Modern Stochastics: Theory and Applications 5 (1) (2018) 99–111
https://doi.org/10.15559/18-VMSTA97

Fractional Cox–Ingersoll–Ross process with non-zero «mean»

Yuliya Mishura, Anton Yurchenko-Tytarenko*

Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Volodymyrska St., 64/13, Kyiv 01601, Ukraine

myus@univ.kiev.ua (Yu. Mishura), antonyurty@gmail.com (A. Yurchenko-Tytarenko)

Received: 26 September 2017, Revised: 29 January 2018, Accepted: 30 January 2018,
Published online: 5 March 2018

Abstract In this paper we define the fractional Cox–Ingersoll–Ross process as $X_t := Y_t^2 \mathbf{1}_{\{t<\inf\{s>0:Y_s=0\}\}}$, where the process $Y = \{Y_t, t \geq 0\}$ satisfies the SDE of the form $dY_t = \frac{1}{2}(\frac{k}{Y_t} - aY_t)dt + \frac{\sigma}{2}dB_t^H$, $\{B_t^H, t \geq 0\}$ is a fractional Brownian motion with an arbitrary Hurst parameter $H \in (0, 1)$. We prove that X_t satisfies the stochastic differential equation of the form $dX_t = (k - aX_t)dt + \sigma\sqrt{X_t} \circ dB_t^H$, where the integral with respect to fractional Brownian motion is considered as the pathwise Stratonovich integral. We also show that for $k > 0$, $H > 1/2$ the process is strictly positive and never hits zero, so that actually $X_t = Y_t^2$. Finally, we prove that in the case of $H < 1/2$ the probability of not hitting zero on any fixed finite interval by the fractional Cox–Ingersoll–Ross process tends to 1 as $k \to \infty$.

Keywords Fractional Cox–Ingersoll–Ross process, stochastic differential equation, Stratonovich integral

2010 MSC 60G22, 60H05, 60H10

1 Introduction

The classical Cox–Ingersoll–Ross (CIR) process, which was proposed and studied by Cox, Ingersoll and Ross in [4–6], is the process $r = \{r_t, t \geq 0\}$ that satisfies the following stochastic differential equation:

$$dr_t = (k - ar_t)dt + \sigma\sqrt{r_t}dW_t, \quad a, k, \sigma > 0. \tag{1}$$

*Corresponding author.

Here a corresponds to the speed of adjustment, k/a is called "the mean", σ is "the volatility", $W = \{W_t, t \geq 0\}$ is a Wiener process and $r_0 > 0$.

The CIR process is widely used for short-term interest rate modeling as well as for stochastic volatility modeling in the Heston model [10]. Therefore, in most cases it is also assumed that $2k \geq \sigma^2$ as, according to [9], if this condition holds, the process is strictly positive and never hits zero.

It is well known that the CIR process is ergodic and has a stationary distribution. Moreover, the distribution of its future values r_{t+T} provided that r_t is known is a non-central chi-square distribution, and the distribution of the limit value r_∞ is a gamma distribution.

However, the real financial models are often characterized by the so-called "memory phenomenon" (see [1, 2, 7, 18] for more detail), while the standard Cox–Ingersoll–Ross process does not display it. Therefore, for a better simulation of interest rates or stochastic volatility it is reasonable to consider a fractional generalization of the Cox–Ingersoll–Ross process. It should be noted that there are several approaches to definition of the fractional Cox–Ingersoll–Ross process. In [11, 12] the fractional CIR process is introduced as a time changed CIR process with inverse stable subordinator, the so-called "rough path approach" is described in [13]. Another way of defining the considered process is presented in [8] as part of the discussion on rough Heston models.

The definition of the fractional CIR process with $k = 0$, based on the pathwise integration with respect to fractional Brownian motion, was also presented in [14] for the case $H > 2/3$. In [16] it was shown that for such definition the fractional CIR process is the square of the fractional Ornstein–Uhlenbeck process until the first zero hitting (for definition and properties of the fractional Ornstein–Uhlenbeck process see [3]). Based on that, the fractional CIR process with $k = 0$ was defined as the square of the fractional Ornstein–Uhlenbeck process before its first zero hitting. It was also shown that such process satisfies the stochastic differential equation of the form

$$dX_t = aX_t dt + \sigma\sqrt{X_t} \circ dB_t^H, \quad t \geq 0, \tag{2}$$

where $X_0 > 0, a \in \mathbb{R}, \sigma > 0, H \in (0, 1)$ and integral with respect to the fractional Brownian motion is the pathwise Stratonovich integral. However, due to positive probability of hitting zero, this process is not suitable for interest rate modeling.

In this paper we introduce a natural generalization of the above model. First, we consider the process $Y = \{Y_t, t \geq 0\}$ which satisfies the SDE of the form

$$dY_t = \frac{1}{2}\left(\frac{k}{Y_t} - aY_t\right)dt + \frac{\sigma}{2}dB_t^H, \quad Y_0 > 0. \tag{3}$$

Then, we define the fractional Cox–Ingersoll–Ross process as the square of Y_t until the first zero hitting moment and show that it satisfies the SDE of the form

$$dX_t = (k - aX_t)dt + \sigma\sqrt{X_t} \circ dB_t^H, \quad t \geq 0, \tag{4}$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral. We also show that for any $k > 0$ and for any Hurst parameter $H > 1/2$ the process is strictly positive and never hits zero.

Next, for the case of $H < 1/2$, we prove that the probability that the fractional CIR process does not hit zero on any fixed finite interval tends to 1 as $k \to \infty$. As an auxiliary result, we prove the analogue of the comparison theorem.

The paper is organized as follows. In Section 2 we define the fractional CIR process and show that it satisfies the SDE (7) with the pathwise Stratonovich integral. In Section 3 we prove that the fractional CIR process is strictly positive for $k > 0$ and $H > 1/2$. In Section 4 we prove the analogue of the comparison theorem and show that in the case of $H < 1/2$ the probability that the fractional CIR process does not hit zero on any fixed finite interval tends to 1 as $k \to \infty$. In Appendix there are simulations that illustrate the results of the paper.

2 Definition of the fractional Cox–Ingersoll–Ross process

Consider the process $Y = \{Y_t, t \geq 0\}$ that satisfies the following SDE until its first zero hitting:

$$dY_t = \frac{1}{2}\left(\frac{k}{Y_t} - aY_t\right)dt + \frac{\sigma}{2}dB_t^H, \quad Y_0 > 0, \tag{5}$$

where $a, k \in \mathbb{R}, \sigma > 0$ and $\{B_t^H, t \geq 0\}$ is a fractional Brownian motion with the Hurst parameter $H \in (0, 1)$.

Definition 1. Let $H \in (0, 1)$ be an arbitrary Hurst index, $\{Y_t, t \geq 0\}$ be the process that satisfies the equation (5) and τ be the first moment of reaching zero by the latter. The *fractional Cox–Ingersoll–Ross* process is the process $\{X_t, t \geq 0\}$ such that for all $t \geq 0, \omega \in \Omega$:

$$X_t(\omega) = Y_t^2(\omega)\mathbf{1}_{\{t<\tau(\omega)\}}. \tag{6}$$

Before moving to the main result of this section, let us give the definition of the pathwise Stratonovich integral.

Definition 2. Let $\{X_t, t \geq 0\}$, $\{Y_t, t \geq 0\}$ be random processes. The pathwise *Stratonovich integral* $\int_0^T X_s \circ dY_s$ is a pathwise limit of the following sums

$$\sum_{k=1}^{n} \frac{X_{t_k} + X_{t_{k-1}}}{2}(Y_{t_k} - Y_{t_{k-1}}),$$

as the mesh of the partition $0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = T$ tends to zero, in case if this limit exists.

Theorem 1. *Let $\tau := \inf\{s > 0 : Y_s = 0\}$. For $0 \leq t \leq \tau$ the fractional CIR process from Definition 1 satisfies the following SDE:*

$$dX_t = (k - aX_t)dt + \sigma\sqrt{X_t} \circ dB_t^H, \tag{7}$$

where $X_0 = Y_0^2 > 0$ and the integral with respect to the fractional Brownian motion is defined as the pathwise Stratonovich integral.

Proof. Let us fix an $\omega \in \Omega$ and consider an arbitrary $t < \tau(\omega)$.

According to (5) and (6),

$$X_t = Y_t^2 = \left(\sqrt{X_0} + \frac{1}{2}\int_0^t \left(\frac{k}{Y_s} - aY_s\right)ds + \frac{\sigma}{2}B_t^H\right)^2. \tag{8}$$

Consider an arbitrary partition of the interval $[0, t]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = t.$$

Using (8), we get

$$\begin{aligned} X_t &= \sum_{i=1}^n (X_{t_i} - X_{t_{i-1}}) + X_0 \\ &= \sum_{i=1}^n \Bigg(\left[\sqrt{X_0} + \frac{1}{2}\int_0^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \frac{\sigma}{2}B_{t_i}^H\right]^2 \\ &\quad - \left[\sqrt{X_0} + \frac{1}{2}\int_0^{t_{i-1}}\left(\frac{k}{Y_s} - aY_s\right)ds + \frac{\sigma}{2}B_{t_{i-1}}^H\right]^2\Bigg) + X_0. \end{aligned}$$

Factoring each summand as the difference of squares, we get:

$$\begin{aligned} X_t = X_0 + \sum_{i=1}^n &\Bigg[2\sqrt{X_0} + \frac{1}{2}\Bigg(\int_0^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds \\ &+ \int_0^{t_{i-1}}\left(\frac{k}{Y_s} - aY_s\right)ds\Bigg) + \frac{\sigma}{2}(B_{t_i}^H + B_{t_{i-1}}^H)\Bigg] \\ &\times \left[\frac{1}{2}\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \frac{\sigma}{2}(B_{t_i}^H - B_{t_{i-1}}^H)\right]. \end{aligned}$$

Expanding the brackets in the last expression, we obtain:

$$\begin{aligned} X_t &= X_0 + \sum_{i=1}^n \sqrt{X_0}\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds \\ &\quad + \frac{1}{4}\sum_{i=1}^n \left(\int_0^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \int_0^{t_{i-1}}\left(\frac{k}{Y_s} - aY_s\right)ds\right) \\ &\quad \times \int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \frac{\sigma}{4}\sum_{i=1}^n (B_{t_i}^H + B_{t_{i-1}}^H)\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds \\ &\quad + \sigma\sqrt{X_0}\sum_{i=1}^n (B_{t_i}^H - B_{t_{i-1}}^H) + \frac{\sigma^2}{4}\sum_{i=1}^n (B_{t_i}^H - B_{t_{i-1}}^H)(B_{t_i}^H + B_{t_{i-1}}^H) \\ &\quad + \frac{\sigma}{4}\sum_{i=1}^n \left(\int_0^{t_i}\left(\frac{k}{Y_s} - aY_s\right)ds + \int_0^{t_{i-1}}\left(\frac{k}{Y_s} - aY_s\right)ds\right)(B_{t_i}^H - B_{t_{i-1}}^H). \end{aligned} \tag{9}$$

Let the mesh Δt of the partition tend to zero. The first three summands

$$\sum_{i=1}^{n}\sqrt{X_0}\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s}-aY_s\right)ds$$
$$+\frac{1}{4}\sum_{i=1}^{n}\left(\int_0^{t_i}\left(\frac{k}{Y_s}-aY_s\right)ds+\int_0^{t_{i-1}}\left(\frac{k}{Y_s}-aY_s\right)ds\right)$$
$$\times\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s}-aY_s\right)ds+\frac{\sigma}{4}\sum_{i=1}^{n}\left(B_{t_i}^H+B_{t_{i-1}}^H\right)\int_{t_{i-1}}^{t_i}\left(\frac{k}{Y_s}-aY_s\right)ds$$
$$\to\int_0^t\left(\frac{k}{Y_s}-aY_s\right)\left(\sqrt{X_0}+\frac{1}{2}\int_0^s\left(\frac{k}{Y_u}-aY_u\right)du+\frac{\sigma}{2}B_s^H\right)ds$$
$$=\int_0^t\left(k-aY_s^2\right)ds=\int_0^t(k-aX_s)ds,\quad \Delta t\to 0, \tag{10}$$

and the last three summands

$$\sigma\sqrt{X_0}\sum_{i=1}^{n}\left(B_{t_i}^H-B_{t_{i-1}}^H\right)+\frac{\sigma^2}{4}\sum_{i=1}^{n}\left(B_{t_i}^H-B_{t_{i-1}}^H\right)\left(B_{t_i}^H+B_{t_{i-1}}^H\right)$$
$$+\frac{\sigma}{4}\sum_{i=1}^{n}\left(\int_0^{t_i}\left(\frac{k}{Y_s}-aY_s\right)ds+\int_0^{t_{i-1}}\left(\frac{k}{Y_s}-aY_s\right)ds\right)\left(B_{t_i}^H-B_{t_{i-1}}^H\right)$$
$$\to\sigma\int_0^t\left(\sqrt{X_0}+\frac{1}{2}\int_0^s\left(\frac{k}{Y_u}-aY_u\right)du+\frac{\sigma}{2}B_s^H\right)\circ dB_s^H$$
$$=\sigma\int_0^t Y_s\circ dB_s^H=\sigma\int_0^t\sqrt{X_s}\circ dB_s^H,\quad \Delta t\to 0. \tag{11}$$

Note that the left-hand side of (9) does not depend on the partition and the limit in (10) exists as the pathwise Riemann integral, therefore the corresponding pathwise Stratonovich integral exists and the passage to the limit in (11) is correct.

Thus, the fractional Cox–Ingersoll–Ross process, introduced in Definition 1, satisfies the SDE of the form

$$X_t=X_0+\int_0^t(k-aX_s)ds+\sigma\int_0^t\sqrt{X_s}\circ dB_s^H, \tag{12}$$

where $\int_0^t\sqrt{X_s}\circ dB_s^H$ is the pathwise Stratonovich integral. □

Remark 1. In the case of $k=0$, the process (5) is the fractional Ornstein–Uhlenbeck process and the definition coincides with the one given in [16].

3 Hitting zero by the fractional CIR process with positive "mean" and H > 1/2

The next natural question regarding the fractional CIR process is finiteness of its zero hitting time moment. It is obvious that it coincides with the respective moment of the process $\{Y_t, t\geq 0\}$, defined by the equation (5).

Before formulating the main result of the section let us give a well-known property of trajectories of fractional Brownian motion (see, for example, [15]).

Proposition 1. *Let $\{B_t^H, t \geq 0\}$ be a fractional Brownian motion with the Hurst index H. Then, $\exists \Omega' \subset \Omega$, $\mathbb{P}\{\Omega'\} = 1$, such that $\forall \omega \in \Omega'$, $\forall T > 0$, $\forall \delta > 0$, $\forall 0 \leq s \leq t \leq T$ $\exists C = C(T, \omega, \delta) \in \mathbb{R}$:*

$$\left|B_t^H - B_s^H\right| \leq C|t-s|^{H-\delta}.$$

Theorem 2. *Let $k > 0$, $H > 1/2$. Then the process $\{Y_t, t \geq 0\}$, defined by the equation (5), is strictly positive a.s.*

Proof. The proof is by contradiction.

Let Ω' be the same as in Proposition 1. First, assume that $a > 0$ and let for some $\omega \in \Omega'$, $\tau(\omega) = \inf\{t > 0 : X_t = 0\} = \inf\{t > 0 : Y_t = 0\} < \infty$.

For all $\varepsilon \in (0, \min(Y_0, \sqrt{\frac{k}{a}}))$ (the condition $\varepsilon < \sqrt{\frac{k}{a}}$ provides the inequality $\frac{k}{\varepsilon} - a\varepsilon > 0$) let us introduce the last moment of hitting the level of ε before the first zero reaching:

$$\tau_\varepsilon := \sup\left\{t \in (0, \tau) : Y_t = \varepsilon\right\}.$$

Consider $\delta > 0$ such that the inequality $H - \delta > 1/2 \Leftrightarrow 1 + \delta - H < 1/2$ holds. According to the definitions of τ, τ_ε and Y, the following equality is true:

$$-\varepsilon = Y_\tau - Y_{\tau_\varepsilon} = \frac{1}{2}\int_{\tau_\varepsilon}^{\tau} \left(\frac{k}{Y_s} - aY_s\right) ds + \frac{\sigma}{2}\left(B_\tau^H - B_{\tau_\varepsilon}^H\right).$$

The process $Y_s \in (0, \varepsilon)$ on the interval (τ_ε, τ), hence $\forall s \in (\tau_\varepsilon, \tau)$:

$$\frac{k}{Y_s} - aY_s \geq \frac{k}{\varepsilon} - a\varepsilon. \tag{13}$$

From this and Proposition 1, it follows that $\exists C = C(\tau(\omega), \omega, \delta)$:

$$\frac{\sigma}{2}C|\tau - \tau_\varepsilon|^{H-\delta} \geq \frac{\sigma}{2}|B_\tau^H - B_{\tau_\varepsilon}^H| \geq \frac{1}{2}\left(\frac{k}{\varepsilon} - a\varepsilon\right)(\tau - \tau_\varepsilon) + \varepsilon.$$

It is clear that there exists $\tilde{\varepsilon} > 0$ such that $\forall \varepsilon < \tilde{\varepsilon}$: $\frac{k}{\varepsilon} - a\varepsilon > \frac{k}{2\varepsilon}$. Then, by choosing an arbitrary $\varepsilon < \tilde{\varepsilon}$, we have:

$$\frac{\sigma}{2}C|\tau - \tau_\varepsilon|^{H-\delta} \geq \frac{k}{4\varepsilon}(\tau - \tau_\varepsilon) + \varepsilon. \tag{14}$$

For $x \geq 0$ consider the function

$$F_\varepsilon(x) = \frac{k}{4\varepsilon}x - \frac{\sigma}{2}Cx^{H-\delta} + \varepsilon. \tag{15}$$

Let us show that there exists $\varepsilon^* \in (0, \tilde{\varepsilon})$ such that, for all $\varepsilon < \varepsilon^*$ and for all $x \geq 0$, $F_\varepsilon(x) > 0$. It is easy to check that $F_\varepsilon(0) = \varepsilon > 0$ and F_ε is convex on $\mathbb{R}^+\backslash\{0\}$ (its second derivative is strictly positive on this set), so it is enough to examine the sign of the function in its critical points.

$$F'(\tilde{x}) = \frac{k}{4\varepsilon} - \frac{\sigma(H-\delta)}{2} C\tilde{x}^{H-\delta-1} = 0$$
$$\Longrightarrow \quad \tilde{x} = \left(\frac{k}{2\sigma\varepsilon C(H-\delta)}\right)^{1/(H-\delta-1)}$$
$$= \left(\frac{2\sigma C(H-\delta)}{k}\right)^{1/(1+\delta-H)} \varepsilon^{1/(1+\delta-H)}.$$

After some calculations we get

$$F(\tilde{x}) = \frac{1}{2}\left(\frac{2(H-\delta)}{k}\right)^{\frac{H-\delta}{1+\delta-H}} (\sigma C)^{\frac{1}{1+\delta-H}} (H-\delta-1)\varepsilon^{\frac{H-\delta}{1+\delta-H}} + \varepsilon.$$

From the choice of δ it follows that $\frac{H-\delta}{1+\delta-H} > 1$, hence $\forall K \in \mathbb{R} \quad \exists \varepsilon^* > 0$:

$$\varepsilon - K\varepsilon^{\frac{H-\delta}{1+\delta-H}} > 0, \quad \forall \varepsilon < \varepsilon^*. \tag{16}$$

Choosing the corresponding ε^* for

$$K := -\frac{1}{2}\left(\frac{2(H-\delta)}{k}\right)^{\frac{H-\delta}{1+\delta-H}} (\sigma C)^{\frac{1}{1+\delta-H}} (H-\delta-1),$$

and choosing an arbitrary $\varepsilon < \min\{\tilde{\varepsilon}, \varepsilon^*\}$ we obtain that

$$F_\varepsilon(x) > 0 \quad \forall x > 0.$$

However, from (14) it follows that

$$F_\varepsilon(\tau - \tau_\varepsilon) \le 0.$$

The contradiction obtained proves the theorem for $a > 0$. If $a \le 0$, instead of (13) the following bound can be used:

$$\frac{k}{Y_s} - aY_s \ge \frac{k}{\varepsilon}. \tag{17}$$

□

4 Hitting zero by the fractional CIR process in the case of $H < 1/2$

The condition of $H > 1/2$ is essential for Theorem 2, as if $H < 1/2$, the condition (16) does not hold. However, it is possible to obtain another result concerning zero hitting by the fractional CIR process in the case of $H < 1/2$.

Let $\{B_t^H, t \ge 0\}$ be the fractional Brownian motion with $H < 1/2$ and let $a \in \mathbb{R}$, $\sigma > 0$ be fixed. Consider the set of processes

$$\mathbb{Y} := \left\{Y^{(k)} = \left\{Y_t^{(k)}, t \ge 0\right\}, k > 0\right\}, \tag{18}$$

each element of which starts from the same level $Y_0 > 0$, satisfies the SDE of the form (5) before hitting zero and remains in zero after that moment:

$$Y_t^{(k)}(\omega) = \begin{cases} Y_0 + \frac{1}{2}\int_0^t (\frac{k}{Y_s^{(k)}(\omega)} - aY_s^{(k)}(\omega))ds + \frac{\sigma}{2} dB_t^H(\omega), & \text{if } t < \tau^{(k)}(\omega), \\ 0, & \text{if } t \geq \tau^{(k)}(\omega), \end{cases}$$

where $\tau^{(k)} := \inf\{t \geq 0 \mid Y_t^{(k)} = 0\}$.

Lemma 1. *Let $k_1 < k_2$. Then $\forall \omega \in \Omega, \forall t \geq 0$:*

(i) $\tau^{(k_1)}(\omega) \leq \tau^{(k_2)}(\omega)$*;*

(ii) $Y_t^{(k_1)}(\omega) \leq Y_t^{(k_2)}(\omega)$*, and the inequality is strict for* $t \in (0, \tau^{(k_2)}(\omega))$.

Remark 2. This lemma holds for an arbitrary Hurst index $H \in (0, 1)$.

Proof. Let $\omega \in \Omega$ be fixed (we will omit ω in brackets in further formulas). Consider the function δ on the interval $[0, \min\{\tau^{(k_1)}, \tau^{(k_2)}\})$, such that $\delta(t) = Y_t^{(k_2)} - Y_t^{(k_1)}$. It is obvious that δ is differentiable, $\delta(0) = 0$ and

$$\delta'_+(0) = \frac{1}{2}\left(\frac{k_2}{Y_0} - aY_0\right) - \frac{1}{2}\left(\frac{k_1}{Y_0} - aY_0\right) = \frac{k_2 - k_1}{2Y_0} > 0.$$

As $\delta(t) = \delta'_+(0)t + o(t)$, $t \to 0+$, it is easy to see that there exists the maximal interval $(0, t^*) \subset (0, \min\{\tau^{(k_1)}, \tau^{(k_2)}\})$ such that $\delta(t) > 0$ for all $t \in (0, t^*)$. It is also clear that

$$t^* = \sup\big\{t \in \big(0, \min\big\{\tau^{(k_1)}, \tau^{(k_2)}\big\}\big) \mid \forall s \in (0, t) : \delta(s) > 0\big\}.$$

Assume that $t^* < \min\{\tau^{(k_1)}, \tau^{(k_2)}\}$. According to the definition of t^*, $\delta(t^*) = 0$. Hence $Y_{t^*}^{(k_2)} = Y_{t^*}^{(k_1)} = Y^* > 0$ and

$$\delta'\big(t^*\big) = \frac{k_2 - k_1}{2Y^*} > 0.$$

As $\delta(t) = \delta'(t^*)(t - t^*) + o(t - t^*)$, $t \to t^*$, there exists $\varepsilon > 0$ such that $\delta(t) < 0$ for all $t \in (t^* - \varepsilon, t^*)$, that contradicts the definition of t^*.

Therefore, $\forall t \in (0, \min\{\tau^{(k_1)}, \tau^{(k_2)}\})$:

$$Y_t^{(k_2)} > Y_t^{(k_1)}. \tag{19}$$

Now it is easy to show that (i) holds: indeed, if $\tau^{(k_1)} > \tau^{(k_2)}$, then

$$0 = Y_{\tau^{(k_2)}}^{(k_2)} < Y_{\tau^{(k_2)}}^{(k_1)}.$$

This means that $\exists t_* < \tau^{(k_2)}$ such that $Y_t^{(k_2)} < Y_t^{(k_1)}$ for all $t \in (t_*, \tau^{(k_2)})$, which contradicts (19).

Finally, as $Y_t^{(k_2)} > Y_t^{(k_1)}$ for all $t \in (0, \tau^{(k_1)})$, $Y_t^{(k_2)} > Y_t^{(k_1)} = 0$ for all $t \in [\tau^{(k_1)}, \tau^{(k_2)})$ and $Y_t^{(k_2)} = Y_t^{(k_1)} = 0$ for all $t \geq \tau^{(k_2)}$, (ii) also holds. □

Now let us move to the main result of the section.

Theorem 3. *For all $T > 0$:*

$$\mathbb{P}\big(\tau^{(k)} > T\big) \to 1, \quad k \to \infty. \tag{20}$$

Proof. The proof is by contradiction.

Assume that $\exists T^* > 0$, $\exists\{k_n, n \geq 1\}$, $k_n \uparrow \infty$ as $n \to \infty$ such that:

$$\mathbb{P}\big(\tau^{(k_n)} \leq T^*\big) \to \alpha > 0, \quad n \to \infty.$$

Let us consider the case of $a > 0$. Let Ω' be from Proposition 1, and for all $\varepsilon \in (0, \min(Y_0, 1, \sqrt{\frac{k_1}{2a}}))$ denote $\tau_\varepsilon^{(k_n)} := \sup\{t \in (0, \tau) : Y_t^{(k_n)} = \varepsilon\}$ and

$$D_{T^*}^{(k_n)} := \big\{\omega \in \Omega' \mid \tau^{(k_n)} \leq T^*\big\}.$$

According to Lemma 1, $\forall n \geq 1 : D_{T^*}^{(k_{n+1})} \subset D_{T^*}^{(k_n)}$, so

$$\mathbb{P}\Big(\bigcap_{n\geq 1} D_{T^*}^{(k_n)}\Big) = \lim_{n\to\infty} \mathbb{P}\big(D_{T^*}^{(k_n)}\big) = \alpha > 0.$$

Just like in Theorem 2, $\forall n \geq 1$, $\forall \omega \in D_{T^*}^{(k_n)}$:

$$-\varepsilon = Y_{\tau^{(k_n)}}^{(k_n)} - Y_{\tau_\varepsilon^{(k_n)}}^{(k_n)} = \frac{1}{2}\int_{\tau_\varepsilon^{(k_n)}}^{\tau^{(k_n)}} \bigg(\frac{k_n}{Y_s^{(k_n)}} - aY_s^{(k_n)}\bigg) ds + \frac{\sigma}{2}\big(B_{\tau^{(k_n)}}^H - B_{\tau_\varepsilon^{(k_n)}}^H\big).$$

The process $Y_s^{(k_n)} \in (0, \varepsilon)$ on the interval $(\tau_\varepsilon^{(k_n)}, \tau^{(k_n)})$, hence

$$\frac{k_n}{Y_s^{(k_n)}} - aY_s^{(k_n)} \geq \frac{k_n}{\varepsilon} - a\varepsilon, \quad \forall s \in \big(\tau_\varepsilon^{(k_n)}, \tau^{(k_n)}\big). \tag{21}$$

Let $\delta > 0$ satisfy the condition $0 < H - \delta < 1/2$.

According to Proposition 1, $\exists C_\omega = C(T^*, \omega, \delta)$, $\forall 0 < s < t < T^*$:

$$\big|B_t^H - B_s^H\big| \leq C_\omega |t - s|^{H-\delta}.$$

As $\varepsilon < \sqrt{\frac{k_1}{2a}}$, the following inequality is true:

$$\frac{k_n}{\varepsilon} - a\varepsilon > \frac{k_n}{2\varepsilon} \quad \forall n \geq 0,$$

so just like in the proof of Theorem 2 we can obtain that $\forall n \geq 1$, $\forall \omega \in D_{T^*}^{(k_n)}$:

$$\frac{\sigma}{2} C_\omega\big(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\big)^{H-\delta} \geq \frac{k_n}{4\varepsilon}\big(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\big) + \varepsilon. \tag{22}$$

According to (22), $\forall n \geq 1$, $\forall \omega \in \bigcap_{n\geq 0} D_{T^*}^{(k_n)}$:

$$\frac{\sigma}{2} C_\omega\big(\tau^{(k_n)} - \tau_\varepsilon^{(k_n)}\big)^{H-\delta} > \varepsilon. \tag{23}$$

However, it is easy to see from (22) that

$$\frac{\sigma}{2}C_\omega\big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big)^{H-\delta} > \frac{k_n}{4\varepsilon}\big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big). \tag{24}$$

Let us transform (24):

$$\frac{\sigma}{2}C_\omega > \frac{k_n}{4\varepsilon}\big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big)^{1-H+\delta},$$
$$\frac{2\sigma\varepsilon}{k_n}C_\omega > \big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big)^{1-H+\delta},$$
$$\left(\frac{2\sigma\varepsilon}{k_n}C_\omega\right)^{\frac{1}{1-H+\delta}} > \tau^{(k_n)}-\tau_\varepsilon^{(k_n)},$$

hence

$$\begin{aligned}\frac{\sigma}{2}C_\omega\big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big)^{H-\delta} &< \frac{\sigma}{2}C_\omega\left(\frac{2\sigma\varepsilon}{k_n}C_\omega\right)^{\frac{H-\delta}{1-H+\delta}}\\ &= \big(2^{\frac{2H-2\delta-1}{1-H+\delta}}\sigma^{\frac{1}{1-H+\delta}}\big)k_n^{-\frac{H-\delta}{1-H+\delta}}C_\omega^{\frac{1}{1-H+\delta}}\varepsilon^{\frac{H-\delta}{1-H+\delta}}\\ &= \tilde{C}k_n^{-\frac{H-\delta}{1-H+\delta}}C_\omega^{\frac{1}{1-H+\delta}}\varepsilon^{\frac{H-\delta}{1-H+\delta}}.\end{aligned}$$

According to [17], $\mathbb{E}(|C_\omega|^p) < \infty$ for all $p \in [1,\infty)$, so C_ω is finite a.s.

Therefore, as $\mathbb{P}(\bigcap_{n\ge 0} D_{T^*}^{(k_n)}) = \alpha > 0$, $\exists M > 0$, $\exists E \subset \bigcap_{n\ge 0} D_{T^*}^{(k_n)}$, $\mathbb{P}(E) > 0$ such that $\forall\omega \in E$:

$$C_\omega < M.$$

Hence, as $\varepsilon < 1$,

$$\begin{aligned}\frac{\sigma}{2}C_\omega\big(\tau^{(k_n)}-\tau_\varepsilon^{(k_n)}\big)^{H-\delta} &< \tilde{C}k_n^{-\frac{H-\delta}{1-H+\delta}}C_\omega^{\frac{1}{1-H+\delta}}\varepsilon^{\frac{H-\delta}{1-H+\delta}}\\ &< \tilde{C}k_n^{-\frac{H-\delta}{1-H+\delta}}M^{\frac{1}{1-H+\delta}} < \varepsilon,\end{aligned}$$

if $k_n > \big(\frac{\tilde{C}M^{\frac{1}{1-H+\delta}}}{\varepsilon}\big)^{\frac{1-H+\delta}{H-\delta}}$, which contradicts (23).

If $a < 0$, the following inequality can be used instead of (21):

$$\frac{k_n}{Y_s^{(k_n)}} - aY_s^{(k_n)} \ge \frac{k_n}{\varepsilon}.$$

□

A Appendix: Simulations of the fractional Cox–Ingersoll–Ross process

Theorems 2 and 3 can be illustrated by numerical simulations.

10000 sample paths of the fractional Cox–Ingersoll–Ross process were simulated on the interval [0, 10] as the square of the process Y defined in (5). The Euler approximation of Y was used until the first zero hitting by the latter with the mesh of the partition of $\Delta t = 0.001$:

$$Y_{t_n} = \begin{cases} Y_{t_{n-1}} + \frac{1}{2}\big(\frac{k}{Y_{t_{n-1}}} - aY_{t_{n-1}}\big)\Delta t + \frac{\sigma}{2}\Delta B_{t_n}^H, & \text{if } Y_{t_{n-1}} > 0,\\ 0, & \text{if } Y_{t_{n-1}} \le 0,\end{cases}$$
$$X_{t_n} = Y_{t_n}^2.$$

There were no zero hitting for 10000 trajectories simulated for four cases that satisfy the conditions of Theorem 2 (see Fig. 1, 2; the amount of trajectories on these and further figures is reduced in order to make them more convenient for the reader).

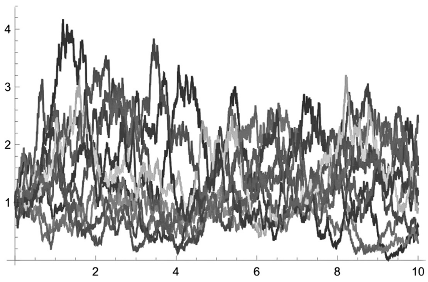

Fig. 1. Case of $a = 1, k = 1, \sigma = 1, H = 0.6$, $X_0 = 1$

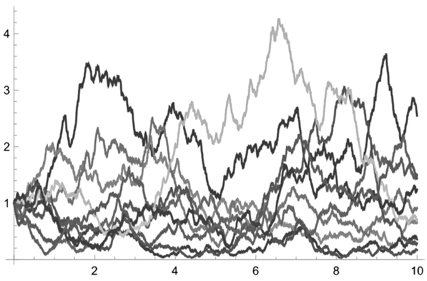

Fig. 2. Case of $a = 1, k = 1, \sigma = 1, H = 0.8$, $X_0 = 1$

However, the behavior of the fractional Cox–Ingersoll–Ross process is not completely clear for the situation of $k > 0$, $H < 1/2$. Simulations for different parameters k, H, σ (see Figures 3–8) show that in this case the process may hit zero with positive probability, however, as stated in Theorem 3, as k gets bigger, the amount of trajec-

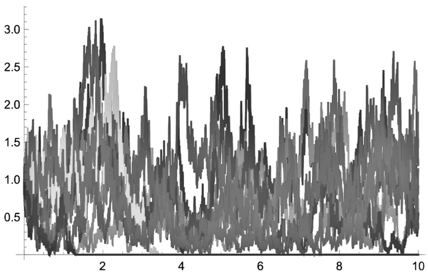

Fig. 3. Case of $a = 1, k = 0.5, \sigma = 1, H = 0.4$, $X_0 = 1$. 17% of sample paths hit zero

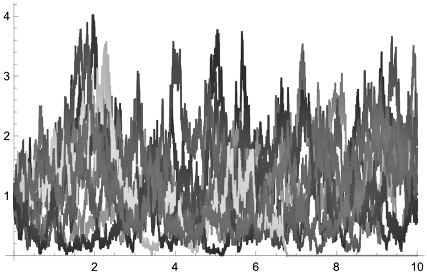

Fig. 4. Case of $a = 1, k = 1, \sigma = 1, H = 0.4$, $X_0 = 1$. Less than 1% of sample paths hit zero

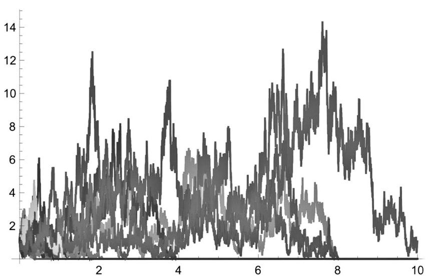

Fig. 5. Case of $a = 1, k = 1, \sigma = 2, H = 0.4$, $X_0 = 1$. 86% of sample paths hit zero

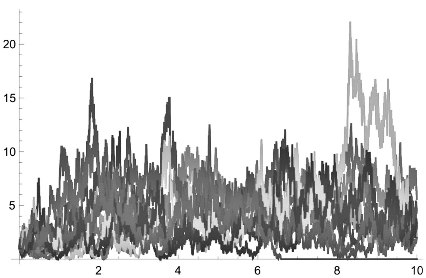

Fig. 6. Case of $a = 1, k = 3, \sigma = 2, H = 0.4$, $X_0 = 1$. Less than 1% of sample paths hit zero

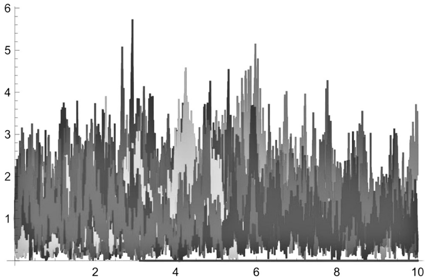

Fig. 7. Case of $a = 1, k = 1, \sigma = 1, H = 0.2$, $X_0 = 1$. 43% of sample paths hit zero

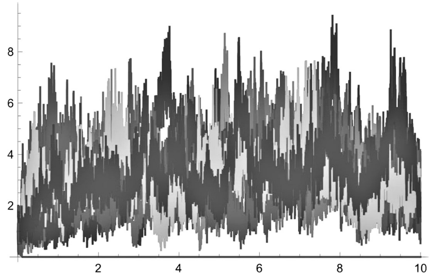

Fig. 8. Case of $a = 1, k = 3, \sigma = 1, H = 0.2$, $X_0 = 1$. Less than 1% of sample paths hit zero

tories that hit zero tends to zero. Moreover, it seems that the less H and the bigger σ are, the bigger is the probability of reaching zero.

References

[1] Anh, V., Inoue, A.: Financial markets with memory I: Dynamic models. Stoch. Anal. Appl. **23**(2), 275–300 (2005). MR2130350

[2] Bollerslev, T., Mikkelsen, H.O.: Modelling and pricing long memory in stock market volatility. J. Econometrics **73**(1), 151–184 (2005). MR0054814

[3] Cheridito, P., Kawaguchi, H., Maejima, M.: Fractional Ornstein-Uhlenbeck processes. Electron. J. Probab **8**(1), 1–14 (2003).

[4] Cox, J.C., Ingersoll, J.E., Ross, S.A.: A re-examination of traditional hypotheses about the term structure of interest rates. J. Finance **36**, 769–799 (1981).

[5] Cox, J.C., Ingersoll, J.E., Ross, S.A.: An intertemporal general equilibrium model of asset prices. Econometrica **53**(1), 363–384 (1985).

[6] Cox, J.C., Ingersoll, J.E., Ross, S.A.: A theory of the term structure of interest rates. J. Finance **53**(2), 385–408 (1985). MR0785475

[7] Ding, Z., Granger, C.W., Engle, R.F.: A long memory property of stock market returns and a new model. J. Empirical Finance **1**(1), 83–106 (1993).

[8] Euch, O., Rosenbaum, M.: The characteristic function of rough Heston models. https://arxiv.org/pdf/1609.02108.pdf. Accessed 24 Sep 2017. arXiv:1609.02108

[9] Feller, W.: Two singular diffusion problems. Annals of Mathematics **54**, 173–182 (1951).

[10] Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies **6**(2), 327–343 (1993).

[11] Leonenko, N., Meerschaert, M., Sikorskii, A.: Correlation structure of fractional Pearson diffusion. Comput. Math. Appl. **66**(5), 737–745 (2013). MR3089382

[12] Leonenko, N., Meerschaert, M., Sikorskii, A.: Fractional Pearson diffusion. J. Math. Anal. Appl. **403**(2), 532–546 (2013).

[13] Marie, N.: A generalized mean-reverting equation and applications. ESAIM Probab. Stat. **18**, 799–828 (2014). MR3334015

[14] Melnikov, A., Mishura, Y., Shevchenko, G.: Stochastic viability and comparison theorems for mixed stochastic differential equations. Methodol. Comput. Appl. Probab. **17**(1), 169–188 (2015). MR3306678

[15] Mishura, Y.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, Berlin (2008).

[16] Mishura, Y., Piterbarg, V., Ralchenko, K., Yurchenko-Tytarenko, A.: Stochastic representation and pathwise properties of fractional Cox-Ingersoll-Ross process (ukr). Theory of Probability and Mathematical Statistics **97**, 157–170 (2017).

[17] Nualart, D., Rascanu, A.: Differential equations driven by fractional Brownian motion. Collectanea Mathematica **53**(1), 177–193 (2002).

[18] Yamasaki, K., Muchnik, L., Havlin, S., Bunde, A., Stanley, H.E.: Scaling and memory in volatility return intervals in financial markets. Proc. Natl. Acad. Sci. USA **102**(26), 9424–9428 (2005).