is not equal to 0. Thus, piecewise differential equation (49) has a unique solution satisfying certain conditions, and that solution is given by (50). Applying arguments similar to those in the proof of Theorem 3 guarantees that the functions $v_j(x)$ we have found coincide with the expected discounted dividend payments until ruin on $[b_{j-1}, b_j]$, which completes the proof. □

Remark 7. In particular, if $k = 2$, then $\tilde{C}_{3,2} = 0$ and the constants $\tilde{C}_{1,1}$, $\tilde{C}_{2,1}$, $\tilde{C}_{3,1}$, $\tilde{C}_{1,2}$ and $\tilde{C}_{2,2}$ are determined from the system of linear equations (57)–(61):

$$\sum_{i=1}^{3}\left(\frac{\bar{\lambda}}{\bar{\mu}\tilde{z}_{i,1}-1}\left(1-e^{(\tilde{z}_{i,1}-1/\bar{\mu})b_1}\right)+d_1\tilde{z}_{i,1}+\lambda+\bar{\lambda}+\delta\right)\tilde{C}_{i,1}$$
$$+\sum_{i=1}^{2}\frac{\bar{\lambda}e^{(\tilde{z}_{i,2}-1/\bar{\mu})b_1}}{\bar{\mu}\tilde{z}_{i,2}-1}\tilde{C}_{i,2} = -\frac{d_1\lambda}{\delta}-\frac{\bar{\lambda}(d_1-d_2)e^{-b_1/\bar{\mu}}}{\delta}, \tag{57}$$

$$\lambda e^{-b_1/\mu}\sum_{i=1}^{3}\frac{\tilde{C}_{i,1}}{\mu\tilde{z}_{i,1}+1}\left(1-e^{(\tilde{z}_{i,1}+1/\mu)b_1}\right)+\sum_{i=1}^{2}\left(\frac{\bar{\lambda}e^{\tilde{z}_{i,2}b_1}}{\bar{\mu}\tilde{z}_{i,2}-1}\right.$$
$$\left.+(d_2\tilde{z}_{i,2}+\lambda+\bar{\lambda}+\delta)e^{\tilde{z}_{i,2}b_1}\right)\tilde{C}_{i,2} = \frac{\lambda(d_1-d_2)}{\delta}-\frac{d_1\lambda e^{-b_1/\mu}}{\delta}, \tag{58}$$

$$\tilde{C}_{1,1}+\tilde{C}_{2,1}+\tilde{C}_{3,1} = -d_1/\delta, \tag{59}$$

$$d_1\sum_{i=1}^{3}\tilde{z}_{i,1}e^{\tilde{z}_{i,1}b_1}\tilde{C}_{i,1}-d_2\sum_{i=1}^{2}\tilde{z}_{i,2}e^{\tilde{z}_{i,2}b_1}\tilde{C}_{i,2} = d_1-d_2 \tag{60}$$

and

$$\sum_{i=1}^{3}e^{\tilde{z}_{i,1}b_1}\tilde{C}_{i,1}-\sum_{i=1}^{2}e^{\tilde{z}_{i,2}b_1}\tilde{C}_{i,2} = \frac{d_2-d_1}{\delta} \tag{61}$$

provided that its determinant is not equal to 0.

6 Numerical illustrations

To present numerical examples for the results obtained in Section 5, we set $\lambda = 0.1$, $\bar{\lambda} = 2.3$, $\mu = 3$, $\bar{\mu} = 0.2$, $b = 5$ and $\delta = 0.01$.

In addition, we denote by $\psi^*(x)$ the ruin probability in the corresponding model without dividend payments. It is given by

$$\psi^*(x) = \frac{\lambda(\mu+\bar{\mu})}{\bar{\mu}(\lambda+\bar{\lambda})}\exp\left(-\frac{(\bar{\lambda}\bar{\mu}-\lambda\mu)x}{\mu\bar{\mu}(\lambda+\bar{\lambda})}\right), \quad x \in [0,\infty),$$

(see [5, 28]). For the parameters chosen above, $\psi^*(x) \approx 0.666667e^{-0.111111x}$.

Table 1. The ruin probabilities without and with dividend payments and the expected discounted dividend payments, $d_1 = 0.05$ and $d_2 = 0.1$

x	$\psi^*(x)$	$\psi(x)$	$v(x)$
0	0.666667	1	0
1	0.596560	0.777184	3.663273
2	0.533825	0.737542	4.283457
5	0.382502	0.636926	5.911685
7	0.306284	0.575029	6.716708
10	0.219462	0.492173	7.623108
15	0.125917	0.379750	8.612682
20	0.072245	0.293007	9.190265
50	0.002577	0.061825	9.967986
70	0.000279	0.021912	9.996285

Moreover, let now $d_1 = 0.05$ and $d_2 = 0.1$. Applying Theorems 3 and 4 as well as Remarks 6 and 7 we can calculate the ruin probability $\psi(x)$ and the expected discounted dividend payments until ruin $v(x)$:

$$\psi_1(x) \approx 0.530821e^{-0.084781x} + 0.179668e^{-43.248552x} + 0.289512, \quad x \in [0, 5],$$

$$\psi_2(x) \approx 0.826718e^{-0.051863x} - 7.043723 \cdot 10^{38} e^{-19.28147x}, \quad x \in [5, \infty);$$

$$v_1(x) \approx 5 - 2.992137e^{-43.470279x} - 4.421273e^{-0.124597x} + 2.41341e^{0.061543x}, \quad x \in [0, 5],$$

$$v_2(x) \approx 10 - 2.198169 \cdot 10^{40} e^{-19.405407x} - 6.97712e^{-0.107684x}, \quad x \in [5, \infty).$$

Table 1 presents the results of calculations for some values of x.

Next, for $d_1 = 0.1$ and $d_2 = 0.05$, we get

$$\psi_1(x) \approx 1.204304e^{-0.051863x} + 0.218067e^{-19.28147x} - 0.42237, \quad x \in [0, 5],$$

$$\psi_2(x) \approx 0.772527e^{-0.084781x} + 1.012903 \cdot 10^{91} e^{-43.248552x}, \quad x \in [5, \infty);$$

$$v_1(x) \approx 10 - 2.219094e^{-19.405407x} - 5.609737e^{-0.107684x} - 2.171169e^{0.079758x}, \quad x \in [0, 5],$$

$$v_2(x) \approx 5 + 5.716149 \cdot 10^{92} e^{-43.470279x} - 2.857069e^{-0.124597x}, \quad x \in [5, \infty).$$

The values of $\psi^*(x)$, $\psi(x)$ and $v(x)$ for some x are given in Table 2.

The results presented in Tables 1 and 2 indicate that dividend payments substantially increase the ruin probability. The first strategy is much more profitable, although the corresponding ruin probability is larger in that case.

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Table 2. The ruin probabilities without and with dividend payments and the expected discounted dividend payments, $d_1 = 0.1$ and $d_2 = 0.05$

x	$\psi^*(x)$	$\psi(x)$	$v(x)$
0	0.666667	1	0
1	0.596560	0.721066	2.611525
2	0.533825	0.663275	2.930525
5	0.382502	0.506845	3.490686
7	0.306284	0.426750	3.805635
10	0.219462	0.330912	4.178134
15	0.125917	0.216577	4.559200
20	0.072245	0.141747	4.763582
50	0.002577	0.011141	4.994372
70	0.000279	0.002044	4.999534

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