On generalized stochastic fractional integrals and related inequalities

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Abstract The generalized mean-square fractional integrals $\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}$ and $\mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}$ of the stochastic process X are introduced. Then, for Jensen-convex and strongly convex stochastic processes, the generalized fractional Hermite–Hadamard inequality is establish via generalized stochastic fractional integrals.

Keywords Hermite–Hadamard inequality, stochastic fractional integrals, convex stochastic process

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1 Introduction

In 1980, Nikodem [11] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [17] obtained some further results on convex stochastic processes.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, .)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

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(i) continuous in probability in interval I, if for all $t_0 \in I$ we have

$$P\text{-}\lim_{t \to t_0} X(t, .) = X(t_0, .),$$

where $P\text{-}\lim$ denotes the limit in probability.

(ii) *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} E\big[\big(X(t) - X(t_0)\big)^2\big] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t, .)$.

Obviously, *mean-square* continuity implies continuity in probability, but the converse implication is not true.

Definition 1. Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{a_{m,0}, \dots, a_{m,n_m}\}$. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the n-th partition tends to zero, i.e.,

$$\lim_{m \to \infty} \sup_{1 \le i \le n_m} |a_{m,i} - a_{m,i-1}| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [18].

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t)^2] < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \dots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a, b]$, if we have

$$\lim_{n \to \infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequences of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \dots, n$. Then, we write

$$Y(\cdot) = \int_a^b X(s, \cdot)ds \text{ (a.e.).}$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

Throughout the paper we will frequently use the monotonicity of the mean-square integral. If $X(t, \cdot) \le Y(t, \cdot)$ (a.e.) in some interval $[a, b]$, then

$$\int_a^b X(t, \cdot)dt \le \int_a^b Y(t, \cdot)dt \text{ (a.e.).}$$

Of course, this inequality is an immediate consequence of the definition of the mean-square integral.

Definition 2. We say that a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is convex, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X\big(\lambda u + (1 - \lambda)v, \cdot\big) \le \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \quad \text{(a.e.)} \tag{1}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is concave if $(-X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in [11, 18].

Now, we present some results proved by Kotrys [6] about Hermite–Hadamard inequality for convex stochastic processes.

Lemma 1. *If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, \cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E[A^2] < \infty$, $E[B^2] < \infty$ and $[a, b] \subset I$, then*

$$\int_a^b X(t, \cdot)dt = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b - a) \text{ (a.e.)}.$$

Proposition 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in int\, I$. Then there exists a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is*

$$X(t, \cdot) \ge A(\cdot)(t - t_0) + X(t_0, \cdot) \text{ (a.e.)}.$$

for all $t \in I$.

Theorem 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex, mean-square continuous in the interval I stochastic process. Then for any $u, v \in I$ we have*

$$X\left(\frac{u + v}{2}, \cdot\right) \le \frac{1}{v - u} \int_u^v X(t, \cdot)dt \le \frac{X(u, \cdot) + X(v, \cdot)}{2} \text{ (a.e.)} \tag{2}$$

In [7], Kotrys introduced the concept of strongly convex stochastic processes and investigated their properties.

Definition 3. Let $C : \Omega \to \mathbb{R}$ denote a positive random variable. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is called strongly convex with modulus $C(\cdot) > 0$, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X\big(\lambda u + (1 - \lambda)v, \cdot\big) \le \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) - C(\cdot)\lambda(1 - \lambda)(u - v)^2 \quad \text{a.e.}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is strongly Jensen-convex with modulus $C(\cdot)$.

In [5], Hafiz gave the following definition of stochastic mean-square fractional integrals.

Definition 4. For the stochastic proces $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals I_{u+}^α and I_{v+}^α of X of order $\alpha > 0$ is defined by

$$I_{u+}^\alpha[X](t) = \frac{1}{\Gamma(\alpha)} \int_u^t (t-s)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t > u,$$

and

$$I_{v-}^\alpha[X](t) = \frac{1}{\Gamma(\alpha)} \int_t^v (s-t)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t < v.$$

Using this concept of stochastic mean-square fractional integrals I_{a+}^α and I_{b+}^α, Agahi and Babakhani proved the following Hermite–Hadamard type inequality for convex stochastic processes:

Theorem 2. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval I. Then for any $u, v \in I$, the following Hermite–Hadamard inequality*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[I_{u+}^\alpha[X](v) + I_{v-}^\alpha[X](u)\right] \leq \frac{X(u,\cdot) + X(v,\cdot)}{2} \quad (a.e.)$$

(3)

holds, where $\alpha > 0$.

For more information and recent developments on Hermite–Hadamard type inequalities for stochastic process, please refer to [2–4, 9–11, 14, 16, 15, 20, 19].

2 Main results

In tis section, we introduce the concept of the generalized mean-square fractional integrals $\mathcal{J}_{\rho,\lambda,u+;\omega}^\sigma$ and $\mathcal{J}_{\rho,\lambda,v-;\omega}^\sigma$ of the stochastic process X.

In [13], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^\sigma(x) = \mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\dots}(x) = \sum_{k=0}^\infty \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < \mathcal{R}), \quad (4)$$

where the cofficents $\sigma(k)$ ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) make a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. For more information on the function (4), please refer to [8, 12]. With the help of (4), we give the following definition.

Definition 5. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process. The generalized mean-square fractional integrals $\mathcal{J}_{\rho,\lambda,a+;\omega}^\alpha$ and $\mathcal{J}_{\rho,\lambda,b-;\omega}^\alpha$ of X are defined by

$$\mathcal{J}_{\rho,\lambda,u+;\omega}^\sigma[X](x) = \int_u^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(x-t)^\rho\right] X(t,\cdot)dt, \quad (a.e.) \ x > u, \quad (5)$$

and

$$\mathcal{J}_{\rho,\lambda,v-;\omega}^\sigma[X](x) = \int_x^v (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(t-x)^\rho\right] X(s,\cdot)dt, \quad (a.e.) \ x < v, \quad (6)$$

where $\lambda, \rho > 0, \omega \in \mathbb{R}$.

Many useful generalized mean-square fractional integrals can be obtained by specializing the coefficients $\sigma(k)$. Here, we just point out that the stochastic mean-square fractional integrals I_{a+}^{α} and I_{b+}^{α} can be established by coosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$.

Now we present Hermite–Hadamard inequality for generalized mean-square fractional integrals $\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}$ and $\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}$ of X.

Theorem 3. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval I. For every $u, v \in I$, $u < v$, we have the following Hermite–Hadamard inequality*

$$X\left(\frac{u+v}{2}, \cdot\right) \tag{7}$$

$$\leq \frac{1}{2(v-u)^{\lambda}\mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]}\left[\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}[X](t)\right]$$

$$\leq \frac{X(u,\cdot) + X(v,\cdot)}{2}. \qquad\qquad a.e.$$

Proof. Since the process X is mean-square continuous, it is continuous in probability. Nikodem [11] proved that every Jensen-convex and continuous in probability stochastic process is convex. Since X is convex, then by Proposition 1, it has a supporting process at any point $t_0 \in int I$. Let us take a support at $t_0 = \frac{u+v}{2}$, then we have

$$X(t, \cdot) \geq A(\cdot)\left(t - \frac{u+v}{2}\right) + X\left(\frac{u+v}{2}, \cdot\right). \quad a.e. \tag{8}$$

Multiplying both sides of (8) by $[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}] + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]]$, then integrating the resulting inequality with respect to t over $[u, v]$, we obtain

$$\int_{u}^{v}\left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}] + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]\right]X(t,\cdot)dt \tag{9}$$

$$\geq A(\cdot)\int_{u}^{v}\left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]\right.$$

$$\left. + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]\right]\left(t - \frac{u+v}{2}\right)dt$$

$$+ X\left(\frac{u+v}{2}, \cdot\right)\int_{u}^{v}\left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]\right.$$

$$\left. + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]\right]dt$$

$$= A(\cdot)\int_{u}^{v}\left[t(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}] + t(t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]\right]dt$$

$$- A(\cdot)\frac{u+v}{2}\int_{u}^{v}\left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}] + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]\right]dt$$

$$+ X\left(\frac{u+v}{2}, \cdot\right) \int_u^v \left[(v-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]\right]$$

$$+ (t-u)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]]dt.$$

Calculating the integrals, we have

$$\int_u^v t(v-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]dt \tag{10}$$

$$= -\int_u^v (v-t)^{\lambda} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]dt + v \int_u^v (v-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(v-t)^{\rho}]dt$$

$$= -(v-u)^{\lambda+1} \mathcal{F}_{\rho,\lambda}^{\sigma_1}[\omega(v-u)^{\rho}] + v(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]$$

and similarly,

$$\int_u^v t(t-u)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]dt \tag{11}$$

$$= \int_u^v (t-u)^{\lambda} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]dt + u \int_u^v (t-u)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(t-u)^{\rho}]dt$$

$$= (v-u)^{\lambda+1} \mathcal{F}_{\rho,\lambda}^{\sigma_1}[\omega(v-u)^{\rho}] + u(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]$$

where $\sigma_1(k) = \frac{\sigma(k)}{\rho k + \lambda + 1}$, $k = 0, 1, 2, \ldots$. Using the identities (10) and (11) in (9), we obtain

$$\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}[X](t)$$

$$\geq A(\cdot)(u+v)(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]$$

$$- A(\cdot)\frac{u+v}{2} 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]$$

$$+ X\left(\frac{u+v}{2}, \cdot\right) 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]$$

$$= X\left(\frac{u+v}{2}, \cdot\right) 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}].$$

That is,

$$X\left(\frac{u+v}{2}, \cdot\right)$$

$$\leq \frac{1}{2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^{\rho}]} \left[\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}[X](t)\right] \text{ a.e.,}$$

which completes the proof of the first inequality in (7).

By using the convexity of X, we get

$$X(t, \cdot) = X\left(\frac{v-t}{v-u}u + \frac{t-u}{v-u}v, \cdot\right) \le \frac{v-t}{v-u}X(u, \cdot) + \frac{t-u}{v-u}X(v, \cdot)$$

$$= \frac{X(v, \cdot) - X(u, \cdot)}{v-u}t + \frac{X(u, \cdot)v - X(v, \cdot)u}{v-u} \quad \text{a.e.}$$

for $t \in [u, v]$. Using the identities (10) and (11), it follows that

$$\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(v-t)^\rho\right] + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(t-u)^\rho\right]\right]X(t, \cdot)dt$$

$$\le \frac{X(v, \cdot) - X(u, \cdot)}{v-u}$$

$$\times \int_u^v \left[t(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(v-t)^\rho\right] + t(t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(t-u)^\rho\right]\right]dt$$

$$+ \frac{X(u, \cdot)v - X(v, \cdot)u}{v-u}$$

$$\times \int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(v-t)^\rho\right] + (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^\sigma\left[\omega(t-u)^\rho\right]\right]dt$$

$$= \frac{X(v, \cdot) - X(u, \cdot)}{v-u}(u+v)(v-u)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma\left[\omega(v-u)^\rho\right]$$

$$+ \frac{X(u, \cdot)v - X(v, \cdot)u}{v-u}2(v-u)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma\left[\omega(v-u)^\rho\right]$$

$$= \left[X(u, \cdot) + X(v, \cdot)\right](v-u)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma\left[\omega(v-u)^\rho\right].$$

That is,

$$\frac{1}{2(v-u)^\lambda \mathcal{F}_{\rho,\lambda+1}^\sigma[\omega(v-u)^\rho]}\left[\mathcal{J}_{\rho,\lambda,u+;\omega}^\sigma[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^\sigma[X](t)\right]$$

$$\le \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad \text{a.e.,}$$

which completes the proof. $\qquad\square$

Remark 1. *i) Choosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (3).*

ii) Choosing $\lambda = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (2).

Theorem 4. *Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process, which is strongly Jensen-convex with modulus $C(\cdot)$ and mean-square continuous in the interval I so that $E[C^2] < \infty$. Then for any $u, v \in I$, we have*

$$X\left(\frac{u+v}{2}, \cdot\right)$$

$$- C(\cdot)\left\{2(v-u)^{\lambda+2}\mathcal{F}_{\rho,\lambda}^{\sigma_2}\big[\omega(v-u)^\rho\big] - 2(v-u)^\lambda\mathcal{F}_{\rho,\lambda}^{\sigma_1}\big[\omega(v-u)^\rho\big]\right.$$

$$\left. + \big(u^2+v^2\big)(v-u)^\lambda\mathcal{F}_{\rho,\lambda+1}^{\sigma}\big[\omega(v-u)^\rho\big] - \left(\frac{u+v}{2}\right)^2\right\}$$

$$\leq \frac{1}{2(v-u)^\lambda\mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^\rho]}\big[\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}[X](t)\big]$$

$$\leq \frac{X(u,\cdot)+X(v,\cdot)}{2} - C(\cdot)\left\{\frac{u^2+v^2}{2} + 2(v-u)^{\lambda+2}\mathcal{F}_{\rho,\lambda}^{\sigma_2}\big[\omega(v-u)^\rho\big]\right.$$

$$\left. - 2(v-u)^\lambda\mathcal{F}_{\rho,\lambda}^{\sigma_1}\big[\omega(v-u)^\rho\big] + \big(u^2+v^2\big)(v-u)^\lambda\mathcal{F}_{\rho,\lambda+1}^{\sigma}\big[\omega(v-u)^\rho\big]\right\} \ a.e.$$

Proof. It is known that if X is strongly convex process with the modulus $C(\cdot)$, then the process $Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$ is convex [7, Lemma 2]. Appying the inequality (7) for the process $Y(t,.)$, we have

$$Y\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{2(v-u)^\lambda\mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^\rho]}\int_u^v \big[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big]$$

$$+ (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(t-u)^\rho\big]\big]Y(t,\cdot)dt$$

$$\leq \frac{Y(u,\cdot)+Y(v,\cdot)}{2} \qquad\qquad a.e.$$

That is

$$X\left(\frac{u+v}{2},\cdot\right) - C(\cdot)\left(\frac{u+v}{2}\right)^2$$

$$\leq \frac{1}{2(v-u)^\lambda\mathcal{F}_{\rho,\lambda+1}^{\sigma}[\omega(v-u)^\rho]}\left\{\int_u^v \big[(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big]\right.$$

$$+ (t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(t-u)^\rho\big]\big]X(t,.)dt$$

$$\left. - C(\cdot)\int_u^v \big[t^2(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big] + t^2(t-u)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(t-u)^\rho\big]\big]dt\right\}$$

$$\leq \frac{X(u,\cdot)-C(\cdot)u^2+X(v,\cdot)-C(\cdot)v^2}{2} \qquad\qquad a.e.$$

Calculating the integrals, we obtain

$$\int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big]dt$$

$$= \int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big]dt + \int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\big[\omega(v-t)^\rho\big]dt$$

$$+ \int_u^v t^2 (v-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v-t)^{\rho}] dt$$

$$= (v-u)^{\lambda+2} \mathcal{F}_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] - 2v(v-u)^{\lambda+1} \mathcal{F}_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}]$$
$$+ v^2 (v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}]$$

and similarly,

$$\int_u^v t^2 (t-u)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t-u)^{\rho}] dt$$

$$= (v-u)^{\lambda+2} \mathcal{F}_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] + 2u(v-u)^{\lambda+1} \mathcal{F}_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}]$$
$$+ u^2 (v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}],$$

where $\sigma_2(k) = \frac{\sigma(k)}{\rho k + \lambda + 2}$, $k = 0, 1, 2, \ldots$. Then it follows that

$$X\left(\frac{u+v}{2}, \cdot\right) - C(\cdot)\left(\frac{u+v}{2}\right)^2$$

$$\leq \frac{1}{2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}]} \left[\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}[X](t) \right]$$

$$- C(\cdot)\left[2(v-u)^{\lambda+2} \mathcal{F}_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] \right.$$

$$\left. - 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] + (u^2+v^2)(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}] \right]$$

$$\leq \frac{X(u,\cdot) + X(v,\cdot)}{2} - C(\cdot)\frac{u^2+v^2}{2} \qquad \text{a.e.}$$

Then

$$X\left(\frac{u+v}{2}, \cdot\right)$$

$$- C(\cdot)\left\{ 2(v-u)^{\lambda+2} \mathcal{F}_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] - 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] \right.$$

$$\left. + (u^2+v^2)(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}] - \left(\frac{u+v}{2}\right)^2 \right\}$$

$$\leq \frac{1}{2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}]} \left[\mathcal{J}_{\rho,\lambda,u+;\omega}^{\sigma}[X](t) + \mathcal{J}_{\rho,\lambda,v-;\omega}^{\sigma}[X](t) \right]$$

$$\leq \frac{X(u,\cdot) + X(v,\cdot)}{2} - C(\cdot)\left\{ \frac{u^2+v^2}{2} + 2(v-u)^{\lambda+2} \mathcal{F}_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] \right.$$

$$\left. - 2(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] + (u^2+v^2)(v-u)^{\lambda} \mathcal{F}_{\rho,\lambda+1}^{\sigma} [\omega(v-u)^{\rho}] \right\} \text{ a.e.}$$

This completes the proof. $\qquad \square$

Remark 2. *Choosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 4, it reduces to Theorem 7 in [1].*

Acknowledgments

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