Modern Stochastics: Theory and Applications 5 (2) (2018) 129–143
https://doi.org/10.15559/18-VMSTA99

Exponential bounds for the tail probability of the supremum of an inhomogeneous random walk

Dominyka Kievinaitė, Jonas Šiaulys[*,1]

Faculty of Mathematics and Informatics, Vilnius University, Naugarduko 24, Vilnius LT-03225, Lithuania

d.kievinaite@gmail.com (D. Kievinaitė), jonas.siaulys@mif.vu.lt (J. Šiaulys)

Received: 13 October 2017, Revised: 10 February 2018, Accepted: 13 February 2018,
Published online: 15 March 2018

Abstract Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent but not necessarily identically distributed random variables. In this paper, the sufficient conditions are found under which the tail probability $\mathbb{P}(\sup_{n\geqslant 0} \sum_{i=1}^{n} \xi_i > x)$ can be bounded above by $\varrho_1 \exp\{-\varrho_2 x\}$ with some positive constants ϱ_1 and ϱ_2. A way to calculate these two constants is presented. The application of the derived bound is discussed and a Lundberg-type inequality is obtained for the ultimate ruin probability in the inhomogeneous renewal risk model satisfying the net profit condition on average.

Keywords Exponential bound, supremum of sums, tail probability, risk model, inhomogeneity, ruin probability, Lundberg's inequality

2010 MSC 62E20, 60E15, 60G50, 91B30

1 Introduction

Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of independent real-valued random variables (r.v.'s), and let

$$M_\infty = \sup_{n\geqslant 0}\left\{\sum_{k=1}^{n} \xi_k\right\}.$$

Here and subsequently, all empty sums are assumed to be zero.

[*]Corresponding author.
[1]The second author was supported by grant No S-MIP-17-72 from the Research Council of Lithuania.

Sgibnev [30] generalized results by Kiefer and Wolfowitz [21] obtaining the upper bound for the submultiplicative moment $\mathbb{E}(\varphi(M_\infty))$ in the case of independent and identically distributed (i.i.d.) r.v.'s. In Theorem 2 of that paper, the following assertion is proved.

Theorem 1. *Let $\{\xi_1, \xi_2, \ldots\}$ be a sequence of i.i.d. r.v.'s with distribution function (d.f.) F, and let φ be a nondecreasing submultiplicative function defined on $[0, \infty)$. Then $\mathbb{E}(\varphi(M_\infty)) < \infty$ under the following conditions:*

- $\mathbb{E}\xi_1 < 0$,
- $\int_0^\infty \varphi(x)\overline{F}(x)\,\mathrm{d}x < \infty$,
- $\mathbb{E}\big(\mathrm{e}^{r\xi_1}\big) < 1 \quad$ *if* $r := \lim_{x\to\infty} \dfrac{\log\varphi(x)}{x} > 0$.

Recall that a function φ defined on the interval $[0, \infty)$ is said to be submultiplicative if

$$\varphi(0) = 1 \quad \text{and} \quad \varphi(x + y) \leqslant \varphi(x)\varphi(y) \quad \text{for all } x, y \in [0, \infty).$$

Theorem 1 was applied several times to find the asymptotic behavior of the ruin probabilities in the homogeneous renewal risk models.

We say that the insurer's surplus process $R(t)$ varies according to the homogeneous renewal risk model if

$$R(t) = u + pt - \sum_{i=1}^{\Theta(t)} Z_i, \qquad t \geqslant 0, \tag{1}$$

where:

- $u \geqslant 0$ *denotes the initial insurer's surplus;*
- $p > 0$ *denotes a constant premium rate;*
- *the claim sizes $\{Z_1, Z_2, \ldots\}$ form a sequence of i.i.d. nonnegative r.v.'s;*
- $\Theta(t) = \sum_{n=1}^{\infty} \mathbb{1}_{\{\theta_1+\theta_2+\cdots+\theta_n \leqslant t\}}$ *is the renewal counting process generated by the inter-occurrence times $\{\theta_1, \theta_2, \ldots\}$, which form another sequence of i.i.d. nonnegative and nondegenerate at* 0 *r.v.'s;*
- *the sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent.*

The ultimate ruin probability

$$\psi(u) = \mathbb{P}\Big(\inf_{t\geqslant 0} R(t) < 0\Big) = \mathbb{P}\bigg(\sup_{n\geqslant 1} \sum_{k=1}^{n} (Z_k - p\theta_k) > u\bigg) \tag{2}$$

and the probability of ruin within time T

$$\psi(u, T) = \mathbb{P}\Big(\inf_{0\leqslant t\leqslant T} R(t) < 0\Big) = \mathbb{P}\bigg(\sup_{1\leqslant n\leqslant \Theta(T)} \sum_{k=1}^{n} (Z_k - p\theta_k) > u\bigg)$$

are the main characteristics of the renewal risk model.

The asymptotic behavior of $\psi(u, T)$ was considered by Tang [31] when random claims in the homogeneous model have d.f. with consistently varying tail. The author of the paper uses the assertion of Theorem 1 for function $\varphi(x) = (1+x)^q$ with $q > 0$ to get the main term of the asymptotics for the probability $\psi(u, T)$. Leipus and Šiaulys considered the asymptotic behavior of $\psi(u, T)$ in [22, 23] but for subexponentially distributed r.v.'s $\{Z_1, Z_2, \dots\}$. In their proofs, the assertion of Theorem 1 was used for function $\varphi(x) = \exp(\rho x)$ with some $\rho > 0$ (see Lemma 3.3 in [22] and Lemma 2.1 in [23]). In the case of exponential function, Theorem 1 implies the following assertion.

Corollary 1. *Let $\{\xi_1, \xi_2, \dots\}$ be a sequence of i.i.d. r.v.'s. If $\mathbb{E}\xi_1 < 0$ and $\mathbb{E}\,\mathrm{e}^{h\xi_1} < \infty$ for some positive h, then there exists a positive constant ϱ such that*

$$\mathrm{e}^{\varrho x}\mathbb{P}(M_\infty > x) \underset{x\to\infty}{\to} 0.$$

The Sgibnev's proof of Theorem 1 is substantially related to the techniques of Banach algebras, while Corollary 1 can be derived using only the probabilistic approach. Wang et al. (see Lemma 4.4 in [32]) demonstrated such a probabilistic way to obtain the assertion of Corollary 1 supposing, in addition, that r.v.'s $\{\xi_1, \xi_2, \dots\}$ follow some dependence structure. Corollary 1 can be applied not only as auxiliary assertion in the consideration of the asymptotic behavior of $\psi(u, T)$. The assertion of Corollary 1 is closely related to the following statement on the upper bound for $\psi(u)$ in the homogeneous renewal risk model.

Theorem 2. *Let the claim sizes $\{Z_1, Z_2, \dots\}$ and the inter-occurrence times $\{\theta_1, \theta_2, \dots\}$ form a homogeneous renewal risk model. Let, in addition, the net profit condition $\mathbb{E}Z_1 - p\,\mathbb{E}\theta_1 < 0$ hold and $\mathbb{E}\,\mathrm{e}^{hZ_1} < \infty$ for some positive h. Then, there exists a positive H such that*

$$\psi(u) \leqslant \mathrm{e}^{-Hu}, \quad u \geqslant 0. \tag{3}$$

If $\mathbb{E}\,\mathrm{e}^{R(Z_1 - p\theta_1)} = 1$ for some positive R, then we can take $H = R$ in (3).

The above assertion is the well-known Lundberg inequality. There exist a lot of different proofs of this inequality. For example, some of the proofs can be found in [4], [14], [15], [16], [25]. The existing proofs of Lundberg's inequality are essentially based on the renewal idea. However, the classical methods used for consideration of the ruin probability in the homogeneous renewal risk model are not applicable in the case of the inhomogeneous model because at any time moment distribution of the future is completely new.

Another way to derive the Lundberg inequality is related to Theorem 1. Namely, the first part of Theorem 2 follows from Theorem 1 and the additional inequality $\psi(0) < 1$. We use this approach to get the inequality similar to the Lundberg inequality but for the inhomogeneous renewal risk model with not necessarily identically distributed claim sizes $\{Z_1, Z_2, \dots\}$ and the inter-occurrence times $\{\theta_1, \theta_2, \dots\}$.

In this paper, we consider a sequence of independent but not necessarily identically distributed r.v.'s $\{\xi_1, \xi_2, \dots\}$. We obtain an assertion similar to that in Corollary 1. We present an algorithm to get the numerical values of the two positive constants in the exponential bound for $\mathbb{P}(M_\infty > x)$ in the case of not necessarily identically distributed r.v.'s $\{\xi_1, \xi_2, \dots\}$. We apply the obtained estimate to derive two

Lundberg-type inequalities similar to that in Theorem 2 but for the inhomogeneous renewal risk model with possibly nonidentically distributed random claim amounts $Z_1, Z_2, \ldots$. Corollaries 2 and 3 below show that the exponential bound for the ruin probability in the inhomogeneous renewal risk model holds if the model satisfies the *net profit condition on average*. This means that the quantity

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}(Z_k - p\theta_k)$$

is negative for all sufficiently large n.

The results of the present paper are complementary to those obtained by Albrecher et al. [1], Ambagaspitiya [2], Bernackaitė and Šiaulys [5, 6], Castañer et al. [7], Cojocaru [9], Constantinescu et al. [10], Czarna and Palmowski [11], Damarackas and Šiaulys [12], Danilenko et al. [13], Grigutis et al. [17, 18], Jordanova and Stehlík [20], Mishura et al. [24], Răducan et al. [26–28], Ragulina [29], Zhang et al. [33], Zhang et al. [34] and other authors who dealt with different inhomogeneous risk models.

The rest of the paper is organized as follows. In Section 2, we present our main result together with its proof. In Section 3, we recall the concept of the inhomogeneous renewal risk model and we present two corollaries from the main theorem, which yield exponential bounds for the ruin probability in this model. Finally, in Section 4, we present some examples which show the applicability of the theorem and corollaries.

2 Main result

In this section, we formulate and prove our main result. The assertion below is a generalization of Lemma 1 by Andrulytė et al. [3]. In that lemma, the exponential bound for $\mathbb{P}(M_\infty > x)$ was established under more restrictive conditions. In addition, the assertion below provides an algorithm to calculate two positive constants establishing this exponential bound. For this reason, the conditions of the main theorem are formulated in an explicit form in contrast to the conditions of Lemma 1 in [3]. It should be noted that the presented proof of the main result has some similarities with the classical approach by Chernoff [8] and Hoeffding [19].

Theorem 3. *Let* $\{\xi_1, \xi_2, \ldots\}$ *be independent r.v.'s such that:*

$$\text{(i)}\quad \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\,\xi_i \leqslant -a \quad \textit{if}\ n \geqslant b,$$

$$\text{(ii)}\quad \sup_{n\geqslant b}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i\leqslant -c\}}\big) \leqslant \varepsilon,$$

$$\text{(iii)}\quad \sup_{n\geqslant b}\frac{1}{n}\sum_{i=1}^{n}\big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\big) \leqslant d_1,$$

$$\text{(iv)}\quad \max_{1\leqslant n\leqslant b-1}\frac{1}{n}\sum_{i=1}^{n}\big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\big) \leqslant d_2,$$

for some $a > 0$*,* $b \in \mathbb{N}$*,* $c > 0$*,* $\varepsilon \geqslant 0$*,* $h > 0$*,* $d_1 \geqslant 1$ *and* $d_2 \geqslant 1$.

If

$$-\Delta := \varepsilon + \delta h d_1 \max\left\{\frac{c^2}{2}, \frac{2}{h^2}\right\} - a < 0, \tag{4}$$

with some $\delta \in (0, 1/2]$*, then*

$$\mathbb{P}\left(\sup_{n\geqslant 0}\sum_{i=1}^{n}\xi_i > x\right) \leqslant \min\bigl\{1,\ c_1 \mathrm{e}^{-\delta h x}\bigr\}, \quad x \geqslant 0,$$

where

$$c_1 = \left(S(b, d_2) + \frac{\exp\{-\delta h \Delta b\}}{1 - \exp\{-\delta h \Delta\}}\right),$$

with

$$S(b, d_2) = \begin{cases} d_2 \frac{d_2^{b-1} - 1}{d_2 - 1} & \textit{if } d_2 > 1, \\ b - 1 & \textit{if } d_2 = 1. \end{cases}$$

Proof. We observe that, for all $x \geqslant 0$,

$$\begin{aligned}\mathcal{P}(x) &:= \mathbb{P}\left(\sup_{n\geqslant 0}\sum_{i=1}^{n}\xi_i > x\right) = \mathbb{P}\left(\bigcup_{n=1}^{\infty}\left\{\sum_{i=1}^{n}\xi_i > x\right\}\right) \\ &\leqslant \sum_{n=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{n}\xi_i > x\right).\end{aligned}$$

Since r.v.'s $\{\xi_1, \xi_2, \ldots\}$ are independent, by the exponential Chebyshev inequality, we get

$$\begin{aligned}\mathcal{P}(x) &\leqslant \mathrm{e}^{-yx}\sum_{n=1}^{\infty}\prod_{i=1}^{n}\mathbb{E}\,\mathrm{e}^{y\xi_i} \\ &= \mathrm{e}^{-yx}\sum_{n=1}^{b-1}\prod_{i=1}^{n}\mathbb{E}\,\mathrm{e}^{y\xi_i} + \mathrm{e}^{-yx}\sum_{n=b}^{\infty}\prod_{i=1}^{n}\mathbb{E}\,\mathrm{e}^{y\xi_i} \\ &:= \mathcal{P}_1(x) + \mathcal{P}_2(x),\end{aligned} \tag{5}$$

for all $x \geqslant 0$ and $0 < y \leqslant h$.

For all $i \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}\mathrm{e}^{y\xi_i} &= 1 + y\mathbb{E}\xi_i + \mathbb{E}\bigl(\bigl(\mathrm{e}^{y\xi_i} - 1\bigr)\mathbb{1}_{\{\xi_i \leqslant -c\}}\bigr) - y\mathbb{E}\bigl(\xi_i \mathbb{1}_{\{\xi_i \leqslant -c\}}\bigr) \\ &\quad + \mathbb{E}\bigl(\bigl(\mathrm{e}^{y\xi_i} - 1 - y\xi_i\bigr)\mathbb{1}_{\{-c < \xi_i \leqslant 0\}}\bigr) \\ &\quad + \mathbb{E}\bigl(\bigl(\mathrm{e}^{y\xi_i} - 1 - y\xi_i\bigr)\mathbb{1}_{\{\xi_i > 0\}}\bigr).\end{aligned}$$

It is obvious that

$$\mathrm{e}^{v} - 1 \leqslant 0 \quad \text{if } v \leqslant 0,$$

$$\mathrm{e}^v - v - 1 \leqslant \frac{v^2}{2} \quad \text{if } v \leqslant 0,$$
$$\mathrm{e}^v - v - 1 \leqslant \frac{v^2}{2}\,\mathrm{e}^v \quad \text{if } v \geqslant 0,$$

and $v^2 \leqslant \mathrm{e}^v$ for nonnegative v. Using these inequalities we get

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{y\xi_i} &\leqslant 1 + y\mathbb{E}\xi_i + y\,\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i \leqslant -c\}}\big) + \frac{y^2}{2}\,\mathbb{E}\big(\xi_i^2\mathbb{1}_{\{-c<\xi_i \leqslant 0\}}\big) \\
&\quad + \frac{y^2}{2}\,\mathbb{E}\big(\xi_i^2\mathrm{e}^{y\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big) \\
&\leqslant 1 + y\mathbb{E}\xi_i + y\,\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i \leqslant -c\}}\big) + \frac{y^2c^2}{2}\,\mathbb{P}(\xi_i \leqslant 0) \\
&\quad + \frac{2y^2}{h^2}\,\mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big),
\end{aligned} \tag{6}$$

if $0 < y \leqslant h/2$, because

$$\mathbb{E}\bigg(\bigg(\frac{h\xi_i}{2}\bigg)^2\mathrm{e}^{y\xi_i}\mathbb{1}_{\{\xi_i>0\}}\bigg) \leqslant \mathbb{E}\bigg(\bigg(\frac{h\xi_i}{2}\bigg)^2\mathrm{e}^{h\xi_i/2}\mathbb{1}_{\{\xi_i>0\}}\bigg) \leqslant \mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big),$$

in this case.

If $n \geqslant b$, then conditions (ii), (iii) and relation (6) together with the inequality $1 + u \leqslant \mathrm{e}^u$, $u \in \mathbb{R}$, imply that

$$\begin{aligned}
\prod_{i=1}^n \mathbb{E}\mathrm{e}^{y\xi_i} &\leqslant \prod_{i=1}^n \bigg(1 + y\mathbb{E}\xi_i + y\,\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i \leqslant -c\}}\big) \\
&\quad + \frac{y^2c^2}{2}\,\mathbb{P}(\xi_i \leqslant 0) + \frac{2y^2}{h^2}\,\mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\bigg) \\
&\leqslant \exp\bigg\{y\sum_{i=1}^n \mathbb{E}\xi_i + ny\,\sup_{n\geqslant b}\frac{1}{n}\sum_{i=1}^n \mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i \leqslant -c\}}\big) \\
&\quad + ny^2\max\bigg\{\frac{c^2}{2}, \frac{2}{h^2}\bigg\}\sup_{n\geqslant b}\frac{1}{n}\sum_{i=1}^n\big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{h\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\big)\bigg\} \\
&\leqslant \exp\bigg\{y\sum_{i=1}^n \mathbb{E}\xi_i + ny\varepsilon + ny^2d_1\max\bigg\{\frac{c^2}{2}, \frac{2}{h^2}\bigg\}\bigg\}.
\end{aligned}$$

Hence, by condition (i)

$$\mathcal{P}_2(x) \leqslant \mathrm{e}^{-yx}\sum_{n=b}^\infty \exp\bigg\{ny\bigg(-a + \varepsilon + yd_1\max\bigg\{\frac{c^2}{2}, \frac{2}{h^2}\bigg\}\bigg)\bigg\}, \tag{7}$$

for all $x \geqslant 0$ and $0 < y \leqslant h/2$.

If $n \leqslant b-1$ and $0 < y \leqslant h$ then, due to the condition (iv), we have

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\mathrm{e}^{y\xi_i} = \frac{1}{n}\sum_{i=1}^{n}\big(\mathbb{E}\big(\mathrm{e}^{y\xi_i}\mathbb{1}_{\{\xi_i\leqslant 0\}}\big) + \mathbb{E}\big(\mathrm{e}^{y\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\big) \leqslant d_2.$$

Therefore,

$$\mathcal{P}_1(x) \leqslant \mathrm{e}^{-yx}\sum_{n=1}^{b-1} d_2^n, \tag{8}$$

because

$$\prod_{i=1}^{n}\mathbb{E}\mathrm{e}^{y\xi_i} \leqslant \left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\mathrm{e}^{y\xi_i}\right)^n,$$

by the inequality of arithmetic and geometric means.

Equality (5) and inequalities (7), (8) imply that

$$\mathcal{P}(x) \leqslant \mathrm{e}^{-yx}\left(S(b,d_2) + \sum_{n=b}^{\infty}\left(\exp\left\{y\left(-a+\varepsilon+yd_1\max\left\{\frac{c^2}{2},\frac{2}{h^2}\right\}\right)\right\}\right)^n\right), \tag{9}$$

for all $x \geqslant 0$ and $0 < y \leqslant h/2$.

Let now $y = \delta h$ with some $\delta \in (0, 1/2]$ satisfying condition (4). For such y, we obtain from (9) that

$$\mathcal{P}(x) \leqslant \mathrm{e}^{-\delta hx}\left(S(b,d_2) + \frac{\exp\{-\delta h\Delta b\}}{1-\exp\{-\delta h\Delta\}}\right).$$

This is the desired inequality. The theorem is proved. □

3 Lundberg-type inequalities

In this section, we present two corollaries from Theorem 3, which yield the Lundberg-type inequalities for the inhomogeneous renewal risk model.

We say that the insurer's surplus process $R(t)$ varies according to the inhomogeneous renewal risk model if equality (1) *holds for all $t \geqslant 0$ with the initial insurer's surplus $u \geqslant 0$, a constant premium rate $p > 0$, a sequence of independent nonnegative and not necessarily identically distributed claim amounts $\{Z_1, Z_2, \ldots\}$ and with the renewal counting process $\Theta(t)$ generated by the inter-occurrence times $\{\theta_1, \theta_2, \ldots\}$, which form a sequence of independent nonnegative nondegenerate at zero and possibly not identically distributed r.v.'s. In addition, sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are supposed to be independent.*

It is obvious that definitions and expressions of the ruin probabilities $\psi(u)$ and $\psi(u, T)$ for the inhomogeneous renewal risk model remain the same as those given in Section 1.

The main requirement to get the Lundberg-type bounds for $\psi(u)$ is the *net profit condition*. In both assertions below, it is supposed that this condition holds on average. Our first corollary follows immediately from Theorem 3 and representation (2).

Corollary 2. *Let us consider the inhomogeneous renewal risk model such that r.v.'s $\xi_k = Z_k - p\theta_k$, $k \in \mathbb{N}$, satisfy conditions* (i)–(iv) *of Theorem 3. Then the ruin probability in the model satisfies the following inequality*

$$\psi(u) \leqslant \min\{1,\ c_1 \mathrm{e}^{-\delta h u}\},\ u \geqslant 0,$$

where constants $h > 0$, $\delta \in (0, 1/2]$ and $c_1 > 0$ are the same as in Theorem 3 for the sequence $\{\xi_1 = Z_1 - p\theta_1, \xi_2 = Z_2 - p\theta_2, \ldots\}$.

Our second corollary is more convenient to use because the requirements are formulated separately for r.v.'s $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ in it. We present the corollary below together with a short proof.

Corollary 3. *Let the inhomogeneous renewal risk model with a sequence of random claim amounts $\{Z_1, Z_2, \ldots\}$, a sequence of random inter-occurrence times $\{\theta_1, \theta_2, \ldots\}$ and premium rate p satisfy the following additional requirements*

$$\text{(i)} \quad \frac{1}{n}\sum_{i=1}^{n}(\mathbb{E}Z_i - p\,\mathbb{E}\theta_i) \leqslant -\alpha \quad \text{if } n \geqslant \beta,$$

$$\text{(ii)} \quad \sup_{n \geqslant \beta} \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\theta_i \mathbb{1}_{\{\theta_i \geqslant \frac{\varkappa}{p}\}}\big) \leqslant \epsilon,$$

$$\text{(iii)} \quad \sup_{n \geqslant \beta} \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\mathrm{e}^{\gamma Z_i} \leqslant \nu_1,$$

$$\text{(iv)} \quad \max_{1 \leqslant n \leqslant \beta - 1} \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\mathrm{e}^{\gamma Z_i} \leqslant \nu_2,$$

for some $\alpha > 0$, $\beta \in \mathbb{N}$, $\varkappa > 0$, $\epsilon \geqslant 0$, $\gamma > 0$, $\nu_1 \geqslant 1$ and $\nu_2 \geqslant 1$.

If

$$-\hat{\Delta} := p\epsilon + \delta\gamma(1 + \nu_1)\max\left\{\frac{\varkappa^2}{2}, \frac{2}{\gamma^2}\right\} - \alpha < 0,$$

for some $\delta \in (0, 1/2]$, then

$$\psi(u) \leqslant \min\{1,\ c_2 \mathrm{e}^{-\delta\gamma u}\}, \quad u \geqslant 0,$$

with the positive constant

$$c_2 = \left(\frac{1 + \nu_2}{\nu_2}\big((1 + \nu_2)^{b-1} - 1\big) + \frac{\exp\{-\delta\gamma\hat{\Delta}\beta\}}{1 - \exp\{-\delta\gamma\hat{\Delta}\}}\right).$$

Proof of Corollary 3. Let $\xi_i = Z_i - p\,\theta_i$ for all $i \in \mathbb{N}$. Then obviously

$$\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\xi_i \leqslant -\alpha \quad \text{if } n \geqslant \beta, \tag{10}$$

by condition (i) of the corollary.

Further, by conditions (iii) and (iv), we have

$$\sup_{n\geqslant\beta}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\mathrm{e}^{\gamma\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\leqslant\sup_{n\geqslant\beta}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\mathrm{e}^{\gamma Z_i}\leqslant\nu_1, \tag{11}$$

$$\max_{1\leqslant n\leqslant\beta-1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\mathrm{e}^{\gamma\xi_i}\mathbb{1}_{\{\xi_i>0\}}\big)\leqslant\nu_2, \tag{12}$$

because of the nonnegativity of the inter-occurrence times θ_i, $i\in\mathbb{N}$.

For the use of Theorem 3, it remains to estimate

$$\sup_{n\geqslant\beta}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i\leqslant-c\}}\big),$$

for some suitable positive c.

Choosing $c=\varkappa$ we get

$$\begin{aligned}\sup_{n\geqslant\beta}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(|\xi_i|\mathbb{1}_{\{\xi_i\leqslant-\varkappa\}}\big)&\leqslant p\sup_{n\geqslant\beta}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\theta_i\mathbb{1}_{\{Z_i-p\,\theta_i\leqslant-\varkappa\}}\big)\\&=p\sup_{n\geqslant1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant\frac{1}{p}(Z_i+\varkappa)\}}\big)\\&\leqslant p\sup_{n\geqslant1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\theta_i\mathbb{1}_{\{\theta_i\geqslant\frac{\varkappa}{p}\}}\big)\\&\leqslant p\,\epsilon.\end{aligned} \tag{13}$$

The obtained inequalities (10), (11), (12) and (13) imply that r.v.'s $\{\xi_1,\xi_2,\dots\}$ satisfy conditions (i)–(iv) of Theorem 3 with

$$a=\alpha,\quad b=\beta,\quad c=\varkappa,\quad h=\gamma,\quad d_1=1+\nu_1,\quad d_2=1+\nu_2\quad\text{and}\quad\varepsilon=p\,\epsilon.$$

The assertion of the corollary follows now from Theorem 3. □

4 Examples

In this section, we present four examples. The first two examples show the applicability of Theorem 3. The third example demonstrates how to get the exponential bound for the ruin probability applying Corollary 3. The last example shows that the Lundberg-type inequality of the form $\psi(u)\leqslant\varrho_1\mathrm{e}^{-\varrho_2 u}$, $u\geqslant0$, with $\varrho_1=1$ and a positive constant ϱ_2 is impossible if the inhomogeneous renewal risk model is "good" only on average.

Example 1. Suppose that $\{\xi_1,\xi_2,\dots\}$ are independent r.v.'s such that:

- ξ_i are uniformly distributed on interval $[0,2]$ for all $i\equiv1\bmod3$;
- ξ_i are uniformly distributed on interval $[-2,0]$ for all $i\equiv2\bmod3$;
- $\overline{F}_{\xi_i}(x)=\mathbb{1}_{(-\infty,-2)}(x)+\mathrm{e}^{-x-2}\mathbb{1}_{[-2,\infty)}(x)$ if $i\equiv0\bmod3$.

We can see that the presented sequence $\{\xi_1, \xi_2, \dots\}$ has three subsequences. Two of them generate random walks with negative drifts, and one subsequence generates random walk with a positive drift. Fortunately, sequence $\{\xi_1, \xi_2, \dots\}$ has a negative drift on average. Therefore, we can use Theorem 3 to get the exponential bound for $\mathbb{P}(M_\infty > x)$.

It is evident that

$$\mathbb{E}\xi_i = \begin{cases} 1 & \text{if } i \equiv 1 \bmod 3, \\ -1 & \text{if } i \equiv 2 \bmod 3 \text{ or } i \equiv 0 \bmod 3. \end{cases}$$

Therefore, after some calculations, we get

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\xi_i \leqslant -\frac{1}{7} \text{ for } n \geqslant 7. \tag{14}$$

Additionally,

$$\sup_{n\geqslant 1} \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\big(|\xi_i| \mathbb{1}_{\{\xi_i \leqslant -2\}}\big) = 0 \tag{15}$$

and

$$\mathbb{E}\big(\mathrm{e}^{\frac{4}{5}\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big) = \begin{cases} \frac{5}{8}(\mathrm{e}^{8/5} - 1) < 2.48 & \text{if } i \equiv 1 \bmod 3, \\ 0 & \text{if } i \equiv 2 \bmod 3, \\ 5/\mathrm{e}^2 < 0.68 & \text{if } i \equiv 0 \bmod 3. \end{cases}$$

The last expression implies

$$\sup_{n\geqslant 7} \frac{1}{n}\sum_{i=1}^{n} \big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{\frac{4}{5}\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big)\big) < 1.79, \tag{16}$$

$$\max_{1\leqslant n\leqslant 6} \frac{1}{n}\sum_{i=1}^{n} \big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{\frac{4}{5}\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big)\big) < 2.48. \tag{17}$$

By (14)–(17), we conclude that conditions of Theorem 3 hold with $a = 1/7$, $b = 7$, $c = 2$, $\varepsilon = 0$, $h = 4/5$, $d_1 = 1.8$ and $d_2 = 2.5$. Therefore,

$$-\Delta = \varepsilon + \delta h d_1 \max\left\{\frac{c^2}{2}, \frac{2}{h^2}\right\} - a = \frac{9}{2}\delta - \frac{1}{7} = -\frac{1}{14},$$

if $\delta = 1/63$. It follows now from Theorem 3 that

$$\begin{aligned} \mathbb{P}(M_\infty > x) &\leqslant \min\left\{1, \left(d_2\frac{d_2^6 - 1}{d_2 - 1} + \frac{\exp\{-\delta h \Delta b\}}{1 - \exp\{-\delta h \Delta\}}\right)\mathrm{e}^{-\delta h x}\right\} \\ &\leqslant \min\big\{1,\ 1502\exp\{-0.01269x\}\big\}, \end{aligned}$$

for all positive x.

The last inequality works if $x \geqslant 578$. Though the obtained bound is not as good as one would prefer, it is still exponential. Its weakest point is the large constant before the main term. By Theorem 3, the value of this constant is closely related to the behavior of the first elements of the sequence $\{\xi_1, \xi_2, \dots\}$. In our case, the first elements of $\{\xi_1, \xi_2, \dots\}$ increase this constant because the subsequence $\{\xi_1, \xi_4, \xi_7, \dots\}$ has a positive drift. The second example shows that the better exponential bound can be obtained from Theorem 3 in the case when only some of the r.v.'s $\{\xi_1, \xi_2, \dots\}$ drag the model to the positive side.

Example 2. Suppose that $\{\xi_1, \xi_2, \dots\}$ are independent r.v.'s such that:

$$\mathbb{P}(\xi_i = -1) = 1 - \frac{1}{i+1} \quad \text{and} \quad \mathbb{P}(\xi_i = 1) = \frac{1}{i+1} \quad \text{for } i \in \{1, 2, \dots\}.$$

For all $i \geqslant 1$, we have

$$\mathbb{E}\xi_i = \frac{2}{i+1} - 1 \quad \text{and} \quad \mathbb{E}\big(\mathrm{e}^{\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big) = \frac{\mathrm{e}}{i+1}.$$

Consequently,

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\xi_i \leqslant -\frac{5}{18} \quad \text{if } n \geqslant 3,$$

$$\sup_{n \geqslant 3} \frac{1}{n}\sum_{i=1}^{n} \big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big)\big) = \frac{13\mathrm{e} + 23}{36} < 1.621,$$

$$\max_{1 \leqslant n \leqslant 6} \frac{1}{n}\sum_{i=1}^{n} \big(\mathbb{P}(\xi_i \leqslant 0) + \mathbb{E}\big(\mathrm{e}^{\xi_i} \mathbb{1}_{\{\xi_i > 0\}}\big)\big) = \frac{\mathrm{e} + 1}{2} < 1.86.$$

Due to the derived bounds conditions of Theorem 3 hold with

$$a = 5/18, \quad b = 3, \quad c = 11/10, \quad \varepsilon = 0, \quad h = 1, \quad d_1 = 1.625 \quad \text{and} \quad d_2 = \frac{\mathrm{e}+1}{2}.$$

In this case, we have

$$-\varDelta = \varepsilon + \delta h d_1 \max\left\{\frac{c^2}{2}, \frac{2}{h^2}\right\} - a = \frac{13}{4}\delta - \frac{5}{18} < 0,$$

for $\delta < 10/117$.

If we chose $\delta = 1/20$, then by Theorem 3, we have

$$\mathbb{P}(M_\infty > x) \leqslant \min\big\{1,\ 178\exp\{-x/20\}\big\}, \quad x \geqslant 0.$$

As was stated before, the next example shows the possibility of the exponential bound for the ruin probability in the case when the inhomogeneous renewal risk model satisfies net profit condition on average.

Example 3. Let us consider the inhomogeneous risk model which is generated by uniformly distributed on $[1,3]$ inter-occurrence times $\theta_1, \theta_2, \ldots$, constant premium rate $p = 2$ and a sequence of the claim amounts $\{Z_1, Z_2, \ldots\}$ such that $Z_1 = Z_2 = 0$, $Z_3 = Z_4 = 4$ with probability 1 and

$$\overline{F}_{Z_i}(x) = \mathbb{1}_{(-\infty,0)}(x) + \mathrm{e}^{-x}\left(1 + \frac{x}{i}\right)\mathbb{1}_{[0,\infty)}(x), \quad i \geqslant 5.$$

In this case, we have

$$\begin{aligned}
\mathbb{E}Z_i &= 1 + \frac{1}{i}, \quad i \geqslant 5,\\
\mathbb{E}\,\mathrm{e}^{\gamma Z_i} &= \frac{i-1}{i}\frac{1}{1-\gamma} + \frac{1}{i}\frac{1}{(1-\gamma)^2}, \quad i \geqslant 5,\ \gamma \in (0,1).
\end{aligned}$$

Consequently,

$$\frac{1}{n}\sum_{i=1}^{n}(\mathbb{E}Z_i - p\mathbb{E}\theta_i) \leqslant -2 \quad \text{if } n \geqslant 1$$

and

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\big(\mathrm{e}^{Z_i/3}\big) \leqslant 2.4.$$

The obtained inequalities imply conditions of Corollary 3 with

$$\alpha = 2, \quad \beta = 1, \quad \varkappa = 6, \quad \epsilon = 0, \quad \gamma = \frac{1}{3}, \quad \nu_1 = 2.4, \quad \nu_2 = 1.$$

Therefore, for the described model,

$$-\widehat{\Delta} = \frac{102}{5}\delta - 2 < 0,$$

if $\delta < 10/102$.

If we choose $\delta = 9/102$, then $\widehat{\Delta} = 1/5$, and the assertion of Corollary 3 implies the following Lundberg-type inequality for the model

$$\psi(u) \leqslant \min\big\{1,\ 170\,\mathrm{e}^{-0.029\,u}\big\}, \quad u \geqslant 0.$$

If we choose $\delta = 5/102$, then $\widehat{\Delta} = 1$, and Corollary 3 implies that

$$\psi(u) \leqslant \min\big\{1,\ 61\,\mathrm{e}^{-0.016\,u}\big\}, \quad u \geqslant 0.$$

Remark 1. It is clear that we can get a lot of different Lundberg-type inequalities for the same model because there exist infinitely many collections of constants $\{\alpha, \beta, \varkappa, \epsilon, \gamma, \nu_1, \nu_2, \delta\}$ satisfying conditions of Corollary 3. It follows from the construction of the bound in Corollary 3 that we get the better bound for the smaller constants $\beta, \varkappa, \epsilon, \nu_1, \nu_2$ and for larger constants α, γ. If the collection of the constants $\{\alpha, \beta, \varkappa, \epsilon, \gamma, \nu_1, \nu_2\}$ is quite “unfriendly”, then we can still get an exponential

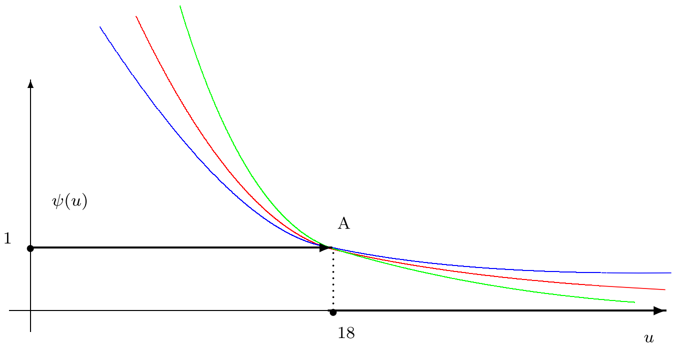

Fig. 1. Ruin probability in the model of Example 4

bound for the ruin probability but with unsatisfiably small δ. All possible exponential bounds for the ruin probability have the form $\varrho_1 \exp\{-\varrho_2 u\}$ with some positive constants ϱ_1 and ϱ_2. Theorem 2 shows that $\varrho_1 = 1$ in the case of the homogeneous renewal risk model satisfying the net profit condition. If the net profit condition holds on average (see condition (i) of Corollaries 2 or 3), then it is impossible to get the exponential bound for $\psi(u)$ with $\varrho_1 = 1$ in general. The following simple example confirms this.

Example 4. Let us consider the inhomogeneous risk model with $p = 1$ such that $Z_1 = Z_2 = 10$, $Z_i = 0$ for $i \geqslant 3$, and $\theta_i = 1$ for $i \geqslant 1$ almost surely.

The model under consideration is inhomogeneous but satisfies the net profit condition on average and all other conditions of, for instance, Corollary 3. On the other hand, the model is degenerate. It is not difficult to obtain the exact values of the ruin probability. Namely, expression (2) implies that

$$\psi(u) = 1 \quad \text{if } 0 \leqslant u < 18,$$
$$\psi(u) = 0 \quad \text{if } u \geqslant 18.$$

We can see the graph of the function ψ in Figure 1. All the best possible exponential bounds for $\psi(u)$ must go through the point A(18,1) (see colored curves in Figure 1). Therefore, the upper bounds of the form $\varrho_1 \exp\{-\varrho_2 u\}$ should satisfy the condition

$$\varrho_1 \mathrm{e}^{-18\varrho_2} \geqslant 1.$$

Hence, it is evident that

$$\varrho_1 \geqslant \mathrm{e}^{18\varrho_2} > 1.$$

5 Concluding remarks

In the paper, the problem of the estimating of the ruin probability for the inhomogeneous renewal risk models is considered. It is evident that this problem is closely related to the bounds for the tail probability of the supremum of an inhomogeneous random walk. The upper bound of the exponential type $\varrho_1 e^{-\varrho_2 u}$ is derived for the renewal risk models satisfying the *net profit condition on average*. The positive constants ϱ_1 and ϱ_2 depend on the constants describing the model. Unfortunately, the obtained estimates are not sharp enough. We guess that it is possible to get sharper exponential bounds for the ruin probabilities but for narrower class of the inhomogeneous renewal risk models.

Acknowledgments

We would like to thank three anonymous referees for the detailed and helpful comments on the previous versions of the manuscript.

References

[1] Albrecher H., Ivanovs J., Zhou X.: Exit identities for Lévy processes observed at Poisson arrival times. *Bernoulli* **22**(3), 1364–1382 (2016). MR3474819

[2] Ambagaspitiya R.S.: Ultimate ruin probability in the Sparre-Andersen model with dependent claim sizes and claim occurrence times. *Insur. Math. Econ.* **44**(3), 464–472 (2009). MR2519090

[3] Andrulytė I.M., Bernackaitė E., Kievinaitė D., Šiaulys J.: A Ludberg-type inequality for an inhomogeneous renewal risk model. *Mod. Stoch.: Theory Appl.* **2**, 173–184 (2015). MR3389589

[4] Asmussen S., Albrecher H.: Ruin Probabilities. Word Scientific Publishing (2010). MR2766220

[5] Bernackaitė E., J. Šiaulys J.: The exponential moment tail of inhomogeneous renewal process. *Stat. Probab. Lett.* **97**, 9–15 (2015). MR3299745

[6] Bernackaitė E., J. Šiaulys J.: The finite-time ruin probability for an inhomogeneous renewal risk model. *J. Ind. Manag. Optim.* **13**(1), 207–222 (2017). MR3576051

[7] Castañer A., Claramunt M.M., Gathy M., Lefèvre Cl., Mármol M.: Ruin problems for a discrete time risk model with non-homogeneous conditions. *Scand. Actuar. J.* **2013**(2), 83–102 (2013). MR3041119

[8] Chernoff H.: A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Stat.* **23**, 493–507 (1952). MR0057518

[9] Cojocaru I.: Ruin probabilities in multivariate risk models with periodic common shock. *Scand. Actuar. J.* **2017**(2), 159–174 (2017). MR3590439

[10] Constantinescu C., Kortschak D., Maume-Deschamps V.: Ruin probabilities in models with a Markov chain dependence structure. *Scand. Actuar. J.* **2013**(6), 453–476 (2013). MR3176014

[11] Czarna I., Palmowski Z.: Ruin probability with Poisson delay for a spectrally negative Lévy process. *J. Appl. Probab.* **48**(4), 984–1002 (2011). MR2896663

[12] Damarackas J., Šiaulys J.: Bi-seasonal discrete time risk model. *Appl. Math. Comput.* **247**, 30–940 (2014). MR3270895

[13] Danilenko S., Markevičiūtė J., Šiaulys J.: Randomly stopped sums with exponential-type distributions. *Nonlinear Anal. Model. Control* **22**(6), 793–807 (2017). MR3724621

[14] Embrechts P., Klüppelberg C., Mikosch T.: Modeling Extremal Events for Insurance and Finance. Springer (1997). MR1458613

[15] Embrechts P., Veraverbeke V.: Estimates for probability of ruin with special emphasis of the possibility of large claims. *Insur. Math. Econ.* **1**, 55–72 (1982). MR0652832

[16] Gerber H.: Martingales in risk theory. *Bull. Swiss Assoc. Actuar.* **1973**, 205–216 (1973).

[17] Grigutis A., Korvel A., Šiaulys J.: Ruin probabilities at a discrete-time multi risk model. *Inf. Technol. Valdym.* **44**(4), 367–379 (2015).

[18] Grigutis A., Korvel A., Šiaulys J.: Ruin probability in the three-seasonal discrete-time risk model. *Mod. Stoch.: Theory Appl.* **2**, 421–441 (2015). MR3456147

[19] Hoeffding W.: Probability inequalities for sums of bounded random variables. *J. Am. Stat. Assoc.* **58**(301), 13–30 (1963). MR0144363

[20] Jordanova P., Stehlík M.: Mixed Poisson process with Pareto mixing variable and its risk applications. *Lith. Math. J.* **56**(2), 189–206 (2016). MR3504255

[21] Kiefer J., Wolfowitz J.: On the characteristics of the general queuing process with applications to random walk. *Ann. Math. Stat.* **27**(1), 147–161 (1956). MR0077019

[22] Leipus R., Šiaulys J.: Asymptotic behavior of the finite-time ruin probability under subexponential claim sizes. *Insur. Math. Econ.* **40**, 498–508 (2007). MR2311546

[23] Leipus R., Šiaulys J.: Asymptotic behavior of the finite-time ruin probability in the renewal risks models. *Appl. Stoch. Models Bus. Ind.* **25**, 309–321 (2009). MR2541109

[24] Mishura Y., Ragulina O., Stroyev O.: Practical approaches to the estimation of the ruin probability in a risk model with additional funds. *Mod. Stoch.: Theory Appl.* **1**(2), 167–180 (2014). MR3316485

[25] Mikosch T.: Non-life Insurance Mathematics. Springer (2009). MR2503328

[26] Răducan A.M., Vernic R., Zbăganu G.: Recursive calculation of ruin probabilities at or before claim instants for non-identically distributed claims. *ASTIN Bull.* **45**(2), 421–443 (2015). MR3394025

[27] Răducan A.M., Vernic R., Zbăganu G.: On the ruin probability for nonhomogeneous claims and arbitrary inter-claim revenues. *J. Comput. Appl. Math.* **290**, 319–333 (2015). MR3370412

[28] Răducan A.M., Vernic R., Zbăganu G.: Upper and lower bounds for a finite-type ruin probability in a nonhomogeneous risks process. *Proc. Rom. Acad., Ser. A: Math. Phys. Tech. Sci. Inf. Sci.* **17**(4), 287–292 (2016). MR3582626

[29] Ragulina O.: The risk model with stochastic premiums, dependence and a threshold dividend strategy. *Mod. Stoch.: Theory Appl.* **4**(4), 315–351 (2017).

[30] Sgibnev M.S.: Submultiplicative moments of the supremum of a random walk with negative drift. *Stat. Probab. Lett.* **32**, 377–383 (1997). MR1602211

[31] Tang Q.: Asymptotics for the finite time ruin probability in the renewal risk model with consistent variation. *Stoch. Models* **20**(3), 281–297 (2004). MR2082126

[32] Wang Y., Cui Z., Wang K., Ma X.: Uniform asymptotics of the finite-time ruin probability for all times. *J. Math. Anal. Appl.* **390**, 208–223 (2012). MR2885767

[33] Zhang T., Fang X.N., Liu J., Yang Y.: Asymptotics for the partial sum and its maximum of dependent random variables. *Lith. Math. J.* **57**(1), 142–153 (2017). MR3621877

[34] Zhang Z., Cheng E.C.K., Yang H.: Lévy insurance risk process with Poissonian taxation. *Scand. Actuar. J.* **2017**(1), 51–87 (2017). MR3592957