Martingale-like sequences in Banach lattices

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Received: 15 April 2018, Revised: 7 October 2018, Accepted: 8 October 2018, Published online: 7 November 2018

Abstract Martingale-like sequences in vector lattice and Banach lattice frameworks are defined in the same way as martingales are defined in [Positivity 9 (2005), 437–456]. In these frameworks, a collection of bounded $X$-martingales is shown to be a Banach space under the supremum norm, and under some conditions it is also a Banach lattice with coordinate-wise order. Moreover, a necessary and sufficient condition is presented for the collection of $E$-martingales to be a vector lattice with coordinate-wise order. It is also shown that the collection of bounded $E$-martingales is a normed lattice but not necessarily a Banach space under the supremum norm.

Keywords Banach lattices, martingales, $E$-martingales, $X$-martingales

2010 MSC Primary 60G48; Secondary 46A40, 46B42

1 Introduction

The classical definition of martingales is extended to a more general case in the space of Banach lattices by V. Troitsky [6]. In the Banach lattice framework, martingales are defined without a probability space and the famous Doob’s convergence theorem was reproduced. Moreover, under certain conditions on the Banach lattice, it was shown that the set of bounded martingales forms a Banach lattice with respect to the point-wise order. In 2011, H. Gessesse and V. Troitsky [2] produced several sufficient
conditions for the space of bounded martingales on a Banach lattice to be a Banach lattice itself. They also provided examples showing that the space of bounded martingales is not necessarily a vector lattice. Several other works have been done by other authors with regard to martingales in vector lattices, such as [4, 3].

In the theory of random processes, not just the study of martingale convergence is important, but the study of convergence of martingale-like stochastic sequences and processes, and the determination of interrelation between them are also crucial. So it is natural to ask if martingale-like sequences can be defined in a vector lattice or Banach lattice framework. In this article, we define and study martingale-like sequences in Banach lattices along the same lines as martingales are defined and studied in [6].

Classically, a martingale-like sequence is defined as follows (for instance, see a paper by A. Melnikov [5]). Consider a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $(\mathcal{F}_n)_{n=1}^\infty$, i.e., an increasing sequence of complete sub-sigma-algebras of $\mathcal{F}$. An integrable stochastic sequence $x = (x_n, \mathcal{F}_n)$ is an $L^1$-martingale if

$$\lim_{n \to \infty} \sup_{m \geq n} E|E(x_m | \mathcal{F}_n) - x_n| = 0.$$ 

An integrable stochastic sequence $x = (x_n, \mathcal{F}_n)$ is an $E$-martingale if

$$P\{\omega : E(x_{n+1} | \mathcal{F}_n) \neq x_n \text{ infinitely often} \} = 0.$$ 

Here we extend the definition of $L^1$-martingales and $E$-martingales in a general Banach lattice $X$ following the same lines as the definition of martingales in Banach lattices in [6]. First we mention some terminology and definitions from the theory of Banach lattices for the reader convenience. For more detailed exploration, we refer the reader to [1]. A vector lattice is a vector space equipped with a lattice order relation, which is compatible with the linear structure. A Banach lattice is a vector lattice with a Banach norm which is monotone, i.e., $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$, and satisfies $\|x\| = \||x||$ for any two vectors $x$ and $y$. A vector lattice is said to be order complete if every nonempty subset that is bounded above has a supremum. We say that a Banach lattice has order continuous norm if $\|x_\alpha\| \to 0$ for every decreasing net $(x_\alpha)$ with $\inf x_\alpha = 0$. A Banach lattice with order continuous norm is order complete. A sublattice $Y$ of a vector lattice is called an (order) ideal if $y \in Y$ and $|x| \leq |y|$ imply $x \in Y$. An ideal $Y$ is called a band if $x = \sup_\alpha x_\alpha$ implies $x \in Y$ for every positive increasing net $(x_\alpha)$ in $Y$. Two elements $x$ and $y$ in a vector lattice are said to be disjoint whenever $|x| \land |y| = 0$ holds. If $J$ is a nonempty subset of a vector lattice, then its disjoint complement $J^d$ is the set of all elements of the lattice, disjoint to every element of $J$. A band $Y$ in a vector lattice $X$ that satisfies $X = Y \otimes Y^d$ is referred to as a projection band. Every band in an order complete vector lattice is a projection band. An operator $T$ on a vector lattice $X$ is positive if $Tx \geq 0$ for every $x \geq 0$. A sequence of positive projections $(E_n)$ on a vector lattice $X$ is called a filtration if $E_n E_m = E_{n \wedge m}$. A sequence of positive contractive projections $(E_n)$ on a normed lattice $X$ is called a contractive filtration if $E_n E_m = E_{n \wedge m}$. A filtration $(E_n)$ in a normed lattice $X$ is called dense if $E_n x \to x$ for each $x$ in $X$. In many articles such as in [6], a martingale with respect to a filtration $(E_n)$ in a vector lattice $X$ is defined as a sequence $(x_n)$ in $X$ such that $E_n x_m = x_n$ whenever $m \geq n$. 
2 Main definitions

Definition 1. A sequence \((x_n)\) of elements of a normed lattice \(X\) is called an \(X\)-martingale relative to a contractive filtration \((E_n)\) if
\[
\lim_{n \to \infty} \sup_{m \geq n} \|E_n x_m - x_n\| = 0.
\]

Definition 2. A sequence \((x_n)\) of elements of a vector lattice \(X\) is called an \(E\)-martingale relative to a filtration \((E_n)\) if there exists \(n \geq 1\) such that \(E_n x_m + 1 = x_m\) for all \(m \geq n\).

Note that Definition 2 is equivalent to saying a sequence \((x_n)\) is an \(E\)-martingale if there exists \(l \geq 1\) such that \(E_n x_m = x_n\) whenever \(m \geq n \geq l\). The symbol “\(E\)” stands for eventual so when we say \((x_n)\) is an \(E\)-martingale, we are saying that after a first few finite elements of the sequence, the sequence becomes a martingale.

Sequences defined by Definition 1 and Definition 2 are collectively called martingale-like sequences. Notice that every martingale \((x_n)\) in a vector lattice \(X\) with respect to a filtration \((E_n)\) is obviously an \(E\)-martingale with respect to the filtration \((E_n)\). Moreover, every \(\mathcal{E}\)-martingale \((x_n)\) in a Banach lattice \(X\) with respect to a contractive filtration \((E_n)\) is an \(X\)-martingale with respect to the contractive filtration \((E_n)\). If \(x\) is in a normed space \(X\) and \((E_n)\) is a contractive filtration, then the sequence \((E_n x)\) is an \(X\)-martingale with respect to the contractive filtration \((E_n)\).

By considering any nonzero martingale \((x_n)\) in a Banach lattice \(X\) with respect to filtration \((E_n)\) where \(x_1\) is nonzero without loss of generality, we can define a sequence \((y_n)\) such that \(y_1 = 2x_1\) and \(y_n = x_n\) for all \(n \geq 2\). Then one can see that \((y_n)\) is an \(E\)-martingale with respect to the filtration \((E_n)\). However, \((y_n)\) is not a martingale.

Note that every sequence which converges to zero is an \(X\)-martingale with respect to any contractive filtration \((E_n)\) because if \(x_n \to 0\) and \(m > n\) then \(\|E_n x_m - x_n\| \leq \|x_m\| + \|x_n\| \to 0\) as \(n \to \infty\). So one can easily create an \(X\)-martingale \((x_n)\) which is not \(E\)-martingale by setting \(x_n = \frac{1}{n} x\) where \(x\) is a nonzero vector in \(X\).

A martingale-like sequence \(A = (x_n)\) with respect to a contractive filtration \((E_n)\) on a normed lattice \(X\) is said to be bounded if its norm defined by \(\|A\| = \sup_n \|x_n\|\) is finite. Given a contractive filtration \((E_n)\) on a normed lattice \(X\), we denote the set of all bounded \(X\)-martingales with respect to the contractive filtration \((E_n)\) by \(M_X = M_X(X, (E_n))\) and the set of all bounded \(E\)-martingales with respect to the contractive filtration \((E_n)\) by \(M_E = M_E(X, (E_n))\). With the introduction of the sup norm in these spaces, one can show that \(M_X\) and \(M_E\) are normed spaces. Keeping the notation \(M\) of [6] for all bounded martingales with respect to the contractive filtration \((E_n)\) and from the preceding arguments, these spaces form a nested increasing sequence of linear subspaces \(M \subset M_E \subset M_X \subset \ell_\infty(X)\), with the norm being exactly the \(\ell_\infty(X)\) norm.

Theorem 3. Let \((E_n)\) be a contractive filtration on a Banach lattice \(X\), then the collection of \(X\)-martingales \(M_X\) is a closed subspace of \(\ell_\infty(X)\), hence a Banach space.
Proof. Suppose a sequence \((A^m) = (x^m_n)\) of \(X\)-martingales converges to \(A\) in \(\ell_\infty(X)\). We show \(A\) is also an \(X\)-martingale. Indeed, from \(\|A^m - A\| = \sup_n \|x^m_n - x_n\| \to 0\) as \(m \to \infty\), we have that for each \(n \geq 1\), \(\|x^m_n - x_n\| \to 0\) as \(m \to \infty\).

Note that for \(l \geq n\),

\[
\|E_nx_l - x_n\| = \|E_nx_l - E_nx^m_l + E_nx^m_l - x^m_l + x^m_l - x_n\| \\
\leq \|E_nx_l - E_nx^m_l\| + \|E_nx^m_l - x^m_l\| + \|x^m_l - x_n\|.
\]

From these inequalities and the contractive property of the filtration, we have

\[
\lim_{n \to \infty} \sup_{l \geq n} \|E_nx_l - x_n\| = 0. \tag{1}
\]

Corollary 1. Let \((E_n)\) be a contractive filtration on a Banach lattice \(X\), then \(ME \subset MX\).

Lemma 1. Let \((E_n)\) be a contractive filtration on a Banach lattice \(X\) and \(A = (x_n)\) be in \(MX\) where \(x_n \to x\). Then

\[
\lim_{n \to \infty} \sup_{m \geq n} \|E_mx - x_m\| = 0.
\]

Proof. Let \(A = (x_n)\) be in \(MX\) where \(x_n \to x\). Thus, for \(m \geq n\)

\[
\|E_nx - x_n\| = \|E_nx - E_nx_m + E_nx_m - x_n\| \leq \|x - x_m\| + \|E_nx_m - x_n\|.
\]

Taking \(\lim_{n \to \infty} \sup_{m \geq n}\) on both sides of the inequality completes the proof. \(\Box\)

The following proposition confirms that for any convergent element \(A = (x_n)\) of \(MX\) we can find a sequence in \(ME\) that converges to \(A\).

Proposition 4. Let \((E_n)\) be a contractive filtration on a Banach lattice \(X\) and \(A = (x_n)\) be a sequence in \(MX\) such that \(x_n \to x\). Then there exists a sequence \(A^m\) in \(ME\) such that \(A^m \to A\) in \(\ell_\infty(X)\).

Proof. Suppose \(x_n \to x\) as \(n \to \infty\). First note that the sequence \((E_nx)\) is in \(M\). Now define \(A^m = (a^m_n)\) such that

\[
a^m_n = \begin{cases} 
  x_n, & \text{for } n \leq m, \\
  E_nx, & \text{for } n > m.
\end{cases}
\]

Then \(A^m \in ME\) and \(A^m \to A\) in \(\ell_\infty(X)\), hence \(A \in ME\). Indeed, by Lemma 1,

\[
\lim_{m \to \infty} \|A^m - A\| = \lim_{m \to \infty} \sup_{j} \|E_{m+j}x - x_{m+j}\| = 0. \tag{2}
\]

In [6] and [2] several sufficient conditions are established where the set of bounded martingales \(M\) is a Banach lattice. In [2], counter examples are provided where \(M\) is not a Banach lattice. So, one may similarly ask when are \(MX\) and \(ME\) Banach spaces and Banach lattices? We start by showing a counter example that illustrates that \(ME\) is not necessarily a Banach space.
Example 5. Let $c_0$ be the set of sequences converging to zero. Consider the filtration $(E_n)$ where $E_n \sum_{i=1}^{\infty} \alpha_i e_i = \sum_{i=1}^{n} \alpha_i e_i$. Thus the sequence $(y_n)$ where $y_n = \sum_{i=1}^{n} \frac{1}{i} e_i$ is an $E$-martingale with respect to this filtration. We define a sequence of $E$-martingales $A^m$ as $A^m = (x^m_n)$ where

$$x^m_n = \begin{cases} \sum_{i=n}^{\infty} \frac{1}{i} e_i, & \text{for } n \leq m, \\ y_n/m, & \text{for } n > m. \end{cases}$$

Define a sequence $A = (x_n)$ where $x_n = \sum_{i=1}^{\infty} \frac{1}{i} e_i$. We can see that $A$ is not an $E$-martingale. But one can show that $A^m$ converges to $A$. Indeed,

$$\|A^m - A\| = \sup_n \|x^m_n - x\| = \sup_{n \in \{m+1, m+2, \ldots\}} \left\|y_n/m - \sum_{i=n}^{\infty} \frac{1}{i} e_i \right\| \to 0$$

as $m \to \infty$.

3 When is $M_E$ a vector lattice?

Given a vector (Banach) lattice $X$ and a filtration (respectively contractive) $(E_n)$ on $X$, we can introduce order structure on the spaces $M_E$ and $M_X$ as follows. For two bounded $E$-martingales (respectively $X$-martingales) $A = (x_n)$ and $B = (y_n)$, we write $A \geq B$ if $x_n \geq y_n$ for each $n$. With this order $M_E$ and $M_X$ are ordered vector spaces and the monotonicity of the norm follows from the monotonicity of the norm of $X$, i.e., for two $E$-martingales (respectively $X$-martingales) with $0 \leq A \leq B$, we have $\|A\| \leq \|B\|$. For two $E$-martingales (respectively $X$-martingales) $A = (x_n)$ and $B = (y_n)$, one may guess that $A \lor B$ (or $A \land B$) can be computed by the formulas $A \lor B = (x_n \lor y_n)$ (or $A \land B = (x_n \land y_n)$). We show in the following theorem that this is in fact the case in order for $M_E$ to be a vector lattice. However, this is not obvious to show in the case of $M_X$.

Theorem 6. Let $X$ be a vector lattice. Then the following statements are equivalent.

(i) $M_E$ is a vector lattice.

(ii) For each $A = (x_n)$ in $M_E$, the sequence $(|x_n|)$ is an $E$-martingale and $|A| = (|x_n|)$.

(iii) $M_E$ is a sublattice of $\ell_\infty(X)$.

Proof. First we show (i) $\implies$ (ii). Suppose $M_E$ is a vector lattice and $A = (x_n)$ is in $M_E$. Since $M_E$ is a vector lattice, $|A|$ exists in $M_E$, say $|A| = B := (y_n)$. Since $\pm A \leq B$, for each $n$, $\pm x_n \leq y_n$. So, $|x_n| \leq y_n$ for each $n$. Since $B$ is in $M_E$, there exists $l$ such that $E_n y_m = y_n$ whenever $m \geq n \geq l$. Now we claim that $y_n = |x_n|$ for each $n$. Fix $k > l$. We show $y_n = |x_n|$ for each $n \leq k$.

Indeed, define an $E$-martingale $C = (z_n)$ where

$$z_n = \begin{cases} |x_n|, & \text{for } n \leq k, \\ y_n, & \text{for } n > k. \end{cases}$$
Since \( k > l \) we can easily see that \( C \) is an \( \mathcal{E} \)-martingale. Moreover, \( C \geq 0 \) and \( \pm A \leq C \leq B \). Since \(|A| = B, C = B\). Thus, for every \( n \leq k, y_n = |x_n|\). This establishes (ii).

(iii) \( \implies \) (i) is straightforward. \( \square \)

Using the equivalence in Theorem 6, the following examples illustrate that \( M_E \) is not always a vector lattice.

**Example 7.** Consider the classical martingale \((x_n)\) in \( L_1[0, 1] \) where \( x_n = 2^n1_{[0,2^{-n}]} - 1 \) with the filtration \((\mathcal{F}_n)\) where \( \mathcal{F}_n \) is the smallest sigma algebra generated by the set

\[ \{ [0, 2^{-n}], (2^{-n}, 2^{-n+1}], \ldots, (1 - 2^{-n}, 1] \}. \]

One can easily show that

\[ E_n|x_{n+1}| = E[|x_{n+1}||x_n] \neq |x_n| \]

for every \( n \) and the sequence \((|x_n|)\) fails to be an \( \mathcal{E} \)-martingale. Hence, Theorem 6 implies that \( M_E \) is not a vector lattice.

**Example 8.** Consider the filtration \((E_n)\) defined on \( c_0 \) as follows. For each \( n = 0, 1, 2, \ldots \)

\[ E_n = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & 1/2 & 1/2 & \\ & & 1/2 & 1/2 & \\ & & & & \\ & & & & 1/2 & 1/2 & \\ & & & & 1/2 & 1/2 & \\ & & & & & & \end{bmatrix} \]

with \( 2n \) ones in the upper left corner. For each \( e_i = (0, \ldots, 0, 1_{ith}, 0, \ldots) \), \( E_ne_i = e_i \) if \( i \leq 2n \) and \( E_ne_{2k-1} = E_ne_{2k} = \frac{1}{2}(e_{2k-1} + e_{2k}) \) if \( n < k \). Now if we define a sequence \( A = (x_n) \) where for each \( n = 0, 1, 2, \ldots, \)

\[ x_n = (-1, 1, \ldots, -1, 1, 0, \ldots), \]

one can show this is a martingale as a result an \( \mathcal{E} \)-martingale. However, \(|A| = (|x_n|)\) where

\[ |x_n| = (1, \ldots, 1, 0, \ldots) \]

is not an \( \mathcal{E} \)-martingale. So, Theorem 6 implies that \( M_E \) is not a vector lattice.

**Proposition 9.** If a filtration \((E_n)\) is a sequence of band projections, then \( M_E \) is a vector lattice with coordinate-wise lattice operations.
Proof. If $A = (x_n) \in M_E$, then there exists $l$ such that $E_n x_m = x_n$ whenever $m \geq n \geq l$. Thus, $E_n |x_m| = |E_n x_m| = |x_n|$. So, $|A| = (|x_n|)$ and thus $M_E$ is a vector lattice. 

**Theorem 10.** If $M_E$ is a normed lattice and the filtration $(E_n)$ is dense in $X$, then for each $x$ in $X$, there exists $l$ such that $|E_n x| = E_n |x|$ whenever $n \geq l$.

Proof. Let $x$ be in $X$. Then $(E_n)$ is dense means $E_n x \to x$. Moreover, $(E_n x)$ is a martingale. Since $M_E$ is a vector lattice, by Theorem 6, $(|E_n x|)$ is an $E$-martingale. Thus there exists $l$ such that for any $m$ and $n$ with $m \geq n \geq l$, $|E_n E_m x| = |E_n x|$ and $E_n |E_m x| = |E_n x|$. So, $|E_n E_m x| = E_n |E_m x|$ and letting $m \to \infty$, we have $|E_n x| = E_n |x|$.

**4 When is $M_X$ a Banach lattice?**

Under the pointwise order structure on $M_X$, for an $X$-martingale $A = (x_n)$, we can refer to Example 8 to show that the sequence $(|x_n|)$ is not necessarily an $X$-martingale. However, under certain assumptions, we can show that $(|x_n|)$ is an $X$-martingale for every $X$-martingale $A = (x_n)$ making $M_X$ a Banach lattice.

**Proposition 11.** If $(E_n)$ is a contractive filtration where $E_n$ is a band projection for every $n$ then $M_X$ is a Banach lattice with coordinate-wise lattice operations.

Proof. Let $A = (x_n)$ be an $X$-martingale. For each $n$ and $m$, $E_n$ is a band projection implies $E_n |x_m| = |E_n x_m|$. Thus, by the fact that $|x| - |y| \leq |x - y|$, for $m \geq n$,

$$\|E_n |x_m| - |x_n|\| = \|E_n x_m| - |x_n|\| \leq \|E_n x_m - x_n\|.$$ 

This implies

$$\lim_{n \to \infty} \sup_{m \geq n} \|E_n |x_m| - |x_n|\| = 0$$

which implies $|A| = (|x_n|)$ is also an $X$-martingale. 

**Question.** From Theorem 6, $M_E$ is a vector lattice if and only if for each $E$-martingale $(x_n)$, the sequence $(|x_n|)$ is also an $E$-martingale. This is the case when the filtration is a sequence of band projections. Can one give a characterization of the filtrations for which $M_E$ is a vector lattice? Or, can one give an example of a filtration which is not a sequence of projections and the corresponding $M_E$ is a vector lattice?

**References**


